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PRESENT VALUE RELATIONS, GRANGER NONCAUSALITY, AND VAR STABILITY

LUCA FANELLI

When in the class of "exact" present value (PV) relations the decision variables do not Granger cause the explanatory variables, and a vector autoregressive (VAR) process is used to derive the cross-equation restrictions, the system embodies explosive roots, which hardly can be reconciled with the typical features observed in most macroeconomic time series. This paper investigates the issue.

1. PROBLEM AND MOTIVATION

An "exact" present value (PV) relation between the $m \times 1$ vector of decision variables, y_t, and the $q \times 1$ vector of explanatory (forcing) variables, z_t, can be formulated as

$$y_t = \omega_0 y_{t-1} + \sum_{j=0}^{\infty} \omega_1^j \gamma \theta E_t z_{t+j} + \varsigma, \tag{1}$$

where ω_h, $h = 0, 1$ and γ are $m \times m$ matrices, θ is $m \times q$, ς is an $m \times 1$ constant, and $E_t \cdot := E(\cdot | \mathcal{F}_t)$ denotes expectations conditional on the sigma algebra $\mathcal{F}_t$, $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$, summarizing agent's information at time t.[1] It is assumed that the coefficients of the system of equations (1) satisfy the following conditions.

A1. For each $i, h = 1, \ldots, m$, the elements γ_{ih} of γ depend on $\varpi_0 = \text{vec}(\omega_0)$ and $\varpi_1 = \text{vec}(\omega_1)$ through differentiable functions $\gamma_{ih} := \gamma_{i,h}(\varpi_0' : \varpi_1')$, $\gamma_{i,h}(\cdot) : S \to R$, with S an open subset of R^{m^2}.

A2. ω_1 is nonsingular and has stable eigenvalues, i.e., lying within the unit circle in the complex plane.

A3. ω_0 has stable eigenvalues.

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The elements of the θ matrix in the relation (1) reflect the link between y_t and z_t and usually depend on agents' preferences and technology. The matrices ω_h, $h = 0, 1$ and γ typically embody adjustment-type parameters. Usually the elements of ω_1 depend on a time-invariant discount factor δ, $(0 < \delta < 1)$, and may depend on the elements of ω_0.

Two important aspects of the PV model specified in (1) are worth stressing. First, we have deliberately left for the moment the stochastic process generating z_t unspecified. In general, however, the process generating z_t plays a key role in determining solution properties and the identification of the PV model. Second, the PV relation reads as an "exact" rational expectations model in the sense of Hansen and Sargent (1991). This means that the model does not include stochastic processes that are unobservable to the econometrician. Examples of "exact" PV models include, inter alia, Sargent (1979), Campbell and Shiller (1987), Baillie (1989), Hansen and Sargent (1991), Engsted and Haldrup (1994), Johansen and Swensen (1999), and Fanelli (2002, 2006a, 2006b).

It might be argued that Assumption A3 rules out some interesting situations, e.g., $\omega_0 = I_m$. Actually, the PV model (1) with A1–A3 is general enough to cover several special cases of interest, provided that the variables are opportunely transformed. For instance, when $\omega_0 = I_m$, one may specify a new PV relation having the form (1), with $y_t^* := \Delta y_t$ serving as the vector of decision variables, and $\omega_0 = 0_{m \times m}$. In general, when ω_0 has eigenvalues equal to one, it is possible to resort to transformations of the system that preserve the structure (1), without violating A1–A3.

A convenient formulation of (1), given the process generating z_t, reads as

$$(I_m + \omega_1 \omega_0) y_t = \omega_1 E_t y_{t+1} + \omega_0 y_{t-1} + \gamma \theta z_t + \varsigma^*, \tag{2}$$

where $\varsigma^* := (I - \omega_1)\varsigma$. The system of Euler equations (2) can be obtained from (1) through the following steps: (i) write (1) at time $t + 1$ and multiply both sides by ω_1; (ii) condition with respect to the information set Ω_t and apply the law of iterated expectations; (iii) subtract the system of equations derived in (ii) from (1) and rearrange terms. Observe that although (1) implies (2), the reverse is not generally true, unless a suitable transversality condition is imposed on, or satisfied by, the stochastic process generating y_t; see Sargent (1987).

