A PRIMITIVE RING ASSOCIATED TO A BURNSIDE 3-GROUP

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ABSTRACT

Let $\mathscr{G}$ be a Burnside 3-group constructed by Gupta–Sidki. The group algebra $\mathbb{Z}_3[\mathscr{G}]$ has a proper ring-quotient which embeds $\mathscr{G}$ and is just-infinite and primitive.

1. *Introduction*

Let $\mathscr{G}$ be the Burnside 3-group introduced in [**5**, **6**] and defined recursively by

$$\langle \gamma, x \,|\, \gamma^3 = x^3 = 1, \gamma = (\gamma, x, x^{-1}) \rangle$$

as a subgroup of automorphisms of a one rooted infinite regular ternary tree. The element x acts on triples by permuting their entries cyclically, and triples are multiplied component-wise. Two properties of this group stand at opposite extremes:

$\mathscr{G}$ contains a copy of every finite 3-group;

every proper quotient of $\mathscr{G}$ is finite (in other words, it is just-infinite).

Another property of importance from a ring theoretic point of view is that the set $\Delta(\mathscr{G}) = \{g \,|\, [\mathscr{G} : C_{\mathscr{G}}(g)] < \infty\}$ of virtually central elements is trivial [**12**]. By a theorem of Connell [**7**, p. 129], the group algebra $k[\mathscr{G}]$ is a prime ring for any field k.

Our purpose in this paper is to construct a proper ring-quotient $\mathscr{P}$ of the group algebra $R = \mathbb{Z}_3[\mathscr{G}]$ which imitates additively the multiplicative structure of the group. The construction is based on the natural k-algebra homomorphism $k[H \times G] \to k[H] \oplus k[G]$, called *summation thinning*, which extends the identity map of the direct product of groups $H \times G$ into the sum of k-algebras $k[H] \oplus k[G]$. An infinite iteration of summation thinning interest applied to $k[G]$ allows us to add triples component-wise. Considering that this process is of independent interest, we describe it for one-rooted infinite regular trees in general.

THEOREM. *The ring $\mathscr{P}$ is a 2-generated infinite dimensional $\mathbb{Z}_3$-algebra with the following properties:*

(1) *the group $\mathscr{G}$ is embedded in the group of units of $\mathscr{P}$, and $\{\gamma, x\}$ generates $\mathscr{P}$ as an algebra;*

(2) *the algebra of polynomials in one variable with coefficients from $\mathbb{Z}_3$ is embedded in $\mathscr{P}$;*

(3) *$\mathscr{P}$ is just-infinite and primitive.*

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A notable and immediate consequence is the following result.

COROLLARY. *The group $\mathscr{G}$ admits a faithful, infinite dimensional, irreducible $\mathbb{Z}_3$-representation.*

Using this fact, Passman–Temple have shown in [**8**] that $\mathscr{G}$ has infinitely many nonequivalent faithful irreducible representations.

We remark that in the Theorem and its Corollary one may replace $\mathbb{Z}_3$ by any field of characteristic 3, provided that the just-infiniteness of $\mathscr{P}$ is restated as just-infinite dimensionality of $\mathscr{P}$. For the sake of a self-contained exposition, we review the construction of the group $\mathscr{G}$ in Section 2, list facts known about it, and take the opportunity to sharpen some of them. The paper ends with some open problems concerning the group $\mathscr{G}$ and its group algebra.

2. *Automorphisms of trees – the Burnside group $\mathscr{G}$*

2.1. *Trees and their automorphisms*

Let $\mathscr{M}$ be a free monoid, freely generated by a set of symbols Y with more than one element. Define on $\mathscr{M}$ the order relation: for $m, m' \in \mathscr{M}$, we set $m \leqslant m'$ if and only if m' is a prefix of m. Then the ordered monoid $\mathscr{T} = (\mathscr{M}, \leqslant)$ is a one rooted infinite regular tree. When $Y = \{0, 1, 2\}$, the tree is illustrated by Figure 1.

