*Mathematika* 58 (2012) 409–418 © University College London
doi:10.1112/S0025579311001847

# RATIONAL DISTANCES WITH RATIONAL ANGLES

RYAN SCHWARTZ, JÓZSEF SOLYMOSI AND FRANK DE ZEEUW

*Abstract.* In 1946 Erdős asked for the maximum number of unit distances, $u(n)$, among $n$ points in the plane. He showed that $u(n) > n^{1+c/\log\log n}$ and conjectured that this was the true magnitude. The best known upper bound is $u(n) < cn^{4/3}$, due to Spencer, Szemerédi and Trotter. We show that the upper bound $n^{1+6/\sqrt{\log n}}$ holds if we only consider unit distances with rational angle, by which we mean that the line through the pair of points makes a rational angle in degrees with the $x$-axis. Using an algebraic theorem of Mann we get a uniform bound on the number of paths between two fixed vertices in the unit distance graph, giving a contradiction if there are too many unit distances with rational angle. This bound holds if we consider rational distances instead of unit distances as long as there are no three points on a line. A superlinear lower bound is given, due to Erdős and Purdy. If we have at most $n^\alpha$ points on a line then we get the bound $O(n^{1+\alpha})$ or $n^{1+\alpha+6/\sqrt{\log n}}$ for the number of rational distances with rational angle depending on whether $\alpha \geq 1/2$ or $\alpha < 1/2$ respectively.

§1. *Introduction.* A famous problem of Erdős from 1946 [3] concerns the maximal number of unit distances among $n$ points in the plane; we will denote this number by $u(n)$. He showed that $u(n) > n^{1+c/\log\log n}$, using a $\sqrt{n} \times \sqrt{n}$ piece of a scaled integer lattice, and conjectured that this was the true magnitude. The best known upper bound is $u(n) < cn^{4/3}$, first proved by Spencer *et al* in 1984 [8]. This bound has several other proofs, the simplest of which was the proof by Székely [9], using a lower bound for the crossing number of graphs. A recent result of Matoušek [7] shows that the number of unit distances is bounded above by $cn \log n \log\log n$ for most norms. As a general reference for work done on the unit distances problem, see [2].

We will show that the upper bound $n^{1+6/\sqrt{\log n}}$ holds if we only consider unit distances that have *rational angle*, by which we mean that the line through the pair of points makes a rational angle in degrees with the $x$-axis (or equivalently, its angle in radians, divided by $\pi$, is rational). Under this restriction, we can use an algebraic theorem of Mann [6] to get a uniform bound on the number of paths between two fixed vertices in the unit distance graph, which will lead to a contradiction if there are too many unit distances with rational angle between the points.

Received 24 September 2010, published online 28 November 2011.
MSC (2000): 52C10 (primary), 52C30 (secondary).

https://doi.org/10.1112/S0025579311001847 Published online by Cambridge University Press

In fact, our proof also shows that the bound $n^{1+6/\sqrt{\log n}}$ holds for the number of rational distances with rational angles, if we have no three points on a line. The lower bound, $n^{1+c/\log\log n}$, of Erdős does not apply in this case as we are restricted to rational angles. However, a construction of Erdős and Purdy gives a superlinear lower bound for unit (and hence rational) distances with rational angles.

If instead we allow up to $n^\alpha$ points on a line where $1/2 \leq \alpha \leq 1$, the number of rational distances with rational angles is bounded by $4n^{1+\alpha}$. This bound is tight up to a constant factor with the lower bound now coming from an $n^{1-\alpha} \times n^\alpha$ square grid. If we allow up to $n^\alpha$ points on a line where $0 < \alpha < 1/2$, the number of rational distances with rational angles is bounded above by $n^{1+\alpha+6/\sqrt{\log n}}$. We get a lower bound of $cn^{1+\alpha}$ from $n^{1-\alpha}$ horizontal lines each containing $n^\alpha$ rational points so that no three points on different lines are collinear.

In §2 we will state our main results and give an outline of the proof. Section 3 contains the algebraic tools that we will use, including, for completeness, a proof of Mann's theorem. In §4 we use the bounds obtained from Mann's theorem and some graph theory to prove our main results. In §5 we give lower bounds for the main results.

§2. *Main results and proof sketch.* We will say that a pair of points in $\mathbb{R}^2$ has *rational angle* if the line segment between them, viewed as a complex number $z = re^{\pi i\gamma}$, has $\gamma \in \mathbb{Q}$. Our first result is the following.

