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CONDITIONAL ORDERING OF GENERALIZED ORDER STATISTICS REVISITED

HONGMEI XIE
Department of Mathematics
Shihezi University
Shihezi, Xinjiang 832003, People's Republic of China

TAIZHONG HU
Department of Statistics and Finance
University of Science and Technology of China
Hefei, Anhui 230026, People's Republic of China
E-mail: thu@ustc.edu.cn

In this article we investigate less restrictive conditions on the model parameters that enable one to establish the likelihood ratio ordering of one generalized order statistic by conditioning on the right tail of another lower-indexed generalized order statistic. One application of the main results is also presented.

1. INTRODUCTION

The concept of generalized order statistics (GOSs) was introduced by Kamps [9,10] as a unification of several models of random variables arranged in ascending order of magnitude with different interpretations and statistical applications. Specifically, let $n \in \mathbb{N} \equiv \{1, 2, \ldots\}$, $k > 0$, and $\widetilde{m}_n = (m_1, \ldots, m_{n-1}) \in \Re^n$ be parameters such that

$$\gamma_{r,n} = k + \sum_{j=r}^{n-1} (m_j + 1) > 0, \qquad r = 1, \ldots, n-1, \tag{1.1}$$

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if $n \geq 2$ ($\tilde{m}_n$ arbitrary if $n = 1$). If the random variables $U_{(r,n,\tilde{m}_n,k)}, r = 1, \ldots, n$, possess a joint density of the form

$$f_{U_{(1,n,\tilde{m}_n,k)},\ldots,U_{(n,n,\tilde{m}_n,k)}}(u_1, \ldots, u_n) = \left(\prod_{j=1}^{n} \gamma_{j,n}\right)\left(\prod_{i=1}^{n-1}(1-u_i)^{m_i}\right)(1-u_n)^{k-1}$$

for all $0 \leq u_1 \leq u_2 \leq \cdots \leq u_n < 1$ with $\gamma_{n,n} = k$, then they are called uniform GOSs. Now, let F be an arbitrary distribution function. The random variables

$$X_{(r,n,\tilde{m}_n,k)} = F^{-1}(U_{(r,n,\tilde{m}_n,k)}), \qquad r = 1, \ldots, n, \tag{1.2}$$

are called the GOSs based on F, where F^{-1} is the right-continuous version of its inverse. In the particular case $m_1 = \cdots = m_{n-1} = m$, $X_{(r,n,\tilde{m}_n,k)}$ is denoted by $X_{(r,n,m,k)}$ for $r = 1, \ldots, n$.

Denote by $[Z|A]$ any random variable whose distribution is the conditional distribution of Z given event A. Hu, Jin, and Khaledi [6] proved, among other things, that if $m \geq -1$ and F is absolutely continuous, then

$$\left[X_{(s,n,m,k)} - y | X_{(r,n,m,k)} > y\right] \leq_{\mathrm{lr}} \left[X_{(s',n',m,k)} - y | X_{(r',n',m,k)} > y\right] \quad \text{for } y \in \Re \tag{1.3}$$

whenever $s > r$ and $s' - s = r' - r \geq \max\{0, n' - n\}$. Under the same conditions as earlier, Hu, Li, Xu, and Zhuang [7] established that

$$\left[X_{(s,n,m,k)} - y | X_{(r,n,m,k)} > y\right] \leq_{\mathrm{lr}} \left[X_{(s+1,n,m,k)} - y | X_{(r,n,m,k)} > y\right] \tag{1.4}$$

and

$$\left[X_{(s,n,m,k)} - y | X_{(r+1,n,m,k)} > y\right] \leq_{\mathrm{lr}} \left[X_{(s,n,m,k)} - y | X_{(r,n,m,k)} > y\right] \tag{1.5}$$

for $r < s$ and any y. Here, $\leq_{\mathrm{lr}}$ denotes the likelihood ratio order. See Müller and Stoyan [15] and Shaked and Shanthikumar [16] for more details on various kinds of univariate and multivariate stochastic orders, their interrelationships, and their properties.

The purpose of this article is to establish analogous results of (1.3)–(1.5) for GOSs with unequal m_i, generalizing the main results in Hu et al. [6,7]. More precisely, we investigate conditions on parameters $(r, s, n, r', s', n', m_i)$ such that

$$\left[X_{(s,n,\tilde{m}_n,k)} - y | X_{(r,n,\tilde{m}_n,k)} > y\right] \leq_{\mathrm{lr}} \left[X_{(s',n',\tilde{m}_{n'},k)} - y | X_{(r',n',\tilde{m}_{n'},k)} > y\right]$$

for $y \in \Re$, where $1 \leq r < s \leq n$ and $1 \leq r' < s' \leq n$. The motivation of such a study can be found in Khaledi and Shaked [12] and Khaledi and Shojaer [13].

The article is organized as follows. The main results and their proofs are given in Section 3. Some useful lemmas, which will be used in Section 3, are presented in Section 2. One application is presented in Section 4.

Throughout the article, the terms 'increasing' and 'decreasing' are used in the weak sense, all ratios are well defined, and $a/0$ is understood to be $+\infty$ whenever $a > 0$.

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2. PRELIMINARIES

Let $\{X_{(r,n,\tilde{m}_n,k)}, r = 1, \ldots, n\}$ be the GOSs based on distribution function F and define

$$\overline{F}_y(x) = \frac{\overline{F}(\max\{y, y+x\})}{\overline{F}(y)}, \qquad x \in \Re,$$

the conditional survival function of $[X - y | X > y]$, where X is a random variable with distribution function F.