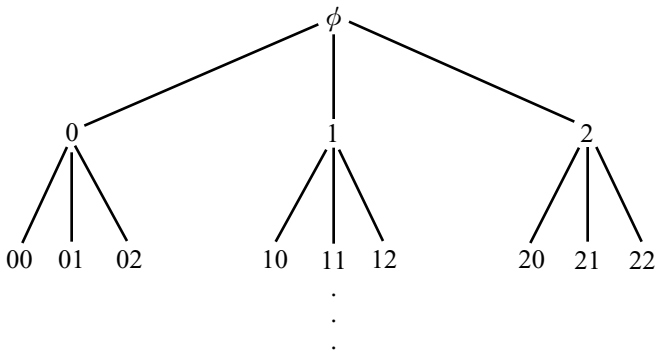


FIG. 1

Let $A = \mathrm{Aut}(\mathscr{T})$ be the automorphism group of $\mathscr{T}$, and let $S = S(Y)$ be the permutation group of the set Y. Extend the action of S to $\mathscr{T}$ as follows: for $s \in S$, $y \in Y$, and $m \in \mathscr{M}$, let $s: \phi \to \phi$, $ym \to (y)s.m$. For $i \geqslant 0$, let A_i be the point-wise stabilizer of words of length i in $\mathscr{M}$. Then, $A_1 = \mathscr{F}(Y, A) = \{f | f: Y \to A\}$; we use the notation $f: y \to (y)f$, for $y \in Y$. The group A is a semi-direct product $A = A_1 \rtimes S$, where for $f \in A$, $s \in S$, we have $sfs^{-1} = s \circ f$. When $Y = \{0, 1, 2\}$ it follows that A_1 is identified with $A \times A \times A$ and S with S_3 which is generated by the permutations $x = (0, 1, 2)$, $t = (0, 1)$ and $A = (A \times A \times A) \rtimes S_3$. For later use, we define the following diagonal elements: given $\alpha \in A$, let $\alpha^{(0)} = \alpha$, $\alpha^{(1)} = (\alpha, \alpha, \alpha)$ and recursively, $\alpha^{(i+1)} = (\alpha^{(i)})^{(1)}$ for $i \geqslant 1$.

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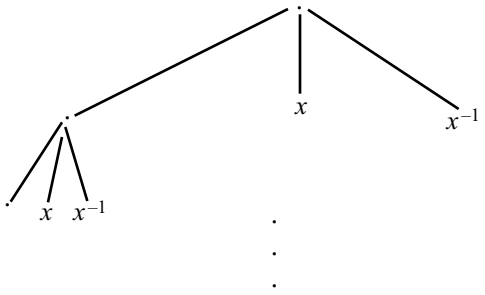


FIG. 2

2.2. *The group $\mathcal{G}$ and its properties*

Let $Y = \{0, 1, 2\}$ and $\mathcal{T}$ be the corresponding tree. Define recursively the tree automorphism $\gamma = (\gamma, x, x^{-1})$ whose action on the tree may be depicted by Figure 2. Our group is $\mathcal{G} = \langle \gamma, x \rangle$.

An element α of $\mathcal{G}$ is described uniquely by $\alpha = \alpha_\phi = (\alpha_0, \alpha_1, \alpha_2)\, x^{j(\phi)}$, where inductively $\alpha_m = (\alpha_{m0}, \alpha_{m1}, \alpha_{m2})\, x^{j(m)}$ for $\alpha_m \in \mathcal{G}$, and where j is some function $j: \mathcal{M} \to Y$.

We collect the most important results about $\mathcal{G}$ in the next theorem (see [**11**, **12**]).

THEOREM 2.2.1. *The group $\mathcal{G}$ is a 2-generated, infinite, residually finite 3-group, and it contains a copy of every finite 3-group. It is just-infinite, and its only finite conjugacy class is $\{e\}$. Moreover, $\mathcal{G}$ is recursively presentable, but does not admit a finite presentation.*

More specific properties of $\mathcal{G}$ will be needed in the sequel.

PROPOSITION 2.2.2. *Let $\mathcal{G}_1 = \mathcal{G} \cap A_1$, and let $\mathcal{G}'$ denote the commutator subgroup of $\mathcal{G}$. Then,*

(1) *given $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in A_1$, a necessary condition for α to belong to $\mathcal{G}$ is that $\alpha_0 \alpha_1 \alpha_2 \in \mathcal{G}' \langle \gamma \rangle$,*

(2) *the diagonal elements $\gamma^{(i)}$ in A are elements of $\mathcal{G}'$ for all $i \geqslant 1$,*

(3) *$\mathcal{G}_1' = \mathcal{G}' \times \mathcal{G}' \times \mathcal{G}'$.*

Yet another property which we need is that $\mathcal{G}$ has a finite depth function (see [**12**, p. 36]).

PROPOSITION 2.2.3. *There exists $d: \mathcal{G} \to \mathbb{Z}_{\geqslant 0}$, called a depth function, such that $d(\alpha)$ is the least non-negative integer for which $(\alpha_{m0}, \alpha_{m1}, \alpha_{m2}) \in \langle \gamma \rangle$ for all $m \in \mathcal{M}$ of length at least $d(\alpha)$.*

We note that α and $(\alpha_0, \alpha_1, \alpha_2)$ have the same depth, and that $d(\alpha) = 0$ holds if and only if $\alpha \in \langle \gamma \rangle \langle x \rangle$. We also note that $d(\alpha) = \max_i \{d(\alpha_i)\} + 1$ if $\alpha \notin \langle \gamma \rangle \langle x \rangle$. In the following we sharpen some known facts about subgroups of $\mathcal{G}$.

PROPOSITION 2.2.4. *We have that*

(1) *$\mathcal{G}$ does not satisfy the* DCC *for normal subgroups, and does not satisfy the* ACC *for subnormal subgroups,*

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(2) *$\mathscr{G}$ contains as a subgroup an infinite direct sum of copies of $\mathscr{G}'$,*

(3) *$\mathscr{G}$ contains a subgroup which is isomorphic to the infinitely iterated restricted wreath product* $((\dots) \operatorname{wr} C_3) \operatorname{wr} C_3$.