Testable implications of (1) or of its counterpart (2) can be derived by assuming that the desired form of the rational expectations solution of (1) is a vector autoregressive (VAR) process for the $p \times 1$ $(p = m + q)$ vector $X_t := (y_t' : z_t')'$ and applying the method of undetermined coefficients; see, e.g., Bekaert and Hodrick (2001). This paper shows that when y_t does not Granger cause z_t, the restrictions that the PV relation entails on the vector autoregression process are at odds with the absence of explosive roots. In other words, the vector autoregression embodies explosive roots if X_t fulfills both the PV and Granger noncausality restrictions. Aside from "rational bubbles" (Diba and Grossman, 1988)

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and other special circumstances and episodes, explosive (unstable) roots usually imply dynamic patterns for the variables that hardly can be reconciled with the typical features observed in most macroeconomic and financial time series.

In general, there are strong economic grounds to expect feedbacks from y_t to z_t in PV relations; see Timmermann (1994).[2] Nonetheless, PV models where z_t behaves (or is treated by the econometrician) as a strongly exogenous vector of variables with respect to the parameters of interest, $(\omega_0, \omega_1, \theta)$, are frequently used in both theoretical and applied research. For instance, in the PV model for stock prices, the finance literature typically treats dividends as given exogenously and specifies univariate autoregressive moving average (ARMA) processes to describe their law of motion (Campbell and Shiller, 1988). In dynamic factor demand models, it is often assumed that firms face stochastic processes where real wages and the rental price of capital, $z_t = (w_t : c_t)'$, are not Granger caused by labor and capital, $y_t = (n_t : k_t)'$; see Meese (1980).

We conclude this section by showing that the PV formulation (1) under A1–A3 covers many of the models typically used in financial and macroeconomic time series. In the expectations theory of interest rates, $y_t = R_t$ $(m = 1)$ is a long-term yield, $z_t = r_t$ $(q = 1)$ is the one-period rate, $\omega_0 = 0$, $\omega_1 = \delta$ is the discount factor, $\gamma = (1 - \omega_1) = (1 - \delta)$, θ is a (scalar) proportionality parameter, and ς is a constant risk premium; see Campbell and Shiller (1987). In modern macroeconomic sticky pricing theories (1) follows from the assumption of forward-looking price-setting firms (Calvo model) (see, e.g., Galì and Gertler, 1999) with $y_t = \pi_t$ $(m = 1)$ being the inflation rate, $z_t = mc_t$ $(q = 1)$ a measure of firm's marginal costs, θ a structural parameter related to the degree of firm's price rigidity, and $\omega_0 = \lambda_1$, $\omega_1 = \lambda_2^{-1}$, $\gamma = (\lambda_1 + \lambda_2)(\lambda_1 \lambda_2^2)^{-1}$, with λ_1 and λ_2 the stable and unstable root of a second-order equation, respectively. In dynamic factor demand models under adjustment costs and rational expectations, assuming that there are no unobservable exogenous stochastic processes affecting the productivity of factor inputs, a PV relation of the form (1) can be obtained by maximizing intertemporally firms' profit streams; see Meese (1980). Here $y_t = (n_t : k_t)'$ $(m = 2)$ contains the stock of production workers and capital, $z_t = (w_t : c_t)'$ $(q = 2)$ is the vector containing real wages and the rental price of capital, ω_0 is a 2×2 nondiagonal nonsymmetric matrix of adjustment parameters, $\omega_1 = \delta I_2$, with δ a discount factor, $\gamma = I_2$, and θ is a 2×2 matrix whose elements depend on the firm-specific parameters related to technology.

2. MAIN RESULT

Let $X_t := (y_t' : z_t')'$ be generated by a VAR(k), $A(L)X_t = \mu + \varepsilon_t$, with $A(L) := I_p - A_1 L - \cdots - A_k L^k$, L being the lag operator, A_i $p \times p$ matrices of parameters, μ a $p \times 1$ constant, and ε_t an $iidN(0, \Sigma)$ $p \times 1$ disturbance term. It is assumed that the following condition holds.

A4. $k \geq 2$.

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The asymptotic stability (hereafter stationarity) of the vector autoregression requires that the following condition be met.

A5. Every root of the characteristic equation $\det(A(s)) = 0$ is such that $|s| > 1$ or $s = 1$, where the symbol $\det(\cdot)$ denotes the determinant of a matrix.

See Fanelli (2002) for an explanation of A4. Condition A5 states that both stationary and integrated processes are allowed.

