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UK pension changes in 2015: some mathematical considerations

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1. *Introduction*

Since April 2015 there has been no legal requirement in the UK to purchase an annuity with pension savings [1] while for those who reach state pension age on or after 6th April 2016 the UK Government changed the state pension deferral arrangements [2]. The latter refers to an arrangement whereby a pensioner can receive an enhanced state pension by deferring its uptake for an arbitrary number of years. These two changes raise certain questions for prospective pensioners which are worthy of some mathematical consideration.

An annuity is a guaranteed income for life in exchange for a certain sum of money: the pension pot. An alternative to the annuity since April 2015 is a 'draw down scheme' in which the pension pot can be used almost like an ordinary bank account and money periodically withdrawn. These two choices arise from 'defined contribution' pension arrangements. By contrast 'defined benefit' work-based (company) pensions allow no such choice and are not considered further here apart from the special case of the UK state pension. With an annuity a further option to consider, and one which predates the 2015 changes, is whether to take payments that are fixed or index-linked to inflation. Only the UK state pension offers a late retirement enhanced pension if its uptake is deferred.

The focus of this paper is to decide mathematically which of the two options offered in each of the three questions below is more financially worthwhile:

(a) annuity or draw down scheme?

(b) inflation proofed (index-linked) income or fixed income?

(c) immediate income or deferred income?

The three questions are addressed in the five sections below and in each case an illustrative numerical example is provided. While Dagpunar [3] has addressed the third question, the key contribution of this article is to provide

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a mathematical model that addresses all three questions: questions which are likely to affect millions of people in the UK in the coming years and decades. It must be emphasised however that many of the assumptions made in this paper may not be realistic and the reader would be advised to take professional financial advice before making any decisions about their own pension arrangements.

2. *Notation and Assumptions*

The following notation is used:

Size of pension pot at time of final contribution	$= P$
Amount of pension pot remaining after i years	$= A_i \ (A_0 = P)$
Initial annual income from pension	$= I_0$
Annual rate of interest on pension pot	$= r$
Annual rate of inflation (for index-linking)	$= R$
Years for which uptake of defined contribution pension is deferred	$= m$
One-off lump sum payment	$= L$
Annual rate at which deferred UK state pension increases	$= r_D$
Years for which uptake of state pension is deferred	$= M$
Years of income from pension after taking I_0	$= N$

The following assumptions are made:

(i) r and R remain constant for the duration of the pensioner's life;

(ii) r_D is a rate set by the UK Government offering enhanced incomes to those pensioners who defer the uptake of their pension upon reaching pension age;

(iii) income I_0 commences m years after the final pension contribution and it is required that $m \geqslant 1$. It is unrealistic for $m = 0$ since then it would be pointless making the final contribution to the pension scheme;

(iv) the metric for deciding which pension option is more financially worthwhile is based on that which gives the greatest total pension receipt (income stream) over the years during which the pension is received and has been chosen purely for reasons of simplicity;

(v) all pension incomes are considered gross of any UK income tax liabilities.

3. *Annuity or draw down scheme*?

With the draw down option, the income I_0 initially drawn must be chosen such that the pension pot does not expire before the pensioner does. The size of I_0 will depend upon (a) the average annual rate of return r from investment of the capital (i.e. the pension pot) and (b) the average annual rate of inflation (index-linking) R to which I_0 is adjusted over the lifetime of the pensioner.

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Deriving the basic pension equation

For a defined contribution pension arrangement, the pension pot P will grow by r per cent during the time that elapses between the final contribution, when $A_0 = P$, and the first income payment I_0 some m years later. On receiving I_0 a one-off lump sum payment L may also be taken. It may be noted that in practice, often $m = 1$ in which case the first pension income is received one year after the final contribution, which often corresponds to the moment of retirement. Hence after m years:

$$A_m = P(1 + r)^m - (L + I_0).$$

In the first year after receiving I_0 the pension income increases with inflation at R per cent per annum for a further n years. So the total time that elapses between final contribution and pension expiry is $m + n$ years with there being $n + 1$ pension payments made. So after $m + 1$ years:

$$A_{m+1} = P(1 + r)^{m+1} - (L + I_0)(1 + r) - I_0(1 + R).$$

After $m + 2$ years:

$$A_{m+2} = P(1 + r)^{m+2} - (L + I_0)(1 + r)^2 - I_0(1 + R)^2$$

and after $m + n$ years:

$$A_{m+n} = P(1 + r)^{m+n} - L(1 + r)^n - I_0(1 + r)^n\left[1 + \left(\frac{1 + R}{1 + r}\right) + \left(\frac{1 + R}{1 + r}\right)^2 + \ldots + \left(\frac{1 + R}{1 + r}\right)^n\right]$$

from which

$$A_{m+n} = (1 + r)^n\left[P(1 + r)^m - L - I_0\left\{\sum_{i=0}^{n}\left(\frac{1 + R}{1 + r}\right)^i\right\}\right]. \quad (1)$$