Proof. (1) The first assertion follows from the fact that the stabilizers $\mathscr{G}_i = \mathscr{G} \cap A_i$, for $i \geqslant 0$, form an infinite descending chain of normal subgroups. Let $\pi_1 : \mathscr{G}_1 \to \mathscr{G}$ be the projection on the first entry. Then π_1 is an epimorphism and $K_1 = \ker(\pi_1)$ is subnormal in $\mathscr{G}$. As $\mathscr{G}_1/K_1 \cong \mathscr{G}$, we may repeat this construction to produce an infinite ascending chain of subnormal subgroups of $\mathscr{G}$.

(2) Since $\mathscr{G}' \times \mathscr{G}' \times \mathscr{G}' < \mathscr{G}_1 < \mathscr{G}'$, we produce the infinite chain of subgroups

$$\mathscr{G}' \times \{1\} \times \mathscr{G}' < \mathscr{G}' \times (\mathscr{G}' \times \{1\} \times \mathscr{G}') \times \mathscr{G}' < \dots < \mathscr{G}' \times (\mathscr{G}' \times (\dots) \times \mathscr{G}') \times \mathscr{G}' < \dots,$$

the union of which is the desired subgroup.

(3) Define the following sequence of elements of $\mathscr{G}$:

$$a_1 = x, \quad a_2 = (1, \gamma^{(1)}, 1), \quad a_i = (1, (1, (1, a_{i-1}, 1), 1), 1) \quad \text{for } i \geqslant 3.$$

Let $H_1 = \langle a_i \mid i \geqslant 1 \rangle$, $H_2 = \langle a_i \mid i \geqslant 2 \rangle$. Then, $H_1 = H_2 \operatorname{wr} \langle a_1 \rangle$. Also, a_i induces a_{i-1} for $i \geqslant 2$ on the subtree headed by $111 \in \mathscr{M}$.

An element of H_2 is expressed as a word

$$u(a_2, a_3, \dots) = (1, (u(\gamma, 1, \dots, 1), (u(\gamma, 1, \dots, 1), u(a_1, a_2, \dots), u(x^{-1}, 1, \dots, 1)), u(\gamma, 1, \dots, 1)), 1).$$

If for some u, we have $u(a_1, a_2, \dots) = 1$, then

$$p = u(a_2, a_3, \dots) = (1, (\gamma^i, (\gamma^i, 1, x^{-i}), \gamma^i), 1)$$

for some integer i, and therefore $pa_2^{-i} = (1, (1, (1, x^{-i}, 1), 1), 1)$ which by Proposition 2.2.2(1) is an element of $\mathscr{G}$ only if $i \equiv 0 \pmod 3$. Hence $H_2 \cong H_1$.

In connection with the above Property (1), we refer to J. Wilson's paper [**13**], which is a study of the subnormal structure of just-infinite groups.

PROPOSITION 2.2.5. *The group of automorphisms of $\mathscr{G}$ has the structure*

$$\operatorname{Aut}(\mathscr{G}) = (\mathscr{G} \rtimes X) \rtimes V,$$

where X is the elementary abelian 3-subgroup of A generated freely by $\{x^{(i)} \mid i \geqslant 1\}$, and V is the elementary abelian 2-group generated by

$$\tau_1 : \gamma \mapsto \gamma, x \mapsto x^{-1}, \quad \tau_2 : \gamma \mapsto \gamma^{-1}, x \mapsto x.$$

We make the new observation that V is realizable in A. Indeed, $\tau_1 = tx^{-1} \cdot \tau_1^{(1)}$ and $\tau_2 = (tx^{-1})^{(1)} \tau_2^{(1)}$. Thus, all automorphisms of $\mathscr{G}$ are induced by conjugation by elements of A; that is, $\operatorname{Aut}(\mathscr{G}) = N_A(\mathscr{G})$.

3. *Thinning the algebra of tree automorphisms*

We go back to the tree $\mathscr{T}$ with a general basis Y. Let k be a field. Since $A = A_1 \rtimes S$, the group algebra $k[A]$ is a *crossed product* of algebras [**9**, p. 526], $k[A] = k[A_1] \rtimes k[S]$. Furthermore, since $A_1 = \mathscr{F}(Y, A)$, we have the inclusion map $v_1 : A \to \mathscr{F}(Y, k[A]) \rtimes k[S]$, and this extends canonically to a k-algebra homomorphism which we indicate by the same symbol $v_1 : k[A] \to \mathscr{F}(Y, k[A]) \rtimes k[S]$; this is the 1st *level summation thinning* of $k[A]$.

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3.1: image (v_1). Given $f \in A_1 = \mathscr{F}(Y, A)$, define $f' \in \mathscr{F}(Y, A-1)$ by $f'\colon y \to (y)f - 1$ for all $y \in Y$. Then, $v_1\colon f \to f = f' + 1$, and clearly,

$$\mathscr{F}(Y, A-1) \leqslant k[A]\, v_1 \leqslant \mathscr{F}(Y, wk[A] + k.1) \rtimes k[S],$$

where w stands for the augmentation ideal. When Y is a finite set, we have equality on the right hand side.

