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Nodal solutions of second order nonlinear boundary value problems

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Abstract

We study the nonlinear boundary value problem (BVP) consisting of the equation $-(p(t)y')'+q(t)y = w(t)f(y)$ on $[a, b]$ and a general separated boundary condition (BC). By comparing it with a linear Sturm–Liouville problem (SLP) we obtain conditions for the existence and nonexistence of nodal solutions of this problem. More specifically, let $\lambda_n, n = 0, 1, 2, \ldots$, be the nth eigenvalue of the corresponding linear SLP. Then the BVP has a pair of solutions with exactly n zeros in (a, b) if λ_n is in the interior of the range of $f(y)/y$; and does not have any solution with exactly n zeros in (a, b) if λ_n is outside this range. These conditions become necessary and sufficient when $f(y)/y$ is monotone on $(-\infty, 0)$ and on $(0, \infty)$. We also discuss the changes of the number of different types of nodal solutions as the equation or the BC changes. Our results are obtained without assuming the global existence and uniqueness of solutions of the corresponding initial value problems.

1. *Introduction*

In this paper, we are concerned with the existence and nonexistence of nodal solutions of the boundary value problem (BVP) consisting of the equation

$$-(p(t)y')' + q(t)y = w(t)f(y),\ t \in (a, b), \tag{1·1}$$

and the separated boundary condition (BC)

$$\begin{aligned} &\cos\alpha\ y(a) - \sin\alpha\ (py')(a) = 0, \quad \alpha \in [0, \pi), \\ &\cos\beta\ y(b) - \sin\beta\ (py')(b) = 0, \quad \beta \in (0, \pi], \end{aligned} \tag{1·2}$$

where $a, b \in \mathbb{R}$ with $a < b$. We assume throughout, and without further mention, that the following conditions hold:

(H1) $p, q, w \in C[a, b]$, $p > 0$ on $[a, b]$ and $w > 0$ a.e. on $[a, b]$;

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(H2) f is continuous on $\mathbb{R}$ and $f(y)y > 0$ for $y \neq 0$;

(H3) there exist extended real numbers f_0, f_∞ such that $0 \leqslant f_0, f_\infty \leqslant \infty$ and

$$\lim_{y\to 0} \frac{f(y)}{y} = f_0 \quad \text{and} \quad \lim_{|y|\to\infty} \frac{f(y)}{y} = f_\infty. \tag{1·3}$$

By a solution of BVP (1·1), (1·2), we mean a function $y \in C[a, b]$ such that $py' \in C^1[a, b]$, and y satisfies Eq. (1·1) and BC (1·2). Moreover, if $y(t)$ changes sign in (a, b), then y is said to be a nodal solution. A simple zero of a nodal solution $y(t)$ is a point $t_0 \in (a, b)$ such that $y(t_0) = 0$ and $(py)'(t_0) \neq 0$.

Special cases of BVP (1·1), (1·2) have been investigated in numerous papers using various methods and techniques, see, for example, [**4–7**, **19**] and the references therein. Most existing results are about the existence of positive solutions and are for special cases of Equation (1·1) such as $p \equiv 1$ and $q \geqslant 0$ ($q \equiv 0$ for many of them) and for special BCs such as Dirichlet and Neumann BCs.

Erbe [**3**] employed the idea to connect BVP (1·1), (1·2) with its corresponding linear Sturm–Liouville problem (SLP). In that paper, using fixed point theory, the existence of positive solutions of BVP (1·1), (1·2) was established by comparing the values of $f(y)/y$, $y \in (0, \infty)$, with the smallest eigenvalue of the corresponding SLP. However, due to the limitation of the approach by the fixed point index theory, results were given only under certain restrictions to the problem, and nothing was found for the existence of solutions with zeros in (a, b).

Recently, with $p \equiv 1$ and $q \equiv 0$ and the Dirichlet BC, Naito and Tanaka [**15**] obtained results on the existence of nodal solutions to BVP (1·1), (1·2) with prescribed numbers of zeros in the interval by comparing the range of $f(y)/y$ with an eigenvalue of the corresponding SLP. This work has been extended by Kong [**8**] to the BVP (1·1), (1·2) with an arbitrary $q \in C^1$ and a general separated BC.