LEMMA 2.1: *For each* i, $0 \le i < n$, *denote* $\tilde{\mu}_i = (\mu_{i,1}, \ldots, \mu_{i,n-i-1}) = (m_{i+1}, \ldots, m_{n-1})$. *Then*

$$\left\{\left[X_{(s,n,\tilde{m}_n,k)} - y | X_{(i+1,n,\tilde{m}_n,k)} > y \ge X_{(i,n,\tilde{m}_n,k)}\right], s = i+1, \ldots, n\right\}$$

$$\stackrel{\text{st}}{=} \left\{X^y_{(s-i,n-i,\tilde{\mu}_i,k)}, s = i+1, \ldots, n\right\} \tag{2.1}$$

$$\stackrel{\text{st}}{=} \left\{\left[X_{(s,n,\tilde{m}_n,k)} - y | X_{(i,n,\tilde{m}_n,k)} = y\right], s = i+1, \ldots, n\right\}, \tag{2.2}$$

where $X_{(0,n,\tilde{m}_n,k)} \equiv -\infty$, $\stackrel{\text{st}}{=}$ *means equality in distribution and* $X^y_{(s-i,n-i,\tilde{\mu}_i,k)}$ *is the* $(s-i)$*th* GOS *based on distribution function* F_y.

PROOF: We give the proof of (2.1); the proof of (2.2) is similar or can be found in Kamps and Cramer [11, Lemma 2]. The proof of (2.7) in Hu et al. [7] for the case $m_1 = \cdots = m_{n-1} = m$ is still valid for the lemma with a minor modification. We give the details. It is well known that $\{X_{(r,n,\tilde{m}_n,k)}, r = 1, \ldots, n\}$ can be regarded as sequential order statistics based on distribution functions $\{F_i, i = 1, \ldots, n\}$, where

$$F_i(x) = 1 - [\overline{F}(x)]^{\alpha_{i,n}}, \qquad x \in \Re,$$

and

$$\alpha_{i,n} = \frac{\gamma_{i,n}}{n-i+1}, \qquad i = 1, \ldots, n; \tag{2.3}$$

see Kamps [10] or Cramer and Kamps [4,5]. Using this observation, we conclude that

$$\left\{\left[X_{(s,n,\tilde{m}_n,k)} - y | X_{(i+1,n,\tilde{m}_n,k)} > y \ge X_{(i,n,\tilde{m}_n,k)}\right], s = i+1, \ldots, n\right\}$$

can be regarded as sequential order statistics of size $n - i$ based on distribution functions $\{G_j, j = 1, \ldots, n-i\}$, where

$$G_j(t) = 1 - \overline{F}_y^{\alpha_{i+j,n}}(t), \qquad t \in \Re.$$

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From (1.1) and (2.3), it follows that

$$\begin{aligned}\alpha_{i+j,n} &= \frac{k+\sum_{l=i+j}^{n-1}(m_l+1)}{n-i-j+1}\\ &= \frac{k+\sum_{l=j}^{(n-i)-1}(m_{i+l}+1)}{(n-i)-j+1}\\ &= \frac{k+\sum_{l=j}^{(n-i)-1}(\mu_{i,l}+1)}{(n-i)-j+1}\\ &\stackrel{\text{def}}{=} \widetilde{\alpha}_{j,n-i},\end{aligned}$$

which implies that $\{X^{y}_{(s-i,n-i,\tilde{\mu}_i,k)}, s=i+1,\ldots,n\}$ can also be regarded as sequential order statistics of size $n-i$ based on distribution functions $\{G_j, j=1,\ldots,n-i\}$. Therefore, (2.1) follows. ■

LEMMA 2.2: *Let $\tilde{\mu}_i$ be as defined in Lemma 2.1. If F is absolutely continuous, then*

$$X_{(s-r,n-r,\tilde{\mu}_r,k)} \leq_{\text{lr}} X_{(s-i,n-i,\tilde{\mu}_i,k)} \tag{2.4}$$

for $0 \leq i < r < s \leq n$.

PROOF: Fix (i,r,s) such that $0 \leq i < r < s \leq n$. Let $\{B_{j,n-r}, j=1,\ldots,n-r\}$ be a sequence of independent random variables such that $B_{j,n-r}$ is exponentially distributed with parameter $\widetilde{\gamma}_{j,n-r}$ given by

$$\widetilde{\gamma}_{j,n-r} = k+\sum_{l=j}^{n-r-1}(m_{r+l}+1) = k+\sum_{l=j+r}^{n-1}(m_l+1).$$

Similarly, let $\{B_{j,n-i}, j=1,\ldots,n-i\}$ be another sequence of independent random variables such that $B_{j,n-i}$ is exponentially distributed with parameter $\widetilde{\gamma}_{j,n-i}$ given by

$$\widetilde{\gamma}_{j,n-i} = k+\sum_{l=j+i}^{n-1}(m_l+1).$$

By Theorem 3.1 in Cramer and Kamps [5] (see also Hu and Zhuang [8, Lemma 2.1]),

$$X_{(s-r,n-r,\tilde{\mu}_r,k)} \stackrel{\text{st}}{=} H_F\left(\sum_{j=1}^{s-r} B_{j,n-r}\right) \tag{2.5}$$

and

$$X_{(s-i,n-i,\tilde{\mu}_i,k)} \stackrel{\text{st}}{=} H_F\left(\sum_{j=1}^{s-i} B_{j,n-i}\right), \tag{2.6}$$

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where $H_F(x) = F^{-1}(1 - e^{-x})$ for $x \in \Re_+$. Since $\widetilde{\gamma}_{j,n-r} = \widetilde{\gamma}_{j+r-i,n-i}$ for each $j = 1, \ldots, s - r$, it follows that

$$\sum_{j=r-i+1}^{s-i} B_{j,n-i} \stackrel{\text{st}}{=} \sum_{j=1}^{s-r} B_{j,n-r}.$$