3.2: $\ker(v_1)$. Given $i \in Y$, let $\mathscr{F}(Y, A)_i$ denote the set of functions with value 1 for all $y \in Y \backslash \{i\}$. Then $\mathscr{F}(Y, A)_i \cong A$. Moreover, let w_i denote the augmentation ideal of $k[\mathscr{F}(Y, A)_i]$. Consider $i, j \in Y$ with $i \neq j$, and let $f_i \in \mathscr{F}(Y, A)_i$, $f_j \in \mathscr{F}(YA)_j$. Then,

$$v_1\colon (f_i - 1)(f_j - 1) \longrightarrow (f_i' + 1)(f_j' + 1) - (f_i' + 1) - (f_j' + 1) + 1 = 0.$$

Thus, the ideal W generated by $\{w_i w_j \mid i \neq j\}$ is contained in $\ker(v_1)$. Again, these are equal when Y is a finite set.

We iterate the algebra thinning as follows. The homomorphism v_1 induces a k-algebra homomorphism $v_2\colon \mathscr{F}(Y, k[A]) \rtimes k[S] \to \mathscr{F}(Y, k[A]\, v_1) \rtimes k[S]$. We call the composition $\tilde{v}_2 = v_1 v_2$, where $\tilde{v}_2\colon k[A] \to \mathscr{F}(Y, k[A]\, v_1) \rtimes k[S]$ the 2nd *level thinning*. We proceed to define inductively for $i \geqslant 3$, the ith *level thinning* $\tilde{v}_i = \tilde{v}_{i-1} v_i$, where

$$v_i\colon \mathscr{F}(Y, k[A]\, \tilde{v}_{i-2}) \rtimes k[S] \longrightarrow \mathscr{F}(Y, k[A]\, \tilde{v}_{i-1}) \rtimes k[S]$$

is natural. Define $T_i = \ker(\tilde{v}_i)$ for $i \geqslant 1$. Then the kernels T_i form an ascending chain of ideals of $k[A]$ such that $(1 + T_i) \cap A = \{1\}$ for all i. Let $T = \bigcup_{i \geqslant 1} T_i$. The *thinned algebra* is $\overline{k[A]} = k[A]/T$. From this we have the following conclusions.

3.3. *With the above notation* $\overline{k[A]} \leqslant \mathscr{F}(Y, k[\overline{A}]) \rtimes k(S)$, *and equality holds if* Y *is finite;*

3.4. *Also* A *and* $k[S]$ *are geometrically embedded in* $\overline{k[A]}$.

4. *The ring* $\mathscr{P}$

For the rest of the paper, let $Y = \{0, 1, 2\}$, $k = \mathbb{Z}_3$, $R = k[\mathscr{G}]$, and let $\mathscr{P}$ be the image of R in $\overline{k[A]}$; that is, $\mathscr{P} = (R + T)/T \cong R/(R \cap T)$. We observe that $R \cap T$ is a characteristic ideal in R, since $\underline{\mathrm{Aut}}(\mathscr{G}) = N_A(\mathscr{G})$. Moreover, as $k\langle x \rangle$ is a subalgebra of $k[S_3]$ which is embedded in $\overline{k[A]}$, it follows that $k\langle x \rangle$ is also embedded in $\mathscr{P}$. As triples in $\mathscr{G}$ can be added component-wise in $\mathscr{P}$, we have $\mathscr{P} \leqslant \mathscr{F}(Y, \mathscr{P}) \rtimes k\langle x \rangle$. The elements of $\mathscr{P}$ have a unique form $u = u_0 + u_1 x + u_2 x^2$, where, for $0 \leqslant i \leqslant 2$, we have $u_i = (u_{i0}, u_{i1}, u_{i2})$ and $u_{ij} \in \mathscr{P}$ for $0 \leqslant j \leqslant 2$. Clearly, $k\langle \gamma \rangle$ is embedded in $\mathscr{P}$ too.

Since by Proposition 2.2.2(3), $\mathscr{F}(Y, \mathscr{G}') = \mathscr{G}' \times \mathscr{G}' \times \mathscr{G}'$ is a subgroup of $\mathscr{G}$, it follows that $k[\mathscr{F}(Y, \mathscr{G}')] \leqslant k[\mathscr{G}]$ and $\ker(v_1) \cap k[\mathscr{G}] \neq \{0\}$. Hence, $\mathscr{P}$ is a proper quotient of R.

4.1. *Special elements*

Given $u \in \mathscr{P}$, define the diagonal elements of $\overline{k[A]}$ by $u^{(0)} = u$, $u^{(1)} = (u, u, u)$, and inductively, $u^{(i+1)} = (u^{(i)})^{(1)}$ for $i \geqslant 1$. We observe that $u^{(i)}$ commutes with x for all $i \geqslant 1$.

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Two elements of the augmentation ideal $w(R)$ will play a special role in our study. These are $x_* = (x-1)^2 = x^2+x+1$ and $\gamma_* = (\gamma-1)^2 = \gamma^2+\gamma+1$. We make use of the component-wise addition in $\mathscr{P}$ to conclude that $\gamma_* = (\gamma_*, x_*, x_*)$.