However, since the work in [**15**] and [**8**] was established using the shooting method which depends heavily on the global existence and uniqueness of solutions of initial value problems, strong assumptions are needed. In fact, it was assumed that $q, w \in C^1$ together with other technical conditions, see assumptions (H1)–(H5) in [**8**]. By this method, the C^1 condition for the coefficient functions cannot be released because the solutions to initial value problems associated with Equation (1·1) may not exist, or may not be unique otherwise, see [**18**, **2**].

In a different direction, several authors used a bifurcation theorem given by Rabinowitz [**16**] to investigate nonlinear BVPs. In particular, Ma [**12**] and Ma and Thompson [**13**, **14**] studied the BVP

$$-y'' = w(t)f(y), \quad y(0) = y(1) = 0, \tag{1·4}$$

where $w\colon [0, 1] \to [0, \infty)$ is continuous. Results on the existence of multiple nodal solutions were given for the following cases:

(i) $f_0 = 0$ and $f_\infty = \infty$ (superlinear);

(ii) $f_0 = \infty$ and $f_\infty = 0$ (sublinear);

(iii) $0 < f_0, f_\infty < \infty$.

Motivated by the work in [**12–14**], in this paper, we utilize the Rabinowitz bifurcation method and the Sturm comparison theorem to study the existence as well as the nonexistence of nodal solutions of BVP (1·1), (1·2) under the general conditions (H1)–(H3). More specifically, let λ_n, $n = 0, 1, 2, \ldots$, be the nth eigenvalues of the corresponding linear SLP.

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We show that BVP (1·1), (1·2) has a pair of solutions with exactly n zeros in (a, b) if λ_n is in the interior of the range of $f(y)/y$; and does not have any solution with exactly n zeros in (a, b) if λ_n is outside the range of $f(y)/y$. These conditions become necessary and sufficient when $f(y)/y$ is strictly between f_0 and f_∞. The existence of multiple and even an infinite number of nodal solutions are derived as consequences.

These results are substantial development of those for BVP (1·4). In fact, the convexity property of solutions and the monotonicity argument, which played key roles in the proofs for BVP (1·4), fail to hold for the case with $q \not\equiv 0$ and the general BC (1·2). This obstacle is overcome by subtle applications of the Sturm comparison theorem for Prüfer angles. Note that our results are obtained without assuming the global existence and uniqueness of the solutions of the corresponding initial value problems, they are substantial improvements of those in [**8**].

In light of the work in [**9–11**] on the dependence of eigenvalues of SLPs on the problem, we also discuss the changes of the number of different types of nodal solutions as the equation or the BC changes.

This paper is organized as follows. After this introduction, we state our main results in Section 2 and give all the proofs in Section 3.

2. *Main results*

It is well known (see, for example, [**17**]) that the linear SLP consisting of the equation

$$-(p(t)y')' + q(t)y = \lambda w(t)y \tag{2·1}$$

and BC (1·2) has a countable number of eigenvalues λ_i, $i = 0, 1, 2, \ldots$, which are bounded below and unbounded above, and can be ordered to satisfy

$$-\infty < \lambda_0 < \lambda_1 < \cdots < \lambda_i < \cdots, \quad \text{and } \lambda_i \longrightarrow \infty \text{ as } i \longrightarrow \infty. \tag{2·2}$$

Furthermore, any eigenfunction associated with λ_i has i simple zeros in (a, b).

In the sequel, we let $\mathbb{N}_k = \{k, k+1, k+2, \ldots\}$ for $k = 0, 1, 2, \ldots$, and let λ_m be the first positive eigenvalue of SLP (2·1), (1·2). The first theorem is about the existence of nodal solutions of BVP (1·1), (1·2).

THEOREM 2·1. *Assume for some $k \in \mathbb{N}_m$ we have*

$$\textit{either} \quad 0 \leqslant f_0 < \lambda_k < f_\infty \leqslant \infty \quad \textit{or} \quad 0 \leqslant f_\infty < \lambda_k < f_0 \leqslant \infty.$$

Then BVP (1·1), (1·2) *has two solutions $u_{k,+}$ and $u_{k,-}$ such that they both have exactly k simple zeros in (a, b), $u_{k,+}$ is positive near a, and $u_{k,-}$ is negative near a.*

The following is an immediate consequence of Theorem 2·1.

