Proof: There is no pair $(a_\ell, b_{\ell'})$ where both elements are (strictly) smaller than c_{n+1} or greater than c_n. Otherwise, there would be $\ell' + (n + 1 - \ell') = n + 1$ numbers smaller than c_{n+1} or greater than c_n in the sequence, a contradiction.

In the case $c_n < c_{n+1}$, each distance $|a_{\ell'} - b_{\ell'}|$ is thus equal to $c_m - c_p$ for different m, p with $m > n \geqslant p$ and the assertion is proven.

We now turn to the case $c_n = c_{n+1} = c$. We can suppose without loss of generality that the terms c_i equal to c appear with increasing indices $j_.$ in the sequence $(a_{\ell'})$ and with greater and decreasing indices $k_.$ in $(b_{\ell'})$ (they may appear in only one sequence). Replace now for $i > n$ the terms c_i equal to c with $c + \varepsilon$, where $\varepsilon > 0$ is small enough to leave the order $c_1 \leqslant \ldots \leqslant c_{2n}$ unchanged. The orders $a_1 \leqslant \ldots \leqslant a_n$ and $b_1 \geqslant \ldots \geqslant b_n$ also remain the same. The sum of the distances $|a_{\ell'} - b_{\ell'}|$ changes by no more than $\pm n\varepsilon$ and is now, as the new c_{n+1} is greater than c_n, the difference between the sums of the upper and lower halves of the new sequence (c_i), which surpasses the old difference by no more than $n\varepsilon$. Letting ε tend to 0^+ proves the result.

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99.32 On the square wave series

The trigonometric series

$$f(x) = \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \ldots \qquad (1)$$

converges in $(0, \pi)$ by Dirichlet's test since the reciprocals form a positive null sequence and the sum of the first n odd sines (that is, $\sin^2 nx / \sin x$) is bounded.

In this note we will show that $f(x) = \pi/4$ in $(0, \pi)$ without the use of Fourier series, thus demonstrating a square wave pattern by periodicity.

Note in passing that the derivative series of unattenuated cosine terms diverges despite its convergence ‘in the mean’.

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The method

With $0 < x < \pi$, $0 < t < 1$, $|\exp ix| = 1$, we sum the convergent geometric progression

$$\exp ix + t^2 \exp i3x + t^4 \exp i5x + t^6 \exp i7x + \ldots$$

$$= \frac{\exp ix}{1 - t^2 \exp i2x} = \frac{\exp ix - t^2 \exp(-ix)}{t^4 - 2t^2 \cos 2x + 1}.$$

Then, integrating the imaginary part with respect to t from 0 to u $(0 < u < 1)$ we get

$$u \sin x + \frac{u^3 \sin 3x}{3} + \frac{u^5 \sin 5x}{5} + \frac{u^7 \sin 7x}{7} + \ldots$$

$$= \sin x \int_0^u \frac{t^2 + 1}{t^4 - 2t^2 \cos 2x + 1}\, dt,$$

whence, letting $u \to 1$,

$$f(x) = \sin x \int_0^1 \frac{t^2 + 1}{t^4 - 2t^2 \cos 2x + 1}\, dt,$$

by Abel's limit theorem.

We next integrate using partial fractions and apply the addition formula for inverse tangents, to obtain

$$f(x) = \sin x \int_0^1 \frac{t^2 + 1}{(t^2 - 2t \cos x + 1)(t^2 + 2t \cos x + 1)}\, dt$$

$$= \frac{1}{2} \int_0^1 \frac{1}{t^2 - 2t \cos x + 1} + \frac{1}{t^2 + 2t \cos x + 1}\, dt$$

$$= \frac{1}{2}\left[\arctan\left(\frac{t - \cos x}{\sin x}\right) + \arctan\left(\frac{t + \cos x}{\sin x}\right)\right]_0^1$$

$$= \frac{1}{2}\left[\arctan\left(\frac{2t / \sin x}{1 - (t^2 - \cos^2 x)/ \sin^2 x}\right)\right]_0^1 = \frac{\pi}{4}.$$

We can now discuss the Gibbs phenomenon, namely the development of the 'overshoot' or 'spike' at each jump discontinuity of such a waveform. This does not appear in the Schaum text [1] and is frequently neglected by writers on Fourier analysis.

The graphical development of the Gibbs spike with an increasing number of terms may be illustrated on a graphical calculator.

The spike

We need to locate the leftmost stationary point in $(0, \pi)$ of the nth partial sum $f_n(x)$ of the square wave series (1). This will be justified later.

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Now, by an easy induction,

$$f_n'(x) = \cos x + \cos 3x + \cos 5x + \cos 7x + \ldots + \cos(2n-1)x$$
$$= \frac{\sin 2nx}{2\sin x},$$

so the first stationary value occurs at $x = \frac{\pi}{2n} = q$ (say), and is given by

$$f_n(q) = \sin q + \frac{\sin 3q}{3} + \frac{\sin 5q}{5} + \frac{\sin 7q}{7} + \ldots + \frac{\sin(2n-1)q}{2n-1}. \qquad (2)$$

Furthermore, since $q \to 0$ and $(2n-1)q \to \pi$ as $n \to \infty$, the right-hand side of (2) takes the form of a Riemann sum of n terms with element $2q$, which tends to the definite integral $\frac{1}{2}\int_0^{\pi} \frac{\sin x}{x}\,dx$ as $n \to \infty$ (see [2]).

Next, integration of the power series for $\sin x/x$ leads to the convergent series

$$\lim_{n\to\infty} f_n(q) = \frac{1}{2}\left(\pi - \frac{\pi^3}{3.3!} + \frac{\pi^5}{5.5!} - \frac{\pi^7}{7.7!} + \ldots\right) \approx 1.17898 \times \frac{\pi}{4},$$

which corrected value indicates an 'overshoot' of roughly 18% at each jump discontinuity.

Other stationary values of $f_n(x)$ invoke the corresponding multiple of π for the upper limit of integration and a graph of the integrand $\sin x/x$ (or $f_{10}(x)$, say) shows these to be insignificant.

A conventional Fourier treatment of the Gibbs phenomenon employing a different waveform is given in reference [3].

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99.33 An integral inequality with applications to probability theory

In [1], a conjecture was proposed concerning the distribution of ages in a closed interval $[0, A]$. The conjecture is stated as follows:

> If $f(x)$ is decreasing for $0 \leqslant x \leqslant A$, $f(A) = 0$ and $\int_0^A f(x)\,dx = 1$ (and $f(x)$ continuous? differentiable?) then $\int_0^A xf(x)\,dx > \frac{1}{2f(0)}$.

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