THEOREM 2.1. *Given $n$ points in $\mathbb{R}^2$, the number of pairs of points with unit distance and rational angle is at most $n^{1+6/\sqrt{\log n}}$.*

Roughly speaking, our proof goes as follows. Given $n$ points in the plane, we construct a graph with the points as vertices, and as edges the unit line segments that have rational angle. We can represent these unit line segments as complex numbers, which must be roots of unity because of the rational angle condition. Then, if this graph has many edges, it should have many cycles of a given length $k$, and each such cycle would give a solution to the equation

$$\sum_{i=1}^{k} \zeta_i = 0,$$

with $\zeta_i$ a root of unity. Using an algebraic theorem of Mann from 1965 [**6**], we could give a uniform bound on the number of such solutions, depending only on $k$ (under the non-degeneracy condition that no subsum vanishes). If the number of non-degenerate cycles goes to infinity with $n$, this would give a contradiction.

However, dealing with cycles of arbitrary length is not so easy, so instead in our proof we count non-degenerate paths (which we will call *irredundant* paths) of length $k$ between two fixed vertices, which correspond to solutions of the

https://doi.org/10.1112/S0025579311001847 Published online by Cambridge University Press

equation

$$\sum_{i=1}^{k} \zeta_i = a,$$

where $a \in \mathbb{C}$, $a \neq 0$ corresponds to the line segment between the two points. We have extended Mann's theorem to this type of equation, giving a similar upper bound and proving our result.

In fact, in our proof it turns out that it is not necessary for the lengths to be 1, but that they only need to be rational. This is because our extension of Mann's theorem also works for equations of the type

$$\sum_{i=1}^{k} a_i \zeta_i = a,$$

where $a_i \in \mathbb{Q}$ and $a \in \mathbb{C}$, $a \neq 0$. This leads to the following results.

THEOREM 2.2. *Suppose we have n points in $\mathbb{R}^2$, no three of which are on a line. Then the number of pairs of points with rational distance and rational angle is at most $n^{1+6/\sqrt{\log n}}$.*

The constant 6 in this theorem and the next is not optimal, but is the smallest integer that followed directly from our proof.

THEOREM 2.3. *Suppose we have n points in $\mathbb{R}^2$, with no more than $n^\alpha$ on a line, where $0 < \alpha < 1/2$. Then the number of pairs of points with rational distance and rational angle is at most $n^{1+\alpha+6/\sqrt{\log n}}$.*

THEOREM 2.4. *Suppose we have n points in $\mathbb{R}^2$, with no more than $n^\alpha$ on a line, where $1/2 \leq \alpha \leq 1$. Then the number of pairs of points with rational distance and rational angle is at most $4n^{1+\alpha}$.*

§3. *Mann's theorem.* For completeness we provide a proof of Mann's theorem. We then prove the extension that we will need to prove the main result in the next section.

THEOREM 3.1 (Mann). *Suppose we have*

$$\sum_{i=1}^{k} a_i \zeta_i = 0,$$

*with $a_i \in \mathbb{Q}$, the $\zeta_i$ roots of unity, and no subrelations $\sum_{i \in I} a_i \zeta_i = 0$ where $\emptyset \neq I \subsetneq [k]$. Then*

$$(\zeta_i / \zeta_j)^m = 1$$

*for all $i$, $j$, with $m = \prod_{p \text{ prime}}^{p \leq k} p$.*

https://doi.org/10.1112/S0025579311001847 Published online by Cambridge University Press

*Proof.* We can assume that $\zeta_1 = 1$ and $a_1 = 1$, so that we have $1 + \sum_{i=2}^{k} a_i \zeta_i = 0$. We take a minimal $m$ such that $\zeta_i^m = 1$ for each $i$.

We will show that $m$ must be squarefree, and that a prime $p$ that divides $m$ must satisfy $p \leq k$. Together these prove the theorem.