The following equations hold in $\overline{k[A]}$:

(1) $x_*(u_0,u_1,u_2) = (u_0,u_1,u_2)+(u_1,u_2,u_0)\,x+(u_2,u_0,u_1)\,x^2$,

(2) $x_*(u_0,u_1,u_2)\,x_* = (u_0+u_1+u_2)^{(1)}\,x_*$,

(3) $x_*(u,v,v)\,x_* = (u-v)^{(1)}\,x_*$.

PROPOSITION 4.1.1. *The group algebra $k[\gamma_* x_*]$ is a polynomial algebra.*

Proof. Let $u_1 = \gamma_* x_*$, $u_2 = x_*\gamma_*$. We calculate powers of u_1 in $\mathscr{P}$ by $u_1^2 = \gamma_* x_* \gamma_* x_* = \gamma_*(\gamma_*-x)^{(1)}\,x_* = (-\gamma_* x_*, x_*\gamma_*, x_*\gamma_*)\,x_* = (-u_1,u_2,u_2)\,x_*$, and, by induction,

$$u_1^{2j-1} = (-1)^{j-1}(u_1^{j-1}\gamma_*, u_2^{j-1}x_*, u_2^{j-1}x_*)\,x_*, \quad u_1^{2j} = (-1)^{j+1}(-u_1^j, u_2^j, u_2^j)\,x_* \quad \text{for all } j \geqslant 1.$$

Suppose that u_1 is a root of some non-zero polynomial $p(y)$ in $\mathbb{Z}_3[y]$ having minimum degree m. Then $0 = p(u_1) = (q,l,l)\,x_*$ and $0 = q = q_0(u_1)+q_1(u_1)\gamma_*$, where $q_0(y)$, $q_1(y)$ are polynomials of degrees at most $[\frac{1}{2}m]$, and at least one has degree $[\frac{1}{2}m]$. Now since $0 = qu_1 = q_0(u_1)\,u_1$ and $0 = qx_* = q_1(u_1)\,u_1$, it follows that u_1 is a root of the polynomials $p(y)$, $q_0(y)\,y$, $q_1(y)\,y$, and thus $m = 1$ or 2. A direct analysis of these cases leads easily to contradictions.

LEMMA 4.1.2. *For all $i \geqslant 0$, we have that $\gamma_*^{(i)}$, $x_*^{(i)}$ are elements of $\mathscr{P}$.*

Proof. Since $\gamma_*^{(i)} = (\gamma^2)^{(i)}+\gamma^{(i)}+1$, and as $\gamma^{(i)} \in \mathscr{G}$, by Proposition 2.2.2(2), it follows that $\gamma_*^{(i)} \in \mathscr{P}$.

Recall that $x^{(i)} \in \mathscr{N}_A(\mathscr{G})$ for all $i \geqslant 0$. Therefore, for all $i \geqslant 0$, we have that $\eta_i = \gamma^{(i)}+(\gamma^{(i)})^{x^{(i)}}+(\gamma^{(i)})^{x^{(i)}x^{(i)}}$ is an element of $k[\mathscr{G}]$, and in $\mathscr{P}$ it has, by induction on i, the form $\eta_i = \gamma^{(i+1)}+x_*^{(i+1)}-1$. Hence, $x_*^{(i)} \in \mathscr{P}$ for all $i \geqslant 0$.

4.2. *x_*-multiplication, x_*-stripping in ideals*

LEMMA 4.2.1. *Let $v = v_0+v_1x+v_2x^2 \in \mathscr{P}$ with $v_i \in \mathscr{F}(Y,\mathscr{P})$ for all i. Then for some $0 \leqslant j \leqslant 2$, there exists a linear combination u of v_0, v_1, v_2 such that $v(x-1)^j = ux_*$, and $u \neq 0$.*

Proof. (i) If $v_0 = v_1 = v_2$, then $u = v_0$.

(ii) If $v_0+v_1+v_2 = 0$ and $v_0 \neq v_1$, then $v(x-1) = ux_*$, $u = v_0-v_1$.

(iii) If $v_0+v_1+v_2 \neq 0$, then $vx_* = ux_*$, $u = v_0+v_1+v_2$.

PROPOSITION 4.2.2. *Let $\mathscr{I}$ be an ideal in $\mathscr{P}$ and let $u = (u_0,u_1,u_2)\,x_* \in \mathscr{I}$. Then, for all $c \in \mathscr{G}'$, we have $((c-1)\,u_i(c-1)^2, 0, 0) \in \mathscr{I}$ for all $0 \leqslant i \leqslant 2$.*

Proof. We calculate the following sequence of elements in $\mathscr{I}$:

$$\begin{aligned} v_1 &= u^{(c,1,1)}-u \\ &= (u_0^c-u_0,0,0)+((c^{-1}-1)\,u_0,0,u_2(c-1))\,x+((c^{-1}-1)\,u_0,u_1(c-1),0)\,x^2, \\ v_2 &= v_1^{(1,c,1)}-v_1 = ((c^{-1}-1)\,u_0(c-1),0,0)\,x+(0,(c^{-1}-1)\,u_1(c-1),0)\,x^2, \\ v_2' &= v_2x^2 = ((c^{-1}-1)\,u_0(c-1),0,0)+(0,(c^{-1}-1)\,u_1(c-1),0)\,x, \end{aligned}$$

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$v_3 = (v_2')^{(1,1,c)} - v_2' = (0, (c^{-1}-1)\,u_1(c^2+c+1), 0)\,x,$
$v_4 = xv_3\,x = ((c^{-1}-1)\,u_1(c-1)^2, 0, 0),$
$v_5 = -c^{(1)}v_4 = ((c-1)\,u_1(c-1)^2, 0, 0).$

Conjugation of u by x replaces u_1 by u_0 in v_5, and clearly u_2 may also be made to replace u_1.