One-step-ahead VAR forecasts of X_t conditional on the sigma algebra $\mathcal{H}_t := \sigma(X_t, X_{t-1}, \ldots, X_1)$, $\mathcal{H}_t \subseteq \mathcal{F}_t$, can be computed as

$$E(X_{t+1}|\mathcal{H}_t) = A_1 X_t + A_2 X_{t-1} + \cdots + A_k X_{t-k} + \mu$$

and used to replace expectations in (2). To this purpose, we introduce some notation. Partition matrices A_i, $i = 1, 2, \ldots, k$ and the constant μ conformably with $X_t := (y_t' : z_t')'$:

$$A_i =: \left[\begin{array}{c|c} A_i^{yy} & A_i^{yz} \\ \hline A_i^{zy} & A_i^{zz} \end{array}\right] \begin{array}{cc} m \times m & m \times q \\ q \times m & q \times q \end{array}, \qquad \mu =: \left[\begin{array}{c} \mu^y \\ \hline \mu^z \end{array}\right] \begin{array}{c} m \times 1 \\ q \times 1 \end{array},$$

where dimensions are reported alongside blocks. It is now clear that

$$E(y_{t+1}|\mathcal{H}_t) = A_1^{yy} y_t + A_1^{yz} z_t + A_2^{yy} y_{t-1} + A_2^{yz} z_{t-1} + \sum_{i=3}^{k} [A_i^{yy} : A_i^{yz}] X_{t-i+1} + \mu^y. \tag{3}$$

Hereafter the absence of Granger causality from y_t to z_t will be denoted by y_t GNC z_t and will correspond to the VAR restrictions

$$A_i^{zy} = 0_{q \times m}, \qquad i = 1, 2, \ldots, k. \tag{4}$$

Likewise, y_t GC z_t means that y_t Granger causes z_t, i.e., that there exists at least a matrix $A_{\tilde{i}}^{zy}$, $1 \leq \tilde{i} \leq k$, such that $A_{\tilde{i}}^{zy} \neq 0_{q \times m}$.

The following proposition and corollary establish the relation between "exact" PV models, Granger causality, and VAR stability.

PROPOSITION 2.1. *Assume that X_t is generated by the VAR process with Assumption A4 and that y_t GNC z_t, i.e., that (4) holds. If X_t satisfies the restrictions implied by the PV model (2) with A1–A3, then X_t does not match A5.*

Proof. Using A2, write (2) as

$$E_t y_{t+1} = \omega_1^{-1}(I_m + \omega_1 \omega_0) y_t - \omega_1^{-1} \omega_0 y_{t-1} - \omega_1^{-1} \gamma \theta z_t + \varsigma^{**},$$

where $\varsigma^{**} := \omega_1^{-1} \varsigma^*$. Condition both sides with respect to $\mathcal{H}_t$, apply the law of iterated expectations, and replace the quantity on the left-hand side by the VAR forecast (3), obtaining the relation

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$$A_1^{yy} y_t + A_1^{yz} z_t + A_2^{yy} y_{t-1} + A_2^{yz} z_{t-1} + \sum_{i=3}^{k} [A_i^{yy} : A_i^{yz}] X_{t-i+1} + \mu^y$$
$$= \omega_1^{-1}(I_m + \omega_1 \omega_0) y_t - \omega_1^{-1} \omega_0 y_{t-1} - \omega_1^{-1} \gamma\theta z_t + \varsigma^*.$$

As $X_t \neq 0_{p\times 1}$ (a.s.) $\forall t$, for the preceding equality to be satisfied the following set of restrictions must hold:

$$A_1^{yy} = \omega_1^{-1}(I_m + \omega_1 \omega_0), \qquad A_1^{yz} = -\omega_1^{-1}\gamma\theta, \tag{5}$$
$$A_2^{yy} = -\omega_1^{-1}\omega_0, \qquad A_2^{yz} = 0_{m\times q},$$
$$A_i^{yy} = 0_{m\times m}, \qquad A_i^{yz} = 0_{m\times q}, \qquad i = 3, \ldots, k,$$
$$\mu^y = \varsigma^*. \tag{6}$$