Let $p$ be a prime dividing $m$. Write $m = p^j \cdot m^*$ with $(p, m^*) = 1$, and use that to factor each $\zeta_i$ as follows:

$$\zeta_i = \rho^{\sigma_i} \cdot \zeta_i^*,$$

with $\rho$ a primitive $p^j$th root of unity so

$$\rho^{p^j} = 1, \qquad (\zeta_i^*)^{p^{j-1} m^*} = 1, \quad 0 \leq \sigma_i \leq p - 1.$$

Now reorganize the equation as follows:

$$0 = 1 + \sum_{i=2}^{k} a_i \zeta_i = 1 + \sum_{l=0}^{p-1} \alpha_\ell \rho^\ell = f(\rho),$$

where the coefficients are of the form

$$\alpha_\ell = \sum_{i \in I_\ell} a_i \zeta_i^* \in \mathbb{Q}(\zeta_2^*, \ldots, \zeta_k^*) = K,$$

with $I_\ell = \{i \in [k] : \sigma_i = \ell\}$. Hence $f$ is a polynomial over the field $K$ of degree $\leq p - 1$ and $f(\rho) = 0$. The polynomial $f$ is not identically zero, since that would give a subrelation containing strictly fewer than $k$ terms. To see this, observe that we must have $\sigma_i \geq 1$ for at least one $i$, otherwise $\zeta_i^{m/p} = 1$ for each $i$, contradicting the minimality of $m$.

However, we can compute the degree of $\rho$ over $K$ to be

$$\deg_K(\rho) = \frac{\phi(m)}{\phi(p^{j-1} m^*)} = \frac{\phi(p^j)}{\phi(p^{j-1})} = \begin{cases} p - 1 & \text{if } j = 1, \\ p & \text{if } j > 1. \end{cases}$$

This is a contradiction unless $j = 1$, which proves that $m$ is squarefree. Knowing that $m$ is squarefree, we have $m = p \cdot m^*$ with $(p, m^*) = 1$, and

$$\zeta_i = \rho^{\sigma_i} \cdot \zeta_i^*, \qquad \rho^p = 1, \qquad (\zeta_i^*)^{m^*} = 1, \quad 0 \leq \sigma_i \leq p - 1.$$

It is still the case that $f(\rho) = 0$ for $f(x)$ a polynomial over $K$, not identically zero. However, we know [5, Ch. VI.3] that the minimal irreducible polynomial of $\rho$ over $K$ is $F(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$, hence we must have $f(x) = cF(x)$ for some $c \in K$. In particular, $f$ has $p$ terms, which implies that our original relation had at least $p$ terms, so $k \geq p$. □

THEOREM 3.2. *Suppose we have*

$$\sum_{i=1}^{k} a_i \zeta_i = a, \qquad \sum_{j=1}^{k} a_j^* \zeta_j^* = a,$$

https://doi.org/10.1112/S0025579311001847 Published online by Cambridge University Press

*with $a \in \mathbb{C}$, $a \neq 0$, $a_i \in \mathbb{Q}$, roots of unity $\zeta_i$, and no subrelations $\sum_{i\in I} a_i\zeta_i = 0$ or $\sum_{j\in J} a_j^*\zeta_j^* = 0$ where $\emptyset \neq I \subsetneq [k]$ and $\emptyset \neq J \subsetneq [k]$. Then for any $\zeta_j^*$ there is a $\zeta_i$ such that*

$$(\zeta_j^*/\zeta_i)^m = 1$$

*with* $m = \prod_{\substack{p\le 2k \\ p \text{ prime}}} p$.

*Proof.* We have $\sum a_i\zeta_i = a = \sum a_j^*\zeta_j^*$, which gives the single equation

$$\sum_{i=1}^{k} a_i\zeta_i - \sum_{j=1}^{k} a_j^*\zeta_j^* = 0. \tag{1}$$

Mann's theorem does not apply immediately, because there might be subrelations. However, we can break the equation up into minimal subrelations

$$\sum_{i\in I_\ell} a_i\zeta_i - \sum_{j\in I_\ell^*} a_j^*\zeta_j^* = 0, \tag{2}$$

where each $I_\ell \neq \emptyset$, $I_\ell^* \neq \emptyset$, and there are no further subrelations.

Given $\zeta_j^*$, there is such a minimal subrelation of length $\le 2k$ in which it occurs, and which must also contain some $\zeta_i$. Applying Mann's theorem to this equation gives $(\zeta_j^*/\zeta_i)^m = 1$ with $m = \prod_{\substack{p\le 2k \\ p \text{ prime}}} p$. □

Note that in the above proof we require $a \neq 0$. If $a = 0$ and there is no proper subrelation as in (2) then (1) still has the subrelations

$$\sum_{i=1}^{k} a_i\zeta_i = 0, \qquad \sum_{j=1}^{k} a_j^*\zeta_j^* = 0,$$

so we cannot use Mann's theorem to get a relation between a $\zeta_i$ and $\zeta_j^*$.