Equation (1) may be termed the basic pension equation from which answers to the three questions posed here can be addressed. Note that if $r = R = 0$ then (1) becomes

$$A_{m+n} = (P - L) - (n + 1)I_0. \quad (2)$$

Equation (2) effectively represents a base line trivial case in which the pension pot P is not invested at all and essentially just stashed under the mattress and then withdrawn in fixed annual amounts. But assuming that $r \neq R$ (1) becomes:

$$A_{m+n} = (1 + r)^n\left[P(1 + r)^m - L - I_0\left(\frac{1 + r}{r - R}\right)\left\{1 - \left(\frac{1 + R}{1 + r}\right)^{n+i}\right\}\right]. \quad (3)$$

Determining the level of income from the pension pot

The amount A_N left after N years (i.e. at death) as measured from the time of taking I_0 might be deliberately set at a certain level $A_N > 0$ either as a legacy to heirs or even just to cover funeral expenses and any financial

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claims on the estate. If however it is desired that the pension pot should last just as long as one lives i.e. one dies penniless but not indebted such that at death $A_N = 0$ then (3) becomes:

$$I_0 = \frac{P(1+r)^m - L}{\left(\frac{1+r}{r-R}\right)\left\{1 - \left(\frac{1+R}{1+r}\right)^{N+1}\right\}}. \tag{4}$$

Of equal interest might be the number of years N needed to live after taking the first payment for an annuity offer to be more financially worthwhile than a draw down scheme. Solving (4) for N gives:

$$N = \left\lfloor \frac{\log\left[\{P(1+r)^m - L\}(R-r) + I_0(1+r)\right] - \log\{I_0(1+r)\}}{\log(1+R) - \log(1+r)} \right\rfloor - 1. \tag{5}$$

The two limiting cases are if:

(i) $R = 0$ (but $r \neq 0$) when the income is fixed rather than index-linked. Then (4) reduces to

$$I_0 = r\{P(1+r)^m - L\}\left[\frac{(1+r)^N}{(1+r)^{N+1} - 1}\right] \tag{6}$$

and (5) becomes

$$N = \frac{\log(I_0) - \log\left[I_0(1+r) - r\{P(1+r)^m - L\}\right]}{\log(1+r)} \tag{7}$$

requiring $I_0 > \frac{r\{P(1+r)^m - L\}}{(1+r)}$.

(ii) $r = 0$ (but $R \neq 0$) i.e. the pension pot 'under the mattress' option. Then (4) reduces to

$$I_0 = \frac{(P-L)R}{(1+R)^{N+1} - 1} \tag{8}$$

and (5) to

$$N = \left\lfloor \frac{\log[I_0 + (P-L)R] - \log(I_0)}{\log(1+R)} \right\rfloor - 1. \tag{9}$$

Example: assume

(i) values for r and R are determined from Bank of England [4] and ONS [5] data respectively, as given in tables 1 and 2;

(ii) years N (i.e. life expectancies) of income from the pension lies within the range $10 \leqslant N \leqslant 30$. According to ONS [6], life expectancy in England and Wales, averaged across males and females, for 2011-2013 at age 65 was 19.6 years and in tables 1 and 2 a range either side of this value is considered;

(iii) nothing remains in the pension pot on death, so that life expiry and income expiry are effectively, and perhaps conveniently, coincident.

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With $m = 5$ and $L = 25\,000$ the results in table 1 are derived from (2), (4) and (6) and in table 2 from (4) and (8). They show the incomes, after 5 years of deferral, that can be drawn down from a £150 000 pension pot P after a £25 000 lump sum withdrawal, for the given time periods N without running out of money. So for an annuity to be more financial advantageous than a draw down, the former would have to offer amounts exceeding the values given in tables 1 and 2 for a given time period N (effectively life expectancy at the moment of taking I_0), inflation rate R and investment rate r.

N	$r = 0$	$r = 0.025$	$r = 0.035$
10	11365	14839	16439
15	7814	10814	12235
20	5953	8723	10068
25	4809	7450	8761
30	4033	6599	7898

TABLE 1: Fixed (non-index-linked, $R = 0$) annual income I_0 (£s) drawn from £150 000 for various investment rates r and time periods N after deduction of £25 000 lump sum.