Note that exponential distributions have log-concave densities. Thus, by Theorem 1.C.9 in Shaked and Shanthikumar [16], we have

$$\sum_{j=1}^{s-r} B_{j,n-r} \leq_{\text{lr}} \sum_{j=r-i+1}^{s-i} B_{j,n-i} + \sum_{j=1}^{r-i} B_{j,n-i}$$

$$= \sum_{j=1}^{s-i} B_{j,n-i}. \tag{2.7}$$

Since F is absolutely continuous, $H_F(x) = \overline{F}^{-1}(e^{-x})$ is increasing and differentiable. Using the fact that the likelihood ratio order is closed under increasing and differentiable transformations (see [16, Thm. 1.C.8]), it follows from (2.5)–(2.7) that (2.4) holds. ■

The next three lemmas (Lemmas 2.3–2.5) are of independent interest. We first recall the definition of the multivariate likelihood ratio order (see [16, Sect. 6.E]). Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional random vectors with density functions f and g, respectively. We say that $\mathbf{X}$ is less than $\mathbf{Y}$ in the *multivariate likelihood ratio order*, denoted by $\mathbf{X} \leq_{\text{lr}} \mathbf{Y}$, if

$$f(\mathbf{x})g(\mathbf{y}) \leq f(\mathbf{x} \wedge \mathbf{y})g(\mathbf{x} \vee \mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in \Re^n,$$

where $\mathbf{x} \wedge \mathbf{y} = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\})$ and $\mathbf{x} \vee \mathbf{y} = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\})$. In the slightly more special case, when $\mathbf{X}$ and $\mathbf{Y}$ are nonnegative, some of the X_i's might be identically zero and the joint distribution of the rest is absolutely continuous or discrete. Suppose that $X_1, \ldots, X_r$ are those that are identically zero for some $0 < r < n$. Also, let f denote the joint density of $(X_{r+1}, \ldots, X_n)$. In this case, $\mathbf{X} \leq_{\text{lr}} \mathbf{Y}$ if

$$\begin{aligned} &f(x_{r+1}, \ldots, x_n)g(y_1, \ldots, y_n) \\ &\quad \leq f(x_{r+1} \wedge y_{r+1}, \ldots, x_n \wedge y_n)g(y_1, \ldots, y_r, x_{r+1} \vee y_{r+1}, \ldots, x_n \vee y_n) \end{aligned} \tag{2.8}$$

for all $(x_{r+1}, \ldots, x_n) \in \Re^{n-r}$ and $\mathbf{y} \in \Re^n$.

LEMMA 2.3: *If $m_1 \geq m_2 \geq \cdots \geq m_{n-1} \geq -1$ and F is absolutely continuous, then*

$$(X_{(1,n,\tilde{m}_n,k)}, X_{(2,n,\tilde{m}_n,k)}, \ldots, X_{(n-1,n,\tilde{m}_n,k)}) \leq_{\text{lr}} (X_{(2,n,\tilde{m}_n,k)}, X_{(3,n,\tilde{m}_n,k)}, \ldots, X_{(n,n,\tilde{m}_n,k)}).$$

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PROOF: First note that the order $\leq_{\mathrm{lr}}$ is preserved under strictly monotone transformations of each individual coordinate of the underlying random vectors (see [16, Thm. 6.E.3]). In view of (1.2) and the fact that F^{-1} is strictly increasing when F is continuous, we thus have to prove that

$$(U_{(1,n,\tilde{m}_n,k)}, U_{(2,n,\tilde{m}_n,k)}, \ldots, U_{(n-1,n,\tilde{m}_n,k)}) \leq_{\mathrm{lr}} (U_{(2,n,\tilde{m}_n,k)}, U_{(3,n,\tilde{m}_n,k)}, \ldots, U_{(n,n,\tilde{m}_n,k)}) \tag{2.9}$$

for the uniform GOSs. Since the multivariate order $\leq_{\mathrm{lr}}$ is closed under marginalization, to prove (2.9) it suffices to verify that

$$(0, U_{(1,n,\tilde{m}_n,k)}, \ldots, U_{(n-1,n,\tilde{m}_n,k)}) \leq_{\mathrm{lr}} (U_{(1,n,\tilde{m}_n,k)}, U_{(2,n,\tilde{m}_n,k)}, \ldots, U_{(n,n,\tilde{m}_n,k)}).$$

Since the joint density of the first r uniform GOSs $(U_{(1,n,\tilde{m}_n,k)}, \ldots, U_{(r,n,\tilde{m}_n,k)})$ is given by

$$\begin{aligned} &f_{U_{(1,n,\tilde{m}_n,k)},\ldots,U_{(r,n,\tilde{m}_n,k)}}(u_1, \ldots, u_r) \\ &\quad = \left(\prod_{j=1}^{r} \gamma_{j,n}\right)\left(\prod_{i=1}^{r-1}(1-u_i)^{m_i}\right)(1-u_r)^{\gamma_{r,n}-1} \end{aligned} \tag{2.10}$$

for all $0 \leq u_1 \leq u_2 \leq \cdots \leq u_r < 1$ (see [10, Lemma 3.1]), by (2.8) we have to prove that for all $0 \leq u_2 \leq \cdots \leq u_n < 1$ and $0 \leq v_1 \leq v_2 \leq \cdots \leq v_n$,

$$\begin{aligned} &\left(\prod_{i=1}^{n-2}(1-u_{i+1})^{m_i}\right)(1-u_n)^{\gamma_{n-1,n}-1} \cdot \left(\prod_{i=2}^{n-1}(1-v_i)^{m_i}\right)(1-v_n)^{k-1} \\ &\quad \leq \left(\prod_{i=1}^{n-2}(1-u_{i+1}\wedge v_{i+1})^{m_i}\right)(1-u_n\wedge v_n)^{\gamma_{n-1,n}-1} \\ &\quad \times \left(\prod_{i=2}^{n-1}(1-u_i\vee v_i)^{m_i}\right)(1-u_n\vee v_n)^{k-1}. \end{aligned} \tag{2.11}$$