For $a \in \mathbb{C}$, $a \neq 0$, $k \in \mathbb{Z}$, $k > 0$ we define $Z_a^k$ to be the set of $k$-tuples of roots of unity $(\zeta_1, \ldots, \zeta_k)$ for which there are $a_i \in \mathbb{Q}$ such that $\sum_{i=1}^k a_i\zeta_i = a$ with no subrelations, *i.e.*

$$Z_a^k = \left\{ (\zeta_1, \ldots, \zeta_k) \;\middle|\; \exists a_i \in \mathbb{Q} : \sum_{i=1}^{k} a_i\zeta_i = a, \sum_{i\in I} a_i\zeta_i \neq 0 \text{ for } \emptyset \neq I \subset [k] \right\}.$$

COROLLARY 3.3. *Let* $C(k) = \prod_{\substack{p\le 2k \\ p \text{ prime}}} p$. *Given* $a \in \mathbb{C}$, $a \neq 0$, $|Z_a^k| \le (k \cdot C(k))^k$.

*Proof.* Fix an element $(\zeta_1, \ldots, \zeta_k) \in Z_a^k$ and let $m = C(k)$ and $M_i = \zeta_i^{-m}$ for $1 \le i \le k$. Then for $\zeta_j^*$ in any element of $Z_a^k$, we have an $i$ such that $M_i(\zeta_j^*)^m = 1$. In other words $\zeta_j^*$ is a solution of $M_i x^m = 1$. Each of these $k$ equations has $m = C(k)$ solutions, hence there are at most $k \cdot m = k \cdot C(k)$ choices for each $\zeta_j^*$. □

https://doi.org/10.1112/S0025579311001847 Published online by Cambridge University Press

§4. *Rational distances and Mann's theorem.* We are now in a position to prove the main results. Suppose we have a graph $G = G(V, E)$ on $v(G) = n$ vertices and $e(G) = cn^{1+\alpha}$ edges. We will denote the minimum degree in $G$ by $\delta(G)$. The following lemma assures us that we can remove low-degree vertices from our graph without greatly affecting the number of edges.

LEMMA 4.1. *Let $G$ be as above. Then $G$ contains a subgraph $H$ with $e(H) = (c/2)n^{1+\alpha}$ edges such that $\delta(H) \geq (c/2)n^{\alpha}$.*

*Proof.* We iteratively remove vertices from $G$ of degree less than $(c/2)n^{\alpha}$. Then, the resulting subgraph $H$ has $\delta(H) \geq (c/2)n^{\alpha}$ and we removed fewer than $(c/2)n^{1+\alpha}$ edges so $H$ contains more than $(c/2)n^{1+\alpha}$ edges. □

Note that the subgraph $H$ constructed above contains at least $v(H) = (c/2)n^{\alpha}$ vertices.

Suppose we are given a path on $k$ edges $P_k = p_0 p_1 \cdots p_k$. We call this path *irredundant* if

$$\sum_{i \in I} \overrightarrow{p_i p_{i+1}} \neq 0$$

for any $\emptyset \neq I \subset \{0, 1, \ldots, k-1\}$.

*Proof of Theorem 2.2.* Let $G$ be the graph with the $n$ points in the plane as vertices and the rational distances with rational angles between pairs of points as edges. Suppose there are $n^{1+f(n)}$ such distances for some positive function $f$. Then $e(G) = n^{1+f(n)}$. We will count the number of irredundant paths $P_k$ in $G$, for a fixed $k$ that we will choose later. By Lemma 4.1 we can assume that $e(G) \geq (1/2)n^{1+f(n)}$, $v(G) \geq (1/2)n^{f(n)}$ and $\delta(G) \geq (1/2)n^{f(n)}$.