4.3. *Depth function on $\mathscr{P}$*

Extend $d: \mathscr{G} \mapsto \mathbb{Z}_{\geqslant 0}$ as follows. For $v = v_0 + v_1 x + v_2 x^2 \in \mathscr{P}$, where $v_i = (v_{i0}, v_{i1}, v_{i2})$ and $v_{ij} \in \mathscr{P}$ for $0 \leqslant i, j \leqslant 2$, define $d(v) = 0$ when $v \in \sum_{i=0}^{2} k\langle\gamma\rangle\, x^i$, otherwise let $d(v) = \max_i\{d(v_i)\}$, where $d(v_i) = \max_j\{d(v_{ij})\} + 1$ for those $v_i \notin k\langle\gamma\rangle$.

Observe that for $v = (v_0, v_1, v_2) \in \mathscr{P}$ not in $k\langle\gamma\rangle$, we have $d(v) > d(v_i)$ for $0 \leqslant i \leqslant 2$, and furthermore $d(v^x) = d(v)$.

4.4. *$\mathscr{P}$ is a just-infinite ring*

We shall prove that the normal subgroup structure of $\mathscr{G}$ controls the ideal structure of $\mathscr{P}$. More precisely, it will be shown that every non-zero 2-sided ideal of $\mathscr{P}$ contains $N-1$ for some non-trivial normal subgroup N of $\mathscr{G}$. Since $\mathscr{G}$ is just-infinite, the same property will hold for $\mathscr{P}$.

PROPOSITION 4.4.1. *Let $\mathscr{I}$ be a 2-sided non-zero ideal in $\mathscr{P}$. Then, there exists $c \in \mathscr{G}'$ with $c \neq 1$, such that $\mathscr{I}$ contains $N-1$, where N is the normal closure of $\langle c\rangle$ in $\mathscr{G}$.*

Proof. Let $v = v_0 + v_1 x + v_2 x^2 \in \mathscr{I}$, with $v \neq 0$, of depth $d(v) = t$.

(1) Suppose that $d(v) = 0$. Then $v_i \in k\langle\gamma\rangle$ for $0 \leqslant i \leqslant 2$. By Lemma 4.2.1, we produce $ux_* \in \mathscr{I}$ with $u \in k\langle\gamma\rangle$ and $u \neq 0$. On writing $u = (u_0, u_1, u_2)$, we have $u_0 = p(\gamma)$, $u_1 = p(x)$, $u_2 = p(x^{-1})$ for some polynomial $p(s) = a_0 + a_1 s + a_2 s^2 \in k[s]$, and $u_i \neq 0$ for all i.

By Proposition 4.2.2, we may strip ux_* of its x_* to produce

$$((c^{(1)}-1)\,u_1(c^{(1)}-1)^2, 0, 0) = ((c^{(1)}-1)^3\,u_1, 0, 0) \in \mathscr{I} \quad \text{for all } c \in \mathscr{G}'.$$

On multiplying this element on the right by $\gamma^{x^{-1}} - 1$ or by $x_*^{(1)}$, this will have the effect of replacing u_1 by $u_1(x-1)$ or by $u_1 x_*$, respectively. Therefore, again by Lemma 4.2.1, we produce $((c^{(1)}-1)^3\,x_*, 0, 0) \in \mathscr{I}$, from which we can strip x_* to produce

$$(((c-1)^6, 0, 0), 0, 0),$$

and thus also

$$(((c-1)^9, 0, 0), 0, 0) = ((c, 1, 1), 1, 1)^9 - 1.$$

Since $\mathscr{G}'$ is of unbounded exponent, this case is done.

(2) Suppose that $d(v) = t > 0$. Then within $\mathscr{I}$ we can produce $ux_* \neq 0$, where $u = (u_0, u_1, u_2)$ is a linear combination of v_0, v_1, v_2, and $u_i \neq 0$ for some i. As $t > 0$, we have $d(u_i) \leqslant t-1$. Let $c \in \mathscr{G}'$ and $c_{t+1} = c^{(t+1)}$. We can produce $((c_t - 1)\,u_i(c_t-1)^2, 0, 0)$, within $\mathscr{I}$, and similar manipulations as in the first case, at some stage j, produce

$$((\ldots(((c_{t-j}-1)^{3^j}, 0, 0), 0, 0), \ldots), 0, 0),$$

which concludes the proof.