Set $E_1 = \{i : u_i > v_i, i = 2, \ldots, n-1\}$. Then (2.11) reduces to

$$\begin{aligned} &\left(\prod_{i\in E_1}(1-u_i)^{m_{i-1}}(1-v_i)^{m_i}\right)(1-u_n)^{m_{n-1}+1} \\ &\quad \leq \left(\prod_{i\in E_1}(1-v_i)^{m_{i-1}}(1-u_i)^{m_i}\right)(1-u_n\wedge v_n)^{m_{n-1}+1}, \end{aligned}$$

which is true if $m_1 \geq m_2 \geq \cdots \geq m_{n-1} \geq -1$. This completes the proof of the lemma. ■

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LEMMA 2.4: *Let* $\{X_{(r,n,\tilde{m}_n,k}), r = 1,\ldots,n\}$ *and* $\{X_{(r,n+1,\tilde{m}_{n+1},k)}, r = 1,\ldots,n+1\}$ *be* GOS*s based on an absolutely continuous distribution* F*, respectively, where* $\tilde{m}_{n+1} = (\tilde{m}_n, m_n)$*. If* $m_n \geq -1$*, then*

$$(X_{(1,n+1,\tilde{m}_{n+1},k)},\ldots,X_{(n,n+1,\tilde{m}_{n+1},k)}) \leq_{\mathrm{lr}} (X_{(1,n,\tilde{m}_n,k)},\ldots,X_{(n,n,\tilde{m}_n,k)}).$$

PROOF: By the same argument as that of Lemma 2.3, we have to prove that for the uniform GOSs,

$$(U_{(1,n+1,\tilde{m}_{n+1},k)},\ldots,U_{(n,n+1,\tilde{m}_{n+1},k)}) \leq_{\mathrm{lr}} (U_{(1,n,\tilde{m}_n,k)},\ldots,U_{(n,n,\tilde{m}_n,k)}).$$

Exploiting (2.10), it is sufficient to check that for all $0 \leq u_1 \leq u_2 \leq \cdots \leq u_n < 1$ and $0 \leq v_1 \leq v_2 \leq \cdots \leq v_n$,

$$\left(\prod_{i=1}^{n-1}(1-u_i)^{m_i}\right)(1-u_n)^{\gamma_{n,n+1}-1}\left(\prod_{i=1}^{n-1}(1-v_i)^{m_i}\right)(1-v_n)^{k-1}$$

$$\leq \left(\prod_{i=1}^{n-1}(1-u_i\wedge v_i)^{m_i}\right)(1-u_n\wedge v_n)^{\gamma_{n,n+1}-1}$$

$$\times \left(\prod_{i=1}^{n-1}(1-u_i\vee v_i)^{m_i}\right)(1-u_n\vee v_n)^{k-1}$$

or, equivalently,

$$(1-u_n)^{\gamma_{n,n+1}}(1-v_n)^k \leq (1-u_n\wedge v_n)^{\gamma_{n,n+1}}(1-u_n\vee v_n)^k,$$

which follows trivially since $m_n \geq -1$ and $\gamma_{n,n+1} = k + (m_n + 1)$. This completes the proof. ■

LEMMA 2.5: *Let* $\{X_{(r,n,\tilde{m}_n,k)}, r = 1,\ldots,n\}$ *and* $\{X_{(r,n+1,\tilde{m}_{n+1},k)}, r = 1,\ldots,n+1\}$ *be the same as in Lemma* 2.4*. If* $m_1 \geq m_2 \geq \cdots \geq m_n$*, then*

$$(X_{(1,n,\tilde{m}_n,k)},\ldots,X_{(n,n,\tilde{m}_n,k)}) \leq_{\mathrm{lr}} (X_{(2,n+1,\tilde{m}_{n+1},k)},\ldots,X_{(n+1,n+1,\tilde{m}_{n+1},k)}).$$

PROOF: By the same argument as that of Lemma 2.3, we have to prove that for the uniform GOSs,

$$(0, U_{(1,n,\tilde{m}_n,k)},\ldots,U_{(n,n,\tilde{m}_n,k)}) \leq_{\mathrm{lr}} (X_{(1,n+1,\tilde{m}_{n+1},k)}, X_{(2,n+1,\tilde{m}_{n+1},k)},\ldots,X_{(n+1,n+1,\tilde{m}_{n+1},k)}).$$

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Exploiting (2.8) and (2.10), it is sufficient to check that for all $0 \le u_2 \le \cdots \le u_{n+1} < 1$ and $0 \le v_1 \le \cdots \le v_{n+1} < 1$,

$$\begin{aligned}
&\left(\prod_{i=1}^{n-1}(1-u_{i+1})^{m_i}\right)(1-u_{n+1})^{k-1}\left(\prod_{i=2}^{n}(1-v_i)^{m_i}\right)(1-v_{n+1})^{k-1} \\
&\quad\le \left(\prod_{i=1}^{n-1}(1-u_{i+1}\wedge v_{i+1})^{m_i}\right)(1-u_{n+1}\wedge v_{n+1})^{k-1} \\
&\quad\times \left(\prod_{i=2}^{n}(1-u_i\vee v_i)^{m_i}\right)(1-u_{n+1}\vee v_{n+1})^{k-1}
\end{aligned}$$

or, equivalently,

$$\prod_{i\in E_2}(1-u_i)^{m_{i-1}}(1-v_i)^{m_i} \le \prod_{i\in E_2}(1-u_i)^{m_i}(1-v_i)^{m_{i-1}}, \tag{2.12}$$

where $E_2 = \{i : u_i > v_i, i = 2, \ldots, n\}$. Since m_j is decreasing in j, (2.12) holds. This completes the proof. ■