The number of irredundant paths $P_k$ starting at any vertex $v$ is at least

$$N = \prod_{\ell=0}^{k-1} (\delta(G) - 2^{\ell} + 1),$$

since, if we have constructed a subpath $P_\ell$ of $P_k$, then at most $2^{\ell} - 1$ of the at least $\delta(G)$ continuations are forbidden. Thus the total number of irredundant paths $P_k$ is at least

$$\frac{nN}{2} \geq (n/2) \prod_{\ell=0}^{k-1} ((1/2)n^{f(n)} - 2^{\ell} + 1) \geq \frac{n^{kf(n)+1}}{2^{2k+1}}$$

if $2^k \leq (1/2)n^{f(n)}$, which is true as long as $k < f(n) \log n / \log 2$. It follows that there are two vertices $v$ and $w$ with at least

$$\frac{N}{n} \geq (1/n) \prod_{\ell=0}^{k-1} ((1/2)n^{f(n)} - 2^{\ell} + 1) \geq \frac{n^{kf(n)-1}}{4^k}$$

irredundant paths $P_k$ between them. We will call the set of these paths $\mathcal{P}_{vw}$, so that we have $|\mathcal{P}_{vw}| \geq n^{kf(n)-1}/4^k$.

https://doi.org/10.1112/S0025579311001847 Published online by Cambridge University Press

Given $P_k \in \mathcal{P}_{vw}$, $P_k = p_0 p_1 \cdots p_k$, consider the $k$-tuple $(\zeta_1, \ldots, \zeta_k)$ where $\zeta_i$ is the root of unity in the direction from $p_{i-1}$ to $p_i$, that is, $\zeta_i = \overrightarrow{p_{i-1}p_i}/|\overrightarrow{p_{i-1}p_i}|$. Note that $(\zeta_1, \ldots, \zeta_k) \in Z_a^k$, because $P_k$ is irredundant. Since there are no three points on a line, this process gives an injective map from $\mathcal{P}_{vw}$ to $Z_a^k$ so $|\mathcal{P}_{vw}| \le (k \cdot C(k))^k$ by Corollary 3.3. Thus

$$\frac{n^{kf(n)-1}}{4^k} \le (k \cdot C(k))^k \Longrightarrow n^{kf(n)-1} \le (4k \cdot C(k))^k.$$

However, this gives

$$e^{(kf(n)-1)\log n} \le e^{k \log(4k \cdot C(k))} \Longrightarrow f(n) \le \frac{\log(4k) + \log(C(k))}{\log n} + \frac{1}{k}.$$

The term $\log(C(k))$ is the log of the product of the primes less than or equal to $2k$. This is a well-known number-theoretic function called the Chebyshev function and denoted by $\vartheta$, specifically $\vartheta(2k) = \log(C(k))$. We use the following bound on $\vartheta$ (for a proof see [1]):

$$\vartheta(x) < 4x \log 2 < 3x \quad \text{for } x \ge 2.$$

This gives

$$f(n) < \frac{\log(4k) + 6k}{\log n} + \frac{1}{k} < \frac{7}{\log n}k + \frac{1}{k}.$$

Let $k$ be an integer such that $f(n)\log n/18 < k < f(n)\log n/14$ (possible since otherwise $f(n) = O(1/\log n)$ giving $n^{f(n)} = O(1)$). Then the condition $k < f(n)\log n/\log 2$ is clearly satisfied, and we get

$$f(n) < \frac{7}{\log n} \cdot \frac{f(n)\log n}{14} + \frac{18}{f(n)\log n} \Longrightarrow f(n) < \frac{6}{\sqrt{\log n}}.$$

This completes the proof. □

*Proof of Theorem 2.1.* In the statement of Theorem 2.1, the requirement that there are no three points on a line is unnecessary. This is because, from any point, there is only one unit distance in any direction. Thus we can apply the same proof as in Theorem 2.2 to Theorem 2.1 without having to worry about multiple points on a line. Thus we also have a proof of Theorem 2.1. □

Consider a path $P_k = p_0 p_1 \cdots p_k$. If the distance from $p_{i-1}$ to $p_i$ is less than the distance from $p_{i-1}$ to any vertex on the line connecting $p_{i-1}$ and $p_i$ and not in $P_{i-1} = p_0 p_1 \cdots p_{i-1}$ then $P_k$ is called a *shortest path*.