COROLLARY 2·2. *Assume either* (i) $f_0 = 0$ *and* $f_\infty = \infty$, *or* (ii) $f_0 = \infty$ *and* $f_\infty = 0$. *Then, for each $k \in \mathbb{N}_m$, BVP* (1·1), (1·2) *has two nodal solutions $u_{k,+}$ and $u_{k,-}$ such that they both have exactly k simple zeros in (a, b), $u_{k,+}$ is positive near a, and $u_{k,-}$ is negative near a.*

Now we present a result on the nonexistence of nodal solutions of BVP (1·1), (1·2).

THEOREM 2·3.

(i) *Assume $f(y)/y < \lambda_n$ for some $n \in \mathbb{N}_0$ and all $y \neq 0$. Then BVP* (1·1), (1·2) *has no solution with exactly i zeros in (a, b) for any $i \geqslant n$.*

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(ii) *Assume* $\lambda_n < f(y)/y$ *for some* $n \in \mathbb{N}_0$ *and all* $y \neq 0$. *Then BVP* (1·1), (1·2) *has no solution with exactly* i *zeros in* (a, b) *for any* $i \leqslant n$.

(iii) *Assume either* $f_0 < \infty$ *or* $f_\infty < \infty$ *and* $f(y)/y \neq \lambda_n$ *for any* $n \in \mathbb{N}_0$ *and all* $y \neq 0$. *Then BVP* (1·1), (1·2) *has no non-trivial solution.*

The combination of Theorems 2·1 and 2·3 leads to the following:

COROLLARY 2·4.

(i) *Assume* $f_0 < f(y)/y < f_\infty$ *for all* $y \neq 0$. *Then, for* $n \in \mathbb{N}_m$, *BVP* (1·1), (1·2) *has two solutions which have exactly* n *simple zeros in* (a, b) *and have opposite signs near* a *if and only if* $f_0 < \lambda_n < f_\infty$.

(ii) *Assume* $f_\infty < f(y)/y < f_0$ *for all* $y \neq 0$. *Then, for* $n \in \mathbb{N}_m$, *BVP* (1·1), (1·2) *has two solutions which have exactly* n *simple zeros in* (a, b) *and have opposite signs near* a *if and only if* $f_\infty < \lambda_n < f_0$.

In the following, we present results on the changes of the number of different types of nodal solutions to BVP (1·1), (1·2) as the problem changes. The first one is about the changes as the interval $[a, b]$ shrinks, more specifically, as $b \to a+$. We discuss both the cases when one of f_0 and f_∞ is infinite and both of them are finite.

THEOREM 2·5.

(i) *Assume either* $f_0 < \infty$ *and* $f_\infty = \infty$, *or* $f_\infty < \infty$ *and* $f_0 = \infty$.

  (a) *For any* $n \in \mathbb{N}_1$, *there exists* $b_n > a$ *such that for any* $b \in (a, b_n)$ *and for any* $i \geqslant n$, *BVP* (1·1), (1·2) *has two solutions which have exactly* i *simple zeros in* (a, b) *and have opposite signs near* a.

  (b) *Let* $\alpha \in [0, \pi)$ *and* $\beta \in (0, \pi]$. *Then*
  for $\alpha < \beta$, *there exists* $b_0 > a$ *such that for any* $b \in (a, b_0)$, *BVP* (1·1), (1·2) *has a positive solution and a negative solution; and*
  for $\beta < \alpha$, *there exists* $b_0 > a$ *such that for any* $b \in (a, b_0)$, *BVP* (1·1), (1·2) *has no positive or negative solution.*

(ii) *Assume* $f_0 < \infty$ *and* $f_\infty < \infty$. *Let* $\alpha \in [0, \pi)$ *and* $\beta \in (0, \pi]$ *with* $\alpha \neq \beta$. *Then there exists* $b_* > a$ *such that for any* $b \in (a, b_*)$, *BVP* (1·1), (1·2) *has no nontrivial solution.*

Remark 2·1.

(i) For the Dirichlet BC $y(a) = y(b) = 0$, we know that $\alpha = 0$ and $\beta = \pi$.