Remark: Theorems 3.1–3.3 in Zhuang and Hu [17] established multivariate likelihood ratio ordering for sequential order statistics, which are analogous to Lemmas 2.3–2.5. Although GOSs can be regarded as sequential order statistics based on special distributions, the conditions imposed on the parameters m_i in Lemmas 2.3–2.5 cannot be deduced from Theorems 3.1–3.3 in Zhuang and Hu [17].

Lemma 2.6 can be used to establish the monotonicity property of a fractional expression in which both numerator and denominator are integrals.

LEMMA 2.6: *(Misra and van der Meulen, [14]): Let Θ and $\mathcal{D}$ be subsets of the real line $\Re$ and let X be a random variable, taking values in $\mathcal{D}$, with a distribution function belonging to the family $\mathcal{P} = \{H(\cdot|\theta), \theta \in \Theta\}$, which satisfies that*

$$H(\cdot|\theta_1) \le_{\mathrm{st}} H(\cdot|\theta_2) \quad \textit{whenever } \theta_1, \theta_2 \in \Theta \textit{ and } \theta_1 < \theta_2.$$

Let $\Psi(x,\theta)$ be a real-valued function defined on $\mathcal{D}\times\Theta$. If $\Psi(x,\theta)$ is increasing [decreasing] in (x,θ), then $\mathsf{E}_\theta[\Psi(X,\theta)]$ is increasing [decreasing] in θ.

3. MAIN RESULTS

Throughout this section, let $\{X_{(i,n,\tilde{m}_n,k)}, i = 1, \ldots, n\}$ and $\{X_{(i,n+1,\tilde{m}_{n+1},k)}, i = 1, \ldots, n+1\}$ be GOSs based on an absolutely continuous distribution function F, where $\tilde{m}_{n+1} = (\tilde{m}_n, m_n)$.

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THEOREM 3.1: *If $m_i \geq -1$ for each i, then*

$$\left[X_{(s,n,\tilde{m}_n,k)} - y | X_{(r+1,n,\tilde{m}_n,k)} > y\right] \leq_{\text{lr}} \left[X_{(s,n,\tilde{m}_n,k)} - y | X_{(r,n,\tilde{m}_n,k)} > y\right] \tag{3.1}$$

whenever $1 \leq r < s \leq n$ and $y \in \Re$.

PROOF: Since the likelihood ratio order is closed under weak convergence (see [16, Thm 1.C.7]), we may assume that all $m_i > -1$; that is, $\gamma_{1,n} > \gamma_{2,n} > \cdots > \gamma_{n,n} = k > 0$. For $x \in \Re_+$, we have

$$\begin{aligned}
&\mathsf{P}\left[X_{(s,n,\tilde{m}_n,k)} - y > x, X_{(r,n,\tilde{m}_n,k)} > y\right] \\
&\quad = \sum_{i=0}^{r-1} \mathsf{P}\left[X_{(s,n,\tilde{m}_n,k)} - y > x, X_{(i+1,n,\tilde{m}_n,k)} > y \geq X_{(i,n,\tilde{m}_n,k)}\right] \\
&\quad = \sum_{i=0}^{r-1} \mathsf{P}\left[X_{(s,n,\tilde{m}_n,k)} - y > x | X_{(i+1,n,\tilde{m}_n,k)} > y \geq X_{(i,n,\tilde{m}_n,k)}\right] \\
&\qquad \times \{\mathsf{P}\left[X_{(i+1,n,\tilde{m}_n,k)} > y\right] - \mathsf{P}[X_{(i,n,\tilde{m}_n,k)} > y]\}
\end{aligned}$$

(see Hu et al. [7, (2.3)]). For any random variable W, denote by F_W and f_W the corresponding distribution and density functions of W, respectively. Thus, by Lemma 2.1 and the identity

$$\mathsf{P}\left[X_{(i+1,n,\tilde{m}_n,k)} > y\right] - \mathsf{P}\left[X_{(i,n,\tilde{m}_n,k)} > y\right] = \frac{\overline{F}(y)}{\gamma_{i+1,n} f(y)} f_{X_{(i+1,n,\tilde{m}_n,k)}}(y)$$

(see [11, Eq. (8)]),

$$\begin{aligned}
&\mathsf{P}[X_{(s,n,\tilde{m}_n,k)} - y > x | X_{(r,n,\tilde{m}_n,k)} > y] \\
&\quad = \sum_{i=0}^{r-1} \overline{F}_{X^y_{(s-i,n-i,\tilde{\mu}_i,k)}}(x) \frac{\overline{F}(y)}{\gamma_{i+1,n} f(y)} \frac{f_{X_{(i+1,n,\tilde{m}_n,k)}}(y)}{\overline{F}_{X_{(r,n,\tilde{m}_n,k)}}(y)},
\end{aligned}$$

so that the conditional density function, $h_{r,s,n,y}(\cdot)$, of $[X_{(s,n,\tilde{m}_n,k)} - y | X_{(r,n,\tilde{m}_n,k)} > y]$ is given by

$$h_{r,s,n,y}(x) = \sum_{i=0}^{r-1} f_{X^y_{(s-i,n-i,\tilde{\mu}_i,k)}}(x) \frac{\overline{F}(y)}{\gamma_{i+1,n} f(y)} \frac{f_{X_{(i+1,n,\tilde{m}_n,k)}}(y)}{\overline{F}_{X_{(r,n,\tilde{m}_n,k)}}(y)}. \tag{3.2}$$