*Proof of Theorem 2.3.* This proof is almost the same as the proof of Theorem 2.2 except that instead of considering all irredundant paths $P_k$, we only consider shortest irredundant paths. Suppose there are $n^{1+\alpha+f(n)}$ edges in the rational distance graph. Since there are at most $n^\alpha$ points on a line, we get that from any vertex $v$ there are at least

$$N = \prod_{\ell=0}^{k-1}\left(\frac{\delta(G)}{n^\alpha} - 2^\ell + 1\right) \ge \frac{n^{kf(n)}}{4^k}$$

https://doi.org/10.1112/S0025579311001847 Published online by Cambridge University Press

shortest irredundant paths $P_k$, if $k < f(n)\log n/\log 2$. For any two vertices $v$, $w$ let $\mathcal{P}_{v,w}$ be the set of shortest irredundant paths $P_k$ between $v$ and $w$. Then there are two vertices $v$, $w$ such that the number of shortest irredundant paths between $v$ and $w$ is at least

$$|\mathcal{P}_{v,w}| \geq \frac{n^{kf(n)-1}}{4^k}.$$

By Mann's theorem, since we are looking at shortest irredundant paths, $|\mathcal{P}_{v,w}| \leq (k \cdot C(k))^k$. Let $k$ be an integer such that $f(n)\log n/18 < k < f(n)\log n/14$. Then

$$\frac{n^{kf(n)-1}}{4^k} \leq (k \cdot C(k))^k \Longrightarrow f(n) < \frac{6}{\sqrt{\log n}}. \qquad \square$$

*Proof of Theorem 2.4.* Assume we have a configuration of $n$ points with at most $n^\alpha$ on a line, $1/2 \leq \alpha \leq 1$, and $n^{1+\alpha+f(n)}$ rational distances with rational angles, for some positive function $f(n)$.

The graph $G$ on these points has $e(G) = n^{1+\alpha+f(n)}$. By Lemma 4.1 we can assume that $e(G) \geq n^{1+\alpha+f(n)}/2$, $v(G) \geq n^{\alpha+f(n)}/2$ and $\delta(G) \geq n^{\alpha+f(n)}/2$. We now count irredundant paths $P_2$ of length 2. Note that an irredundant path on two edges is just a non-collinear path.

For any vertex $v$, since we have at most $n^\alpha$ points on a line, $v$ is the midpoint of at least

$$N = \delta(G)(\delta(G) - n^\alpha) \geq \frac{n^{2(\alpha+f(n))}}{8}$$

paths $P_2$ if $f(n) \geq \log 4/\log n$ (if $f(n) < \log 4/\log n$ then $n^{f(n)} < 4$, completing the proof). Thus there are two vertices $v$ and $w$ with at least $(1/8)n^{2(\alpha+f(n))-1}$ non-collinear paths $P_2$ between them.

However, by Corollary 3.3 there is a constant number of directions from each of $v$ and $w$. Since we are looking at non-collinear paths $P_2$, the direction from $v$ and the direction from $w$ uniquely determine the midpoint for a path $P_2$. Thus there are at most $(k \cdot C(k))^k = 144$ non-collinear paths $P_2$ between $v$ and $w$, since $k = 2$.

Putting the upper and lower bounds together we get that $n^{2(\alpha+f(n))-1} \leq 2^7 3^2$. This gives

$$f(n) \leq \frac{7\log 2 + 2\log 3}{2\log n} + \frac{1}{2} - \alpha \leq \frac{7\log 2 + 2\log 3}{2\log n} < \frac{4}{\log n},$$

since $\alpha \geq 1/2$. However, this gives $n^{f(n)} < 4$, completing the proof. $\square$

§5. *Lower bounds.* In this section we give lower bounds for the theorems given in §2.

The bounds in Theorems 2.1 and 2.2 are not far from optimal as the following construction of Erdős and Purdy [**4**] shows.

Suppose we have $n$ points, no three on a line, with the maximum possible number of unit distances with rational angles; we call this number $f(n)$. Consider these points as the set $\{z_1, \ldots, z_n\}$ of complex numbers. For any $a \in \mathbb{C}$

https://doi.org/10.1112/S0025579311001847 Published online by Cambridge University Press

with $|a| = 1$, $a \neq z_i - z_j$ for any $i \neq j$, the set $\{z_1, \ldots, z_n, z_1 + a, \ldots, z_n + a\}$ contains at least $2f(n) + n$ unit distances since there are $f(n)$ amongst each of the sets $\{z_1, \ldots, z_n\}$ and $\{z_1 + a, \ldots, z_n + a\}$ and $|z_i - (z_i + a)| = 1$ for each $i$. This new set may have three points on a line, but we show that we can choose $a$ appropriately so this is not the case.