  (a) If either $f_0 < \infty$ and $f_\infty = \infty$, or $f_\infty < \infty$ and $f_0 = \infty$, then there exists $b_0 > a$ such that for any $b \in (a, b_0)$ and for any $i \geqslant 0$, BVP (1·1), (1·2) has two solutions which have exactly i simple zeros in (a, b) and have opposite signs near a.

  (b) If $f_0 < \infty$ and $f_\infty < \infty$, then there exists $b_* > a$ such that for any $b \in (a, b_*)$, BVP (1·1), (1·2) has no nontrivial solution.

(ii) The case when $\alpha = \beta \in (0, \pi)$ is much more complicated. Further results can be obtained for the existence of solutions with no zero in (a, b) based on the work in [**10**]. We omit the details.

The theorem below is about the nonexistence of positive and negative solutions and solutions with one zero in (a, b) for some values of α and β in BC (1·2). This can be regarded as a supplement of the work in [**3**].

THEOREM 2·6. *Let Equation* (1·1) *be fixed.*

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(i) *For each* $\beta \in (0, \pi]$, *there exists* $\alpha_* \in [0, \pi)$ *such that BVP* (1·1), (1·2) *has no positive or negative solution for any* $\alpha \in (\alpha_*, \pi)$.

(ii) *For each* $\alpha \in [0, \pi)$, *there exists* $\beta_* \in (0, \pi]$ *such that BVP* (1·1), (1·2) *has no positive or negative solution for any* $\beta \in (0, \beta_*)$.

(iii) *There exist* $\alpha_* \in [0, \pi)$ *and* $\beta_* \in (0, \pi]$ *such that BVP* (1·1), (1·2) *has no positive or negative solution nor solution with one zero in* (a, b) *for any* $\alpha \in (\alpha_*, \pi)$ *and* $\beta \in (0, \beta_*)$.

We then state results on the changes of the number of different types of nodal solutions to BVP (1·1), (1·2) as the functions $1/p$, q and w change in a given direction. In Theorems 2·7–2·9 we assume $w(t) > 0$ on $[a, b]$.

Let $s \in \mathbb{R}$ and $h \in C[a, b]$ such that $h > 0$ on $[a, b]$, and consider the equation

$$-(p(t)y')' + [q(t) + sh(t)]y = w(t)f(y) \quad \text{on } [a, b]. \tag{2·3}$$

THEOREM 2·7.

(i) *For any* $n \in \mathbb{N}_0$, *there exists* $s_n \leqslant 0$ *such that for any* $s < s_n$ *and for any* $i \leqslant n$, *BVP* (2·3), (1·2) *has no solution with exactly* i *zeros in* (a, b).

(ii) *Assume either* $f_0 < \infty$ *and* $f_\infty = \infty$, *or* $f_\infty < \infty$ *and* $f_0 = \infty$. *Then for any* $n \in \mathbb{N}_0$, *there exists* $s_n \geqslant 0$ *such that for any* $s > s_n$ *and for any* $i \geqslant n$, *BVP* (2·3), (1·2) *has two solutions which have exactly* i *simple zeros in* (a, b) *and have opposite signs near* a.

(iii) *Assume* $f_0 < \infty$ *and* $f_\infty < \infty$. *Then there exists* $s_* \geqslant 0$ *such that for any* $s > s_*$, *BVP* (2·3), (1·2) *has no nontrivial solution.*

Let $s \geqslant 0$ and $h \in C[a, b]$ such that $h > 0$ on $[a, b]$, and consider the equation

$$-(p(t)y')' + q(t)y = [w(t) + sh(t)]f(y) \quad \text{on } [a, b]. \tag{2·4}$$

THEOREM 2·8. *Assume* $f(y)/y \geqslant f_* > 0$ *for all* $y \neq 0$. *Then for any* $n \in \mathbb{N}_0$, *there exists* $s_n \geqslant 0$ *such that for any* $s > s_n$ *and for any* $i \leqslant n$, *BVP* (2·4), (1·2) *has no solution with exactly* i *zeros in* (a, b).