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To prove (3.1), it suffices to verify that $h_{r+1,s,n,y}(x)/h_{r,s,n,y}(x)$ is decreasing in x. From (3.2), it follows that

$$\begin{aligned}\frac{h_{r+1,s,n,y}(x)}{h_{r,s,n,y}(x)} &\propto \frac{\sum_{i=0}^{r}[1/(\gamma_{i+1,n})]f_{X^y_{(s-i,n-i,\tilde{\mu}_i,k)}}(x)f_{X_{(i+1,n,\tilde{m}_n,k)}}(y)}{\sum_{i=0}^{r-1}[1/(\gamma_{i+1,n})]f_{X^y_{(s-i,n-i,\tilde{\mu}_i,k)}}(x)f_{X_{(i+1,n,\tilde{m}_n,k)}}(y)}\\ &= 1+\frac{[1/(\gamma_{r+1,n})]f_{X^y_{(s-r,n-r,\tilde{\mu}_r,k)}}(x)f_{X_{(r+1,n,\tilde{m}_n,k)}}(y)}{\sum_{i=0}^{r-1}[1/(\gamma_{i+1,n})]f_{X^y_{(s-i,n-i,\tilde{\mu}_i,k)}}(x)f_{X_{(i+1,n,\tilde{m}_n,k)}}(y)}\\ &= 1+\frac{[1/(\gamma_{r+1,n})]f_{X_{(r+1,n,\tilde{m}_n,k)}}(y)}{\sum_{i=0}^{r-1}[1/(\gamma_{i+1,n})]f_{X_{(i+1,n,\tilde{m}_n,k)}}(y)\left(f_{X^y_{(s-i,n-i,\tilde{\mu}_i,k)}}(x)/f_{X^y_{(s-r,n-r,\tilde{\mu}_r,k)}}(x)\right)}\end{aligned}$$

is decreasing in x if and only if

$$\frac{f_{X^y_{(s-i,n-i,\tilde{\mu}_i,k)}}(x)}{f_{X^y_{(s-r,n-r,\tilde{\mu}_r,k)}}(x)} \text{ is increasing in } x,$$

which is true by Lemma 2.2. This completes the proof of the theorem. ■

THEOREM 3.2: *If* $m_1 \geq m_2 \geq \cdots \geq m_{n-1} \geq -1$, *then*

$$[X_{(s,n,\tilde{m}_n,k)} - y | X_{(r,n,\tilde{m}_n,k)} > y] \leq_{\text{lr}} [X_{(s+1,n,\tilde{m}_n,k)} - y | X_{(r+1,n,\tilde{m}_n,k)} > y] \tag{3.3}$$

whenever $1 \leq r < s < n$ *and* $y \in \Re$.

PROOF: For any $y \in \Re$ and each pair (i,j), $1 \leq i < j \leq n$, the conditional density function of $\left[X_{(j,n,\tilde{m}_n,k)} - y | X_{(i,n,\tilde{m}_n,k)} > y\right]$ is given by

$$h_{i,j,n,y}(x) = \frac{1}{\mathsf{P}[X_{(i,n,\tilde{m}_n,k)} > y]}\int_y^\infty f_{X_{(i,n,\tilde{m}_n,k)},X_{(j,n,\tilde{m}_n,k)}}(u, x+y)du, \quad x \in \Re_+, \tag{3.4}$$

where $f_{X_{(i,n,\tilde{m}_n,k)},X_{(j,n,\tilde{m}_n,k)}}(\cdot,\cdot)$ is the joint density function of $(X_{(i,n,\tilde{m}_n,k)}, X_{(j,n,\tilde{m}_n,k)})$. To prove (3.3) for $1 \leq r < s < n$, it suffices to verify that

$$\Delta_1(\theta) \equiv \frac{h_{r+1,s+1,n,y}(\theta)}{h_{r,s,n,y}(\theta)} \text{ is increasing in } \theta \in \Re_+.$$

From (3.4), we get

$$\begin{aligned}\Delta_1(\theta) &\propto \frac{\int_y^\infty f_{X_{(r+1,n,\tilde{m}_n,k)},X_{(s+1,n,\tilde{m}_n,k)}}(u,\theta+y)du}{\int_y^\infty f_{X_{(r,n,\tilde{m}_n,k)},X_{(s,n,\tilde{m}_n,k)}}(u,\theta+y)du}\\ &= \mathsf{E}_\theta[\Psi_1(U,\theta)],\end{aligned} \tag{3.5}$$

where

$$\Psi_1(u,\theta) = \frac{f_{X_{(r+1,n,\tilde{m}_n,k)},X_{(s+1,n,\tilde{m}_n,k)}}(u,\theta+y)}{f_{X_{(r,n,\tilde{m}_n,k)},X_{(s,n,\tilde{m}_n,k)}}(u,\theta+y)} \quad \text{for } u \leq \theta+y$$

and $\Psi_1(u,\theta) = \Psi_1(\theta+y,\theta)$ for $u > \theta+y$, and the random variable U has a distribution belonging to the family $\mathcal{P} = \{H(\cdot|\theta), \theta \in \Re_+\}$ with corresponding densities

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given by

$$h(u|\theta) = d(\theta) f_{X_{(r,n,\tilde{m}_n,k)},X_{(s,n,\tilde{m}_n,k)}}(u, \theta + y) \quad \text{for } u > y.$$

Here, $d(\theta)$ is the normalizing constant.