COROLLARY 4.4.2. *The ring $\mathscr{P}$ is a prime ring.*

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Proof. Suppose that I_1, I_2 are non-zero ideals and $I_1 I_2 = 0$. Let N_1, N_2 be nontrivial normal subgroups of $\mathscr{G}$ such that $N_1 - 1 \subseteq I_1$, $N_2 - 1 \subseteq I_2$, and let $L = N_1 \cap N_2$. Then, as N_1 and N_2 have finite indices in $\mathscr{G}$, it follows that L also is of finite index in $\mathscr{G}$ and therefore L is finitely generated. Now, from $(L-1)^2 = 0$ it follows that L is commutative and therefore is finite; a contradiction is reached.

Let $w(\mathscr{P})$ denote the image of the augmentation ideal $wk[\mathscr{G}]$ module T, and $\mathscr{J}(\mathscr{P})$ denote the Jacobson radical of $\mathscr{P}$. Recall that for a one-sided ideal M, the maximal 2-sided ideal contained in M is called the core(M).

COROLLARY 4.4.3. *With the above notation $\mathscr{J}(\mathscr{P})$ is either the zero ideal or the augmentation ideal $w(\mathscr{P})$.*

Proof. Since $w(\mathscr{P})$ is a maximal right ideal, $\mathscr{J}(\mathscr{P}) \leqslant w(\mathscr{P})$. Suppose that $\mathscr{J}(\mathscr{P}) \neq \{0\}$. Then it contains $N-1$ for some non-trivial normal subgroup of $\mathscr{G}$. Thus, modulo $\mathscr{J}(\mathscr{P})$, we have that $\mathscr{G}$ is a finite 3-group, and, as $k = \mathbb{Z}_3$, it follows that $w(\mathscr{P})/\mathscr{J}(\mathscr{P})$ is nilpotent. Hence, $w(\mathscr{P}) = \mathscr{J}(\mathscr{P})$.

COROLLARY 4.4.4. *Suppose that M is a maximal right ideal in $\mathscr{P}$ with non-zero core. Then $M = w(\mathscr{P})$.*

5. *$\mathscr{P}$ is a primitive ring*

The essential step here is to prove that $\mathscr{J}(\mathscr{P})$ is the zero ideal. This, of course, ensures the existence of a maximal right ideal M, with null core, which in turn provides us with a faithful irreducible representation of $\mathscr{P}$ on the k-space $\mathscr{P}/M$. As $\mathscr{J}(\mathscr{P})$ is a quasi-invertible ideal [**9**, p. 180] and as $\gamma x_* \in w(\mathscr{P})$, it suffices by Corollary 4.4.3, to prove that $1 + \gamma x_*$ is not invertible in $\mathscr{P}$.

PROPOSITION 5.1. *Suppose that $u = 1 + v x_*$ is an invertible element in $\overline{k[A]}$, where $v = (v_0, v_1, v_2) \in \mathscr{F}(Y, \overline{k[A]})$. Then*

$$u^{-1} = 1 + v' x_* \quad \text{for some } v' = (v_0', v_1', v_2') \in \mathscr{F}(Y, \overline{k[A]}),$$
$$v(1 + v_0' + v_1' + v_2')^{(1)} = -v', \quad v'(1 + v_0 + v_1 + v_2)^{(1)} = -v,$$
$$(1 + v_0 + v_1 + v_2)^{-1} = 1 + v_0' + v_1' + v_2'.$$

Proof. Let $u^{-1} = r_0 + r_1 x + r_2 x^2$, where $r_i \in \mathscr{F}(Y, \bar{A})$ for $0 \leqslant i \leqslant 2$. Then

$$\begin{aligned} u^{-1}u &= u^{-1} + u^{-1} v x_* \\ &= (r_0 + r_0 v + r_1 v^{x^{-1}} + r_2 v^x) + (r_1 + r_0 v + r_1 v^{x^{-1}} + r_2 v^x) x + (r_2 + r_0 v + r_1 v^{x^{-1}} + r_2 v^x) x^2 \\ &= 1. \end{aligned}$$

We conclude that $r_2 = r_1 = r_0 - 1$ and $u^{-1} = 1 + (r_0 - 1) x_*$.

Again, from $u^{-1}u = (1 + v' x_*)(1 + v x_*) = 1$, we derive that $v'(1 + v_0 + v_1 + v_2)^{(1)} = -v$; similarly, from $uu^{-1} = 1$, we obtain that $v(1 + v_0' + v_1' + v_2')^{(1)} = -v'$.

On multiplying the last equation on both sides by x_*, we have

$$(v_0 + v_1 + v_2)^{(1)} (1 + v_0' + v_1' + v_2')^{(1)} x_* = -(v_0' + v_1' + v_2')^{(1)} x_*,$$

from which it follows that

$$(1 + v_0 + v_1 + v_2)(1 + v_0' + v_1' + v_2') = 1.$$

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PROPOSITION 5.2. *We have that* $\eta = 1+\gamma x_*$ *is not invertible in* $\mathscr{P}$.