N	$r = 0$	$r = 0.025$	$r = 0.035$
10	9761	12838	14263
15	6202	8718	9924
20	4360	6561	7652
25	3243	5234	6254
30	2500	4335	5308

TABLE 2: Index-linked annual income I_0 (£s) drawn from £150 000 at 3 per cent ($R = 0.03$) inflation for various investment rates r and time periods N after deduction of £25 000 lump sum.

Using the same values of r and R the results in Table 3 calculated from (5) and (9) and in Table 4 calculated from (2) and (7), show the years N needed to live after the first pension payment for it to be more financially worthwhile to take the annuity offer rather than just drawing down the given incomes.

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annual income offered from annuity	$r = 0$	$r = 0.025$	$r = 0.035$
3000	40.7	∞	∞
3500	34.7	∞	∞
4000	30.3	85.7	∞
4500	26.8	61.1	∞
5000	24.0	48.6	∞

TABLE 3: Years N needed to live after retirement to make fixed income annuity ($R = 0$) more worthwhile than draw down for a £150 000 pension pot P after deduction of £25 000 lump sum.

annual income offered from annuity	$r = 0$	$r = 0.025$	$r = 0.035$
3000	26.4	42.4	57.5
3500	23.6	36.8	48.0
4000	21.4	32.4	41.2
4500	19.5	28.9	36.1
5000	17.9	26.1	32.1

TABLE 4: Years N needed to live after retirement to make index-linked annuity ($R = 0.03$) more worthwhile than draw down for a £150 000 pension pot P after deduction of £25 000 lump sum.

All the results in Tables 3 and 4 indicate the need to live for a considerable number of years after the first pension payment for the annuity option to be financially worthwhile. This is particularly the case when income is fixed when, as Table 3 demonstrates, it becomes both mathematically and demographically impossible in certain cases to make a financial case for the annuity option for the given values of r.

4. *Fixed income annuity or index-linked annuity*?

Section 3 dealt with the difference between an annuity and draw down. In the following situation, it is assumed that an annuity has already been chosen over draw down. The decision to be made is whether to take I_0 as a fixed annual income I_F (say) or an index-linked annual income I_I (say) where in practice the former may be as much as twice the latter. Using the same approach as in section 3, it is required to find the time N when the total income streams with, and without index-linking, have equal value. This situation occurs when

$$I_I \sum_{i=0}^{N} (I + R)^i = I_F(N + 1). \tag{10}$$

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Letting $\frac{I_F}{I_I} = \alpha$ (say) with $\alpha > 1$, (10) becomes:

$$(1 + R)^{N+1} - (N + 1)\alpha R - 1 = 0. \quad (11)$$

Solving for N, a trivial and indeed unrealistic solution exists when $N = -1$ but, as suggested by Figure 1, there is also a non-trivial real solution that can be found either graphically or iteratively through the Newton-Raphson method using

$$N_{k+1} = N_k - \frac{(1 + R)^{N+1} - (N + 1)\alpha R - 1}{(1 + R)^{N+1} \ln(1 + R) - \alpha R}$$

and starting with $N_0 = 30$ (say).

Example: Taking $R = 0.03$ and α values of 1.5, 1.75 and 2.0, approximate solutions for N in (11) are found from Figure 1.

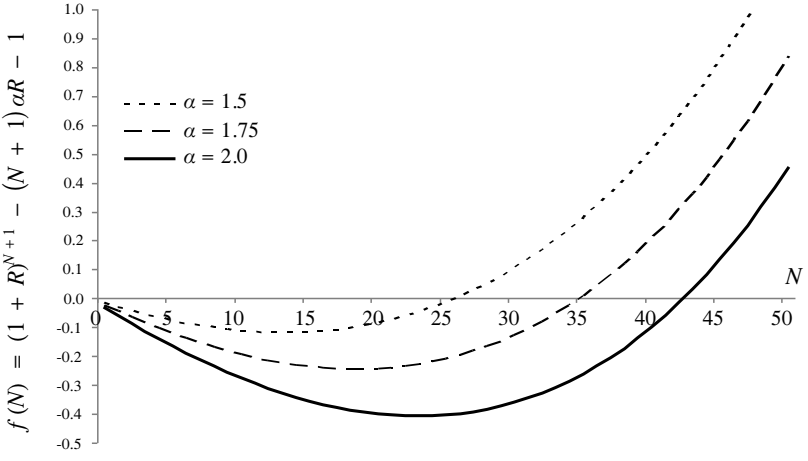


FIGURE 1: Graphical solutions for N in $f(N) = (1 + R)^{N+1} - (N + 1)\alpha R - 1$ for different α values and inflation rate of 3 per cent ($R = 0.03$).