Next, we will prove that

$$H(\cdot|\theta_1) \leq_{\text{lr}} H(\cdot|\theta_2) \text{ whenever } 0 < \theta_1 < \theta_2. \tag{3.6}$$

From (2.2), it follows that

$$h(u|\theta) = d(\theta) f_{X^u_{(s-r,n-r,\tilde{\mu}_r,k)}+u}(\theta + y) f_{X_{(r,n,\tilde{m}_n,k)}}(u).$$

For any $u_2 > u_1 > y$, by Theorem 1.C.27 in Shaked and Shanthikumar [16] we get

$$[X|X > u_1] \leq_{\text{lr}} [X|X > u_2],$$

where X is a random variable with distribution F. Observe that $X^u_{(s-r,n-r,\tilde{\mu}_r,k)} + u$ is the $(s-r)$th GOS based on the distribution of $[X|X > u]$ and that the multivariate order $\leq_{\text{lr}}$ is closed under marginalization. Thus, by Theorem 3.6 in Belzunce, Mercader, and Ruiz [3], we get that if $m_i \geq -1$ for each i, then

$$X^{u_1}_{(s-r,n-r,\tilde{\mu}_r,k)} + u_1 \leq_{\text{lr}} X^{u_2}_{(s-r,n-r,\tilde{\mu}_r,k)} + u_2, \qquad y < u_1 < u_2,$$

which implies that

$$\frac{h(u_2|\theta)}{h(u_1|\theta)} = \frac{f_{X^{u_2}_{(s-r,n-r,\tilde{\mu}_r,k)}+u_2}(\theta + y)}{f_{X^{u_1}_{(s-r,n-r,\tilde{\mu}_r,k)}+u_1}(\theta + y)} \frac{f_{X_{(r,n,\tilde{m}_n,k)}}(u_2)}{f_{X_{(r,n,\tilde{m}_n,k)}}(u_1)}$$

is increasing in $\theta \in (u_1 - y, \infty)$; that is, (3.6) holds and, hence,

$$H(\cdot|\theta_1) \leq_{\text{st}} H(\cdot|\theta_2) \text{ whenever } 0 < \theta_1 < \theta_2. \tag{3.7}$$

Finally, by Lemma 2.3, we get

$$(X_{(r,n,\tilde{m}_n,k)}, X_{(s,n,\tilde{m}_n,k)}) \leq_{\text{lr}} (X_{(r+1,n,\tilde{m}_n,k)}, X_{(s+1,n,\tilde{m}_n,k)}),$$

implying that $\Psi_1(u, \theta)$ is increasing both in u and in θ. Therefore, applying Lemma 2.6 in (3.5) yields that $\Delta_1(\theta)$ is increasing in $\theta \in \Re_+$. This completes the proof. ■

An immediate consequence of Theorems 3.1 and 3.2 is the following.

COROLLARY 3.3: *If* $m_1 \geq m_2 \geq \cdots \geq m_{n-1} \geq -1$, *then*

$$[X_{(s,n,\tilde{m}_n,k)} - y|X_{(r,n,\tilde{m}_n,k)} > y] \leq_{\text{lr}} [X_{(s+1,n,\tilde{m}_n,k)} - y|X_{(r,n,\tilde{m}_n,k)} > y]$$

whenever $1 \leq r < s < n$ *and* $y \in \Re$.

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THEOREM 3.4: *If $m_i \geq -1$ for each i, then*

$$[X_{(s,n+1,\tilde{m}_{n+1},k)} - y | X_{(r,n+1,\tilde{m}_{n+1},k)} > y] \leq_{\text{lr}} [X_{(s,n,\tilde{m}_n,k)} - y | X_{(r,n,\tilde{m}_n,k)} > y] \tag{3.8}$$

whenever $1 \leq r < s \leq n$ and $y \in \Re$.

PROOF: We use the same notations as in the proof of Theorem 3.2. It suffices to prove that

$$\Delta_2(\theta) \equiv \frac{h_{r,s,n+1,y}(\theta)}{h_{r,s,n,y}(\theta)} \text{ is decreasing in } \theta \in \Re_+.$$

From (3.4), we get that

$$\begin{aligned} \Delta_2(\theta) &\propto \frac{\int_y^\infty f_{X_{(r,n+1,\tilde{m}_{n+1},k)},X_{(s,n+1,\tilde{m}_{n+1},k)}}(u, \theta + y) du}{\int_y^\infty f_{X_{(r,n,\tilde{m}_n,k)},X_{(s,n,\tilde{m}_n,k)}}(u, \theta + y) du} \\ &= \mathsf{E}_\theta[\Psi_2(U, \theta)], \end{aligned} \tag{3.9}$$

where

$$\Psi_2(u, \theta) = \frac{f_{X_{(r,n+1,\tilde{m}_{n+1},k)},X_{(s,n+1,\tilde{m}_{n+1},k)}}(u, \theta + y)}{f_{X_{(r,n,\tilde{m}_n,k)},X_{(s,n,\tilde{m}_n,k)}}(u, \theta + y)} \quad \text{for } u \leq \theta + y,$$

$\Psi_2(u, \theta) = \Psi_2(\theta + y, \theta)$ for $u > \theta + y$, and U is the same as that in the proof of Theorem 3.2 with property (3.7). By Lemma 2.4, we get

$$(X_{(r,n+1,\tilde{m}_{n+1},k)}, X_{(s,n+1,\tilde{m}_{n+1},k)}) \leq_{\text{lr}} (X_{(r,n,\tilde{m}_n,k)}, X_{(s,n,\tilde{m}_n,k)}),$$

implying that $\Psi_2(u, \theta)$ is decreasing both in u and in θ. Therefore, applying Lemma 2.6 in (3.9) yields that $\Delta_2(\theta)$ is decreasing in $\theta \in \Re_+$. This completes the proof. ■

THEOREM 3.5: *If $m_1 \geq m_2 \geq \cdots \geq m_n \geq -1$, then*

$$[X_{(s,n,\tilde{m}_n,k)} - y | X_{(r,n,\tilde{m}_n,k)} > y] \leq_{\text{lr}} [X_{(s+1,n+1,\tilde{m}_{n+1},k)} - y | X_{(r+1,n+1,\tilde{m}_{n+1},k)} > y] \tag{3.10}$$

whenever $1 \leq r < s \leq n$ and $y \in \Re$.