Proof. Suppose that η is invertible in $\overline{k[A]}$. Then, applying to it τ_2, the automorphism of $\mathscr{G}$, we get that $\mu = \eta^{\tau_2} = 1+\gamma^{-1}x_*$ is also invertible. We may by the previous proposition write the inverses in the form $\eta^{-1} = 1-\rho x_*$, $\mu^{-1} = 1-\sigma x_*$ with $\rho, \sigma \in \mathscr{F}(Y, \overline{k[A]})$, and conclude that $\rho(\gamma + x_*)^{(1)} = \gamma$, $\sigma(\gamma^{-1}+x_*)^{(1)} = \gamma^{-1}$.

On multiplying the first equation on the right by $(1-\sigma x_*)^{(1)}$, we obtain that $\rho\gamma^{(1)} = \gamma(1-\sigma x_*)^{(1)}$, and from it the (ρ,σ)-dependence equation $\rho = \gamma(1-\sigma x_*)^{(1)}\gamma^{(1)-1}$. Similarly, from the second equation, we obtain the (σ,ρ)-dependence equation $\sigma = \gamma^{-1}(1-\rho x_*)^{(1)}\gamma^{(1)}$. On substituting the expression of σ in ρ, we have

$$\rho = \gamma\gamma^{(1)-1} - \gamma\gamma^{(1)-1}(1-\rho x_*)^{(2)}\gamma^{(2)}x_*^{(1)}\gamma^{(1)-1}.$$

Suppose that $\rho \in \mathscr{F}(Y, \mathscr{P})$, and let $\rho = (\rho_0, \rho_1, \rho_2)$. Then, $d(\rho) \geqslant d(\rho_0)$, and

$$\rho_0 = 1-(1-\rho x_*)^{(1)}\gamma^{(1)}x_*\gamma^{-1} = 1-(1-\rho x_*)^{(1)}\gamma^{(1)}(\gamma^{-1}+\gamma^{-x^{-1}}x+\gamma^{-x}x^2).$$

Write $\rho_0 = u_0+u_1 x+u_2 x^2$ with $u_i \in \mathscr{F}(Y, \mathscr{P})$, for $0 \leqslant i \leqslant 2$. Then $d(\rho_0) \geqslant d(u_0)$, and $u_0 = 1-(1-\rho x_*)^{(1)}\gamma^{(1)}\gamma^{-1} = (u_{00}, u_{01}, u_{02})$, where $u_{00} = \rho x_*$, and $d(u_0) \geqslant d(u_{00}) = d(\rho)$. Thus, $d(\rho) \geqslant d(\rho_0) \geqslant d(u_0) \geqslant d(\rho)$, $\rho \in k\langle\gamma\rangle$, $\rho = \rho_0 = u_0 = u_{00} = \rho x_*$, $\rho = 0$; a contradiction is reached.

6. *Some open problems*

(1) Does $\mathscr{G}$ contain maximal subgroups which are not normal? In this respect, let $c \in \mathscr{G}'$ and let $\mathscr{H} = \langle c\gamma, x\rangle$. Decide whether or not $\mathscr{H} = \mathscr{G}$. If $\mathscr{H} \neq \mathscr{G}$ for some c, then obviously a maximal subgroup of $\mathscr{G}$ containing $\mathscr{H}$ will exist and will not be normal in $\mathscr{G}$.

(2) Use the explicit recursive presentation for $\mathscr{G}$ given in [**11**] to decide whether or not $\mathscr{G}$ admits non-trivial central extensions. In other terms, is the Schur Multiplier of $\mathscr{G}$ different from zero. Is it finitely generated? This last question was posed by G. Baumslag in [**2**, p. 28]. We recall in this context the construction in [**1**] of a 3-generated infinitely-related metabelian group with zero Schur Multiplier.

(3) It was shown by R. I. Grigorchuck in [**4**] that 'arboreal' Burnside p-groups defined by him have growth functions a_n asymptotic to $e^{\sqrt{n}}$. He also showed that $a_n(\mathscr{G}) \leqslant e^{n^{2/3}}$ and posed the question of determining the exact asymptotic nature of $a_n(\mathscr{G})$. J. Fabrykowski and N. Gupta proved in [**3**] that a certain non-periodic variant of $\mathscr{G}$ has sub-exponential growth.

(4) Does the ring $\mathscr{P}$ contain an algebra of polynomials of rank 2? A free algebra of rank 2?

(5) Exhibit explicitly a faithful irreducible $\mathscr{P}$-module. Are there infinitely many nonequivalent such modules?

(6) Is $R = \mathbb{Z}_3[\mathscr{G}]$ a primitive ring? Does R contain an algebra of polynomials of rank 2? A free algebra of rank 2?

(7) Comparing $\mathscr{G}$ with the Golod–Shafarevich group [**14**], does there exist a quotient of R which embeds $\mathscr{G}$ and where the image of the augmentation ideal $w(R)$ is nil? Clearly, $w(R)$ itsef is not nil. As a matter of fact, in the case of the Golod–Shafarevich groups, P. N. Siderov [**10**] has shown that for 'most' of these, the augmentation ideal of the group algebra is different from the Golod algebra.

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