For these three α values it is necessary to receive the pension income for around 26, 35 and 42 years respectively before an index-linked annuity becomes worthwhile. Essentially these results show the need to live for a long time after retirement to make the index-linked option financially attractive. The only caveat to this conclusion is likely future inflation: should this be greater than given in this example then the payback times would be somewhat reduced.

5. *Immediate income or deferred income*?

To answer this question, which only applies to the UK state pension, it is necessary to determine the number of years N that it would take to recoup the loss of income (pension payments) for the years for which it has been deferred and increased at r_D percent per annum. As the state pension benefit

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is not directly dependent on contributions and the benefits are defined, M rather than m is used to refer to the deferral years. If I_M (say) is the state pension income after M years of deferral then:

$$I_M = I_0(1 + r_D)^M.$$

If index-linked, as the UK state pension is, the pension income would then increase with annual index-linking R for the subsequent $N - M$ years. Equating the total income stream with M years deferment, to that with no deferment gives

$$I_0(1 + r_D)^M \sum_{i=0}^{N-M} (I + R)^i = I_0 \sum_{i=0}^{N} (I + R)^i \tag{12}$$

from which

$$N = \left\lfloor \frac{\log\{(1 + r_D)^M - 1\} - \log\{(1 + r_D)^M - (1 + R)^M\}}{\log(1 + R)} \right\rfloor + (M - 1), \tag{13}$$

noting the requirement that $r_D > R$ and ideally $r_D >> R$.

Should however the income be fixed, i.e. $R = 0$, rather than index-linked then (12) becomes

$$I_0(1 + r_D)^M(N - M + 1) = I_0(N + 1)$$

giving

$$N = \frac{(M - 1)(1 + r_D)^M + 1}{(1 + r_D)^M - 1}. \tag{14}$$

Example: Up to April 2016 in the UK, an extra 10.4 per cent per annum could be gained on the state pension income by deferring it – 'state pension deferral' as it is termed. However after April 2016, this deferral rate reduces to 5.8 per cent. Table 5, calculated from (14) gives the years it would take before benefiting from pension deferment at 10.4 and 5.8 per cent for periods between 1 and 5 years with, unlike the UK state pension, a fixed income from the pension i.e. no index-linking.

years of deferment (M)	10.4 % per annum	5.8 % per annum
1	9.6	17.2
2	10.1	17.8
3	10.7	18.3
4	11.2	18.8
5	11.8	19.4

TABLE 5: Years before financial gain from pension deferment at 10.4 and 5.8 per cent per annum with no index-linking

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In the case of index-linking as with the UK state pension, then again using $R = 0.03$ from (13) the results in table 6 are derived for 1 – 5 years of pension deferment.

Years of deferment (M)	10.4 % per annum	5.8 % per annum
1	11.5	24.6
2	12.0	25.1
3	12.6	25.7
4	13.1	26.2
5	13.7	26.7

TABLE 6: Years before financial gain from pension deferment at 10.4 and 5.8 per cent per annum with annual index-linking of 3.0 per cent per annum.

Tables 5 and 6 show that the payback times are considerable. Almost 14 years and 27 years for an index-linked pension at 10.4% and 5.8% per annum respectively, if deferred for 5 years. This means that on retiring at 65 one would have to live to be almost either 79 (10.4 per cent) or 92 (5.8 per cent) before reaping any financial gain assuming the given rate of inflation.

6. *Conclusion*

In deciding which option to choose in the above three questions, the three determining variables used have been life expectancy, interest rates on investments and rates of inflation. Not knowing how these three variables will play out over the future makes it impossible to give precise answers to the questions. Nevertheless the calculations do arguably produce plausible answers and, if nothing else, the work has demonstrated mathematically the need for caution over taking index-linked rather than fixed income offers and deferred rather than immediate pension incomes. Again, however, it is emphasised that the above calculations should not replace the need for professional financial advice in retirement planning.

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10.1017/mag.2016.55

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The answers to the *Nemo* page from March on mathematics as solace were:

1.	Anthony Trollope	The Last Chronicle of Barset (Chapter 61)
2.	William Blake	Annotations to the Works of Sir Joshua Reynolds
3.	Robert Browning	A Toccata of Galuppi's
4.	Emily Dickinson	As by the dead we love to sit
5.	William Wordsworth	The Prelude (Book VI)
6.	Walter Scott	Guy Mannering (Chapter 8)

Congratulations to Henry Ricardo for finding all of these quotations.

In this issue the focus is on spheres. The quotations, which are on page 212, are to be identified by reference to author and work. Solutions are invited to the Editor by 30th September 2016.

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