PROOF: We use the same notations as in the proof of Theorem 3.2. It suffices to prove that

$$\Delta_2(\theta) \equiv \frac{h_{r+1,s+1,n+1,y}(\theta)}{h_{r,s,n,y}(\theta)} \text{ is increasing in } \theta \in \Re_+.$$

From (3.4), we get that

$$\begin{aligned} \Delta_2(\theta) &\propto \frac{\int_y^\infty f_{X_{(r+1,n+1,\tilde{m}_{n+1},k)},X_{(s+1,n+1,\tilde{m}_{n+1},k)}}(u, \theta + y) du}{\int_y^\infty f_{X_{(r,n,\tilde{m}_n,k)},X_{(s,n,\tilde{m}_n,k)}}(u, \theta + y) du} \\ &= \mathsf{E}_\theta[\Psi_3(U, \theta)], \end{aligned} \tag{3.11}$$

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where

$$\Psi_3(u,\theta) = \frac{f_{X_{(r+1,n+1,\tilde{m}_{n+1},k)},X_{(s+1,n+1,\tilde{m}_{n+1},k)}}(u,\theta+y)}{f_{X_{(r,n,\tilde{m}_n,k)},X_{(s,n,\tilde{m}_n,k)}}(u,\theta+y)} \quad \text{for } u \le \theta + y,$$

$\Psi_3(u,\theta) = \Psi_3(\theta+y,\theta)$ for $u > \theta + y$, and U is the same as that in the proof of Theorem 3.2 with property (3.7). By Lemma 2.5, we get

$$(X_{(r,n,\tilde{m}_n,k)}, X_{(s,n,\tilde{m}_n,k)}) \le_{\text{lr}} (X_{(r+1,n+1,\tilde{m}_{n+1},k)}, X_{(s+1,n+1,\tilde{m}_{n+1},k)}),$$

implying that $\Psi_3(u,\theta)$ is increasing both in u and in θ. Therefore, applying Lemma 2.6 in (3.11) yields that $\Delta_3(\theta)$ is increasing in $\theta \in \Re_+$. This completes the proof. ■

4. ONE APPLICATION

In a progressive type II censoring scheme, N units are placed on a lifetime test. The failure times are described by independent and identically distributed random variables with a common distribution F. A number n $(n \le N)$ of units are observed to fail. A predetermined number R_i of surviving units at the time of the ith failure are randomly selected and removed from further testing. Thus, $\sum_{i=1}^{n} R_i$ units are progressively censored; hence, $N = n + \sum_{i=1}^{n} R_i$. The n observed failure times are called progressive type II censored order statistics based on F, denoted by $X^{\mathbf{R}}_{1:n,N} \le X^{\mathbf{R}}_{2:n,N} \le \cdots \le X^{\mathbf{R}}_{n:n,N}$, where $\mathbf{R} = (R_1, \ldots, R_n)$ is a censoring policy. For more details on this model, we refer to Balakrishnan and Aggarwala [2] and Balakrishnan [1].

Note that n progressive type II censored order statistics $\{X^{\mathbf{R}}_{i:n,N}, i = 1, \ldots, n\}$ based on F correspond to n GOSs based on F with parameters $k = R_n + 1 \ge 1$ and $m_r = R_r \ge 0$ for $r = 1, \ldots, n-1$. From Theorems 3.1 and 3.2, Corollary 3.3, and Lemma 2.3, we have the following.

COROLLARY 4.1: *Let F be absolutely continuous.*

(1) For $y \in \Re_+$ and $1 \le r < s \le n$, we have

$$[X^{\mathbf{R}}_{s:n,N} - y | X^{\mathbf{R}}_{r+1:n,N} > y] \le_{\text{lr}} [X^{\mathbf{R}}_{s:n,N} - y | X^{\mathbf{R}}_{r:n,N} > y].$$

(2) If $R_1 \ge R_2 \ge \cdots \ge R_{n-1}$, then for $y \in \Re_+$ and $1 \le r < s < n$,

$$[X^{\mathbf{R}}_{s:n,N} - y | X^{\mathbf{R}}_{r:n,N} > y] \le_{\text{lr}} [X^{\mathbf{R}}_{s+1:n,N} - y | X^{\mathbf{R}}_{r+1:n,N} > y],$$
$$[X^{\mathbf{R}}_{s:n,N} - y | X^{\mathbf{R}}_{r:n,N} > y] \le_{\text{lr}} [X^{\mathbf{R}}_{s+1:n,N} - y | X^{\mathbf{R}}_{r:n,N} > y].$$

(3) If $R_1 \ge R_2 \ge \cdots \ge R_{n-1}$, then

$$(X^{\mathbf{R}}_{1:n,N}, \ldots, X^{\mathbf{R}}_{n-1:n,N}) \le_{\text{lr}} (X^{\mathbf{R}}_{2:n,N}, \ldots, X^{\mathbf{R}}_{n:n,N}).$$

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