Consider a pair of points $z_i$ and $z_j$. For each $z_k$, the set of points $\{z_k + a : |a| = 1\}$ intersects the line through $z_i$ and $z_j$ in at most two points. So there are at most two values of $a$ that will give three points on a line. There are $\binom{n}{2}$ pairs of points and $n$ choices for $z_k$ so there are at most $2n\binom{n}{2} = n^2(n-1)$ values of $a$ that make a point $z_k + a$ collinear with two points $z_i$ and $z_j$. Similarly we have $n^2(n-1)$ values of $a$ that make a point $z_k$ collinear with two points $z_i + a$ and $z_j + a$. Thus there are only finitely many values of $a$ that give three points on a line. There are infinitely many choices for $a$ so we are done.

This shows that $f(2n) \geq 2f(n) + n$ for $n > 2$ and clearly $f(2) = 1$. From this we get that $f(2^k) \geq 2^{k-1}(k-1) = 2^{k-1}\log_2(2^{k-1})$. Taking $2^k \leq n < 2^{k+1}$ we get that $f(n) \geq cn \log n$ for all $n$. This construction gives a lower bound for Theorems 2.1 and 2.2.

The bound in Theorem 2.3 is not far from optimal. In fact we can get a lower bound of $cn^{1+\alpha}$. Consider $n^{1-\alpha}$ lines parallel to the $x$-axis, and choose $n^\alpha$ rational points on each line such that no three points on different lines are collinear (this can always be done since there are infinitely many rational points to choose from). There are $cn^{2\alpha}$ rational distances on each horizontal line and $n^{1-\alpha}$ such lines giving at least $cn^{1+\alpha}$ rational distances with rational angles (all the angles are zero).

The bound in Theorem 2.4 is tight up to a constant factor as can be seen by considering an $n^{1-\alpha} \times n^\alpha$ square grid. Then there are at least $cn^{2\alpha}$ rational distances on each of the $n^{1-\alpha}$ horizontal lines in the grid containing $n^\alpha$ points. This gives at least $cn^{1+\alpha}$ rational distances with rational angles (the angles are all zero).

*Acknowledgements.* The authors would like to thank Jirka Matoušek for making us aware of the lower bound for unit distances with no three points on a line. We are indebted to the anonymous referee for helpful comments and suggestions. The second author was supported by a Sloan Fellowship and NSERC and OTKA grants.

## *References*

1. T. M. Apostol, *Introduction to Analytic Number Theory*, Springer (New York, NY, 1976), Ch. 4, 82–85.
2. P. Brass, W. Moser and J. Pach, *Research Problems in Discrete Geometry*, Springer (2006), Ch. 5, 183–257.
3. P. Erdős, On sets of distances of $n$ points. *Amer. Math. Monthly* **53**(5) (1946), 248–250.
4. P. Erdős and G. B. Purdy, Some extremal problems in geometry, IV. In *Proc. Seventh Southeastern Conference on Combinatorics, Graph Theory, and Computing (Louisiana State University, Baton Rouge, LA, 1976)* (*Congressus Numerantium* **XVII**), Utilitas Math. (Winnipeg, Manitoba, 1976).
5. S. Lang, *Algebra*, 3rd edn., Addison-Wesley (Reading, MA, 1994).
6. H. B. Mann, On linear relations between roots of unity. *Mathematika* **12** (1965), 107–117.

https://doi.org/10.1112/S0025579311001847 Published online by Cambridge University Press

**7.** J. Matoušek, The number of unit distances is almost linear for most norms. *Adv. Math.* **226** (2011), 2618–2628.
**8.** J. Spencer, E. Szemerédi and W. Trotter, Unit distances in the Euclidean plane. In *Graph Theory and Combinatorics: Proceedings of the Cambridge Combinatorial Conference, in Honour of Paul Erdős* (ed. B. Bollobás), Academic Press (New York, 1984), 293–303.
**9.** L. Székely, Crossing numbers and hard Erdős problems in discrete geometry. *Combin. Probab. Comput.* **6**(3) (1997), 353–358.

Ryan Schwartz,
Department of Mathematics,
University of British Columbia,
Vancouver, B.C., V6T1Z2,
Canada
E-mail: ryano@math.ubc.ca

Frank de Zeeuw,
Department of Mathematics,
University of British Columbia,
Vancouver, B.C., V6T1Z2,
Canada
E-mail: fdezeeuw@math.ubc.ca

József Solymosi,
Department of Mathematics,
University of British Columbia,
Vancouver, B.C., V6T1Z2,
Canada
E-mail: solymosi@math.ubc.ca

https://doi.org/10.1112/S0025579311001847 Published online by Cambridge University Press