Let $s \geqslant 0$ and $h \in C[a, b]$ such that $h > 0$ on $[a, b]$, and consider the equation

$$-\left[\frac{1}{1/p(t) + sh(t)}y'\right]' + q(t)y = w(t)f(y) \quad \text{on } [a, b]. \tag{2·5}$$

THEOREM 2·9. *Let* $q^* = \max\{q(t)/w(t) : t \in [a, b]\}$ *and assume* $f(y)/y \geqslant f_* > q^*$ *for all* $y \neq 0$. *Then for any* $n \in \mathbb{N}_0$, *there exists* $s_n \geqslant 0$ *such that for any* $s > s_n$ *and for any* $i \leqslant n$, *BVP* (2·5), (1·2) *has no solution with exactly* i *zeros in* (a, b).

Remark 2·2. Theorems 2·5–2·9 show that we can "create" or "eliminate" certain types of nodal solutions by changing the interval $[a, b]$, the BC (1·2), or the coefficient functions $1/p$, q, w. Since the eigenvalues of SLP (2·1), (1·2) can be easily computed using computer software, we are able to construct specific BVPs (1·1), (1·2) which have or do not have nodal solutions with prescribed zeros in $[a, b]$.

3. *Proof of the main results*

The proof of the main result, Theorem 2·1, is based on a well known bifurcation theorem by Rabinowitz [**16**] on nonlinear BVPs. To present Rabinowitz's result, we first introduce some notations below.

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Let

$$E = \{y \in C[a, b]:\ py' \in C[a, b] \text{ and } y \text{ satisfies BC (1·2)}\}.$$

Then E with the norm

$$\|y\|_1 = \max_{a \leqslant t \leqslant b} (\|y\| + \|py'\|),$$

where $\|y\| = \max_{a \leqslant t \leqslant b} |y(t)|$, is a Banach space. For $k \in \mathbb{N}_0$, let S_k^+ be the set of functions in E which have exactly k simple zeros on (a, b) and are positive near a, and let $S_k^- = -S_k^+$ and $S_k = S_k^+ \cup S_k^-$. We also define $\Phi_k^\pm = \mathbb{R} \times S_k^\pm$ and $\Phi_k = \mathbb{R} \times S_k$. Let $\mathcal{P} = \mathbb{R} \times E$ under the product topology. As in [**16**], we say that a continuum C of $\mathcal{P}$ meets ∞ if C is unbounded.

Consider the equation

$$-(p(t)y')' + q(t)y = \lambda w(t)y + w(t)g(y), \tag{3·1}$$

where $g \in C(\mathbb{R}, \mathbb{R})$ such that $g(y)y > 0$ for $y \neq 0$, and $g(y) = o(y)$ as $y \to 0$. The following lemma is [**16**, theorem 2·3].

LEMMA 3·1. *For $i \in \mathbb{N}_0$ and $\nu \in \{+, -\}$, there exists a continuum C_i^ν of solutions $(\lambda, y(t))$ of Eq. (3·1) in $\Phi_i^\nu \cup \{(\lambda_i, 0)\}$ that meets $(\lambda_i, 0)$ and ∞ in $\mathcal{P}$, where λ_i is the ith eigenvalue of SLP (2·1), (1·2).*

From the remark after [**16**, theorem 2·3], we have

Remark 3·1. Lemma 3·1 can be extended to the case where the function g in Equation (3·1) is replaced by a continuous operator $G: C[a, b] \to C[a, b]$ satisfying the following condition:

(i) $G(y(t)) = o(\|y\|)$ as $\|y\| \to 0$ uniformly for $t \in [a, b]$.

(ii) If (λ, y) is a solution of the modified equation (3·1) with y having a double zero in $[a, b]$, then $y \equiv 0$.

In addition to Lemma 3·1 and Remark 3·1, we need the following lemma in the proof of Theorem 2·1.