Given (4) and (5)–(6), the characteristic polynomial associated with the vector autoregression, $\det(A(s))$, reduces to

$$\det(\tilde{A}(s)) = \det\left(\begin{bmatrix} \tilde{A}_{yy}(s) & \tilde{A}_{yz}(s) \\ 0_{q\times m} & A_{zz}(s) \end{bmatrix}\right) = \det(\tilde{A}_{yy}(s))\det(A_{zz}(s)), \tag{7}$$

where $\sim$ indicates that the VAR coefficients are constrained, $\tilde{A}_{yy}(s) := I_m - (\omega_1^{-1} + \omega_0)s + \omega_1^{-1}\omega_0 s^2$, $\tilde{A}_{yz}(s) := -\omega_1^{-1}\gamma\theta s$, and $A_{zz}(s) := I_q - \sum_{i=1}^{k} A_i^{zz} s^i$. It can be recognized that $\tilde{A}_{yy}(s) = (I_m - \omega_1^{-1}s)(I_m - \omega_0 s)$ and that $\det(\tilde{A}_{yy}(0)) \neq 0$. Moreover, $\det(\tilde{A}_{yy}(1)) \neq 0$, as $\det(I_m - \omega_1^{-1}) \neq 0$ given A2, and $\det(I_m - \omega_0) \neq 0$, given A3 and a Jordan decomposition of ω_0. If λ is a nonzero eigenvalue of the square matrix P, then $s = \lambda^{-1}$ is a root of $\det(I - Ps) = 0$; hence it turns out that $\det(I_m - \omega_1^{-1}s) = 0$ has exactly m (nonzero) roots inside the unit circle because of A2, whereas $\det(I_m - \omega_0 s) = 0$ has roots outside the unit circle (when $\omega_0 \neq 0$) because of A3. Thus the characteristic equation (7) of the constrained VAR process has m roots inside the unit circle, and the system does not match A5. ■

COROLLARY. *Consider the PV relation (2) with A1–A3 and the VAR process with A4. A necessary condition for X_t to match the stability condition A5 under the PV restrictions is that y_t GC z_t.*

Remark 1. Proposition 2.1 holds for every process satisfying A5. Thus, it also holds for I(1), possibly cointegrated, processes. This situation corresponds to the case where the number of roots of the VAR characteristic equation at $s = 1$ is exactly equal to $p - r$, r is the cointegration rank of the system, $r \geq m$, and $[I_m : -\theta]' \in sp(\beta)$, where $sp(\cdot)$ is the space spanned by the columns of the $p \times r$ cointegration matrix β; see Johansen (1996).[3]

Remark 2. The result easily can be extended to the situation where the specified vector autoregression is of the form $X_t := (y_t' : z_t' : w_t')'$, where w_t is a

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vector of "additional" variables that help to forecast z_t. Indeed, define the new vector $z_t^* := (z_t' : w_t')$ of dimension $q^* \times 1$ $(q^* > q)$; for a suitable definition of the $m \times q^*$ matrix θ, the PV model (1) and its counterpart (2) can be rewritten, other things remaining fixed, by replacing z_t by z_t^*.

A natural fix to the shortcoming sketched in Proposition 2.1 is to appeal to "inexact" formulations of (1), other than looking outside VAR processes. For instance, one may add an exogenous $m \times 1$ martingale difference sequence, u_t, on the right-hand side of (1). In principle, it is possible to interpret such a component as a process capturing temporary unexplained deviations from the theory; however, there are also circumstances where a precise role for u_t stems from theory itself.[4] In general, Proposition 2.1 does not apply to the class of "inexact" PV models; see, e.g., Hansen and Sargent (1981).

NOTES

1. Further deterministic terms might be included in (1) without any significant change to the concepts and results that follow.

2. Feedback mechanisms can be interpreted as proxies for equilibrium forces that should be modeled directly. Timmermann (1994) provides a thorough analysis of the impact of feedbacks on the solutions to PV models. Granger causality from y_t to z_t is often regarded as a "weak implication" of the PV model; see, e.g., Campbell and Shiller (1987) and Engsted and Haldrup (1994).

3. The condition $[I_m: -\theta]' \in sp(\beta)$ implies that at least $[I_m: -\theta]X_t = y_t - \theta z_t$ is I(0), covering a well-known empirical implication of PV models with I(1) variables (Campbell and Shiller, 1987).

4. For instance, in the term structure of interest rates u_t might capture a time-varying risk premium; in the permanent income theory of consumption u_t might proxy a transitory consumption component; in New Keynesian pricing theory u_t might represent a shock in firms' markup. Also, u_t might account for measurement errors affecting the PV relation.

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