LEMMA 3·2. *Let $h \in C(\mathbb{R}, \mathbb{R})$ such that $h(y)y > 0$ for $y \neq 0$, and $q_n \in C[a, b]$ such that $q_n(t) \to q_\infty(t) \in C([a, b], \mathbb{R})$ as $n \to \infty$ uniformly on $[a, b]$. Let $\{(c_n, d_n)\}_{n=1}^\infty$ be a sequence of subintervals of (a, b) such that $(c_n, d_n) \to (c, d)$ as $n \to \infty$. For $n \in \mathbb{N}$, assume $y_n(t)$ is a solution of the BVP consisting of the equation*

$$-(p(t)y')' + q_n(t)y = w(t)h(y) \tag{3·2}$$

and the BC

$$\begin{aligned} &\cos \bar{\alpha}\ y(c_n) - \sin \bar{\alpha}\ (py')(c_n) = 0, \quad \bar{\alpha} \in [0, \pi), \\ &\cos \bar{\beta}\ y(d_n) - \sin \bar{\beta}\ (py')(d_n) = 0, \quad \bar{\beta} \in (0, \pi], \end{aligned} \tag{3·3}$$

such that on (c_n, d_n)

$$y_n(t) > 0 \text{ for all } n \in \mathbb{N} \text{ or } y_n(t) < 0 \text{ for all } n \in \mathbb{N}.$$

Then, for any closed interval $I \subset (c, d)$, there exists $r > 0$ such that for sufficiently large $n \in \mathbb{N}$

$$|y_n(t)| \geqslant r \max_{s \in I} |y_n(s)| \text{ for } t \in I. \tag{3·4}$$

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Proof. Without loss of generality we only prove the case when $y_n(t) > 0$ on (c_n, d_n), $n \in \mathbb{N}$. For $n \in \mathbb{N}$ let $u_n(t)$ and $v_n(t)$ be the solutions of the equation

$$-(p(t)y')' + q_n(t)y = 0 \tag{3·5}$$

satisfying the initial conditions

$$u_n(c_n) = \sin \bar{\alpha},\ (pu_n')(c_n) = \cos \bar{\alpha} \tag{3·6}$$

and

$$v_n(d_n) = \sin \bar{\beta},\ (pv_n')(d_n) = \cos \bar{\beta}, \tag{3·7}$$

and let θ_n, ϕ_n, ψ_n be the Prüfer angles of y_n, u_n, v_n, respectively, satisfying

$$\theta_n(c_n) = \phi_n(c_n) = \bar{\alpha} \text{ and } \theta_n(d_n) = \psi_n(d_n) = \bar{\beta}.$$

Noting that $w(t)h(y_n(t))/y_n(t) > 0$ for $t \in (c_n, d_n)$, by the Sturm comparison theorem for Prüfer angles (see, for example, [**17**]), we have

$$0 < \phi_n(t) < \theta_n(t) \leqslant \pi \text{ for } t \in (c_n, d_n]. \tag{3·8}$$

In particular,

$$0 < \phi_n(d_n) < \theta_n(d_n) = \psi_n(d_n) \leqslant \pi.$$

This shows that u_n and v_n are linearly independent on $[c_n, d_n]$. Moreover, from (3·8) we see that $0 < \phi_n(t) < \pi$ for $t \in (c_n, d_n]$ and hence $u_n(t) > 0$ on $(c_n, d_n]$. With a similar argument we find that $v_n(t) > 0$ on $[c_n, d_n)$.

Let $\rho_n := p(t)[u_n'v_n - u_nv_n'](t)$. Then ρ_n is a nonzero constant on $[c_n, d_n]$. Define

$$H_n(t, s) = \frac{1}{\rho_n} \begin{cases} u_n(t)v_n(s), & c_n \leqslant t \leqslant s \leqslant d_n, \\ u_n(s)v_n(t), & c_n \leqslant s \leqslant t \leqslant d_n. \end{cases}$$

Then $H_n(t, s)$ is the Green's function for BVP (3·5), (3·3), $H_n(t, s) > 0$ for $t, s \in (c_n, d_n)$, and

$$\frac{H_n(t, s)}{H_n(s, s)} = \begin{cases} \dfrac{u_n(t)}{u_n(s)}, & c_n \leqslant t \leqslant s \leqslant d_n, \\ \dfrac{v_n(t)}{v_n(s)}, & c_n \leqslant s \leqslant t \leqslant d_n. \end{cases}$$

For $t \in [c_n, d_n]$ denote

$$\bar{l}_n(t) = \max \left\{ \sup_{t<s\leqslant d_n} \frac{u_n(t)}{u_n(s)}, \sup_{c_n\leqslant s<t} \frac{v_n(t)}{v_n(s)} \right\} \tag{3·9}$$

and

$$\underline{l}_n(t) = \min \left\{ \inf_{t<s\leqslant d_n} \frac{u_n(t)}{u_n(s)}, \inf_{c_n\leqslant s<t} \frac{v_n(t)}{v_n(s)} \right\}. \tag{3·10}$$

It is easy to see that $\bar{l}_n$ and $\underline{l}_n$ are well defined and for $t \in I$

$$0 < \min_{s\in I} \underline{l}_n(s) \leqslant \underline{l}_n(t) \leqslant \bar{l}_n(t) \leqslant \max_{s\in I} \bar{l}_n(s) < \infty. \tag{3·11}$$

Moreover,

$$\underline{l}_n(t)H_n(s, s) \leqslant H_n(t, s) \leqslant \bar{l}_n(t)H_n(s, s) \text{ for } t, s \in [c_n, d_n]. \tag{3·12}$$

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Since $y_n(t)$ is a solution of BVP (3·2), (3·3),

$$y_n(t) = \int_{c_n}^{d_n} H_n(t, s)w(s)h(y_n(s))\,ds.$$

From (3·12) we have that for $t \in I$

$$\max_{t\in I} y_n(t) \leqslant \max_{t\in I} \bar{l}_n(t) \int_{c_n}^{d_n} H_n(s, s)w(s)h(y_n(s))\,ds$$

and

$$y_n(t) \geqslant \min_{t\in I} \underline{l}_n(t) \int_{c_n}^{d_n} H_n(s, s)w(s)h(y_n(s))\,ds.$$

Therefore, for $t \in I$

$$y_n(t) \geqslant \left(\min_{t\in I} \underline{l}_n(t)/\max_{t\in I} \bar{l}_n(t)\right) \max_{t\in I} y_n(t). \tag{3·13}$$

Now we show that there exists $r > 0$ such that $\min_{t\in I} \underline{l}_n(t)/\max_{t\in I} \bar{l}_n(t) \geqslant r$ for all sufficiently large $n \in \mathbb{N}$. Without loss of generality we may assume that $\lambda = 0$ is not an eigenvalue of the SLPs consisting of the equation

$$-(p(t)y')' + q_\infty(t)y = \lambda w(t)y \tag{3·14}$$

and one of the BCs

$$\cos\bar{\alpha}\ y(c) - \sin\bar{\alpha}\ (py')(c) = 0, \quad y(d) = 0 \tag{3·15}$$

and

$$y(c) = 0, \quad \cos\bar{\beta}\ y(d) - \sin\bar{\beta}\ (py')(d) = 0.$$

For otherwise, we can replace $q_n(t)$ and $h(y)$ by $q_n(t) + \epsilon w(t)$ and $h(y) + \epsilon y$, respectively in (3·2), for small $\epsilon > 0$. Then for the modified problem, $\lambda = 0$ is no longer an eigenvalue, and the above argument applies exactly the same way.

Let $u(t)$ and $v(t)$ be the solutions of Equation (3·5) satisfying (3·6) and (3·7) with q_n, c_n, d_n replaced by q_∞, c, d, respectively. Then for $t \in (c, d)$

$$\lim_{n\to\infty} u_n(t) = u(t) \ \text{ and } \ \lim_{n\to\infty} v_n(t) = v(t).$$

We claim that $u(t) > 0$ on $(c, d]$ and $v(t) > 0$ on $[c, d)$. In fact, $u_n(t) > 0$ on $(c_n, d_n]$ for $n \in \mathbb{N}$ implies that $u(t) \geqslant 0$ on $(c, d]$. If $u(t^*) = 0$ for some $t^* \in (c, d)$, Then t^* is a double zero and hence $u(t) \equiv 0$ on $[c, d]$ by the uniqueness of solutions of the linear initial value problems; if $u(d) = 0$, then $\lambda = 0$ is an eigenvalue of SLP (3·14), (3·15) with associated eigenfunction $u(t)$, contradicting the assumption. Similarly for $v(t)$.

We define $\bar{l}(t)$ and $\underline{l}(t)$ in the same way as $\bar{l}_n(t)$ and $\underline{l}_n(t)$ in (3·9) and (3·10), where u_n, v_n, c_n, d_n are replaced by u, v, c, d, respectively. Then, with the same argument as for $\bar{l}_n(t)$ and $\underline{l}_n(t)$, we have

$$0 < \min_{s\in I} \underline{l}(s) \leqslant \underline{l}(t) \leqslant \bar{l}(t) \leqslant \max_{s\in I} \bar{l}(s) < \infty, \ t \in I.$$

Since

$$\lim_{n\to\infty} \bar{l}_n(t) = \bar{l}(t) \ \text{ and } \ \lim_{n\to\infty} \underline{l}_n(t) = \underline{l}(t)$$

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uniformly on I, we see that

$$\lim_{n\to\infty} \frac{\min_{t\in I} \underline{l}_n(t)}{\max_{t\in I} \bar{l}_n(t)} = \frac{\min_{t\in I} \underline{l}(t)}{\max_{t\in I} \bar{l}(t)}.$$

Let $r = (1/2)(\min_{t\in I} \underline{l}(t)/\max_{t\in I} \bar{l}(t))$. Then for sufficiently large $n \in \mathbb{N}$, (3·4) follows from (3·13).

Proof of Theorem 2·1. We use the notation $Ly := -(p(t)y')' + q(t)y$ throughout the proof.

(I) We first prove the case where $0 \leqslant f_0 < \lambda_k < f_\infty \leqslant \infty$. Let $g(y) := f(y) - f_0 y$. Then Equation (1·1) is equivalent to

$$Ly = f_0 w(t) y + w(t) g(y) \tag{3·16}$$

and $g(y) = o(y)$ as $y \to 0$. We consider the equation

$$Ly = \lambda w(t) y + w(t) g(y). \tag{3·17}$$

Then, by Lemma 3·1, for $\nu \in \{+, -\}$, there exists a continuum C_k^ν of solutions of (3·17) in $\Phi_k^\nu \cup \{(\lambda_k, 0)\}$ that meets $(\lambda_k, 0)$ and ∞ in $\mathcal{P}$. It is easy seen that any solution of (3·17) in Φ_k^ν of the form (f_0, y) yields a solution y of BVP (1·1), (1·2) with k simple zeros. In the following we show that C_k^ν crosses the hyperplane $\{f_0\} \times E$ in $\mathcal{P}$. Let $\{\mu_n, u_n\}_{n=1}^\infty \subset C_k^\nu$ satisfy

$$\mu_n + \|y_n\|_1 \longrightarrow \infty \quad \text{as } n \longrightarrow \infty. \tag{3·18}$$

Our purpose is to show that $\mu_n \leqslant f_0$ for some $n \in \mathbb{N}$, and hence we reach the conclusion by the connectedness of C_k^ν and the fact that $\lambda_k > f_0$.

Assume the contrary, i.e., $\mu_n > f_0$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $\tau(0, n) = a$ and $\tau(k+1, n) = b$, and let

$$\tau(1, n) < \tau(2, n) < \cdots < \tau(k, n)$$

be the zeros of y_n. Then, after taking a subsequence if necessary,

$$\lim_{n\to\infty} \tau(i, n) := \tau(i), \quad i = 0, \ldots, k+1.$$

Since

$$b - a = \tau(k+1, n) - \tau(0, n) = \sum_{i=0}^{k} (\tau(i+1, n) - \tau(i, n)),$$

we have

$$b - a = \sum_{i=0}^{k} (\tau(i+1) - \tau(i)).$$

Then, there exists $i_0 \in \{0, 1, \ldots, k\}$ such that $\tau(i_0) < \tau(i_0+1)$. Let I be a closed subinterval of $(\tau(i_0), \tau(i_0+1))$ with positive length. Since

$$(-1)^{i_0} \nu y_n(t) > 0 \quad \text{for all } n \in \mathbb{N} \text{ and } t \in (\tau(i_0, n), \tau(i_0+1, n)),$$

there exists a positive integer N_0 such that

$$(-1)^{i_0} \nu y_n(t) > 0 \quad \text{for all } n \geqslant N_0 \text{ and } t \in I. \tag{3·19}$$

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Now we show that $\{\mu_n\}_{n=1}^{\infty}$ is bounded above, i.e., there exists $K > 0$ such that

$$f_0 < \mu_n \leqslant K \quad \text{for all } n \in \mathbb{N}. \tag{3·20}$$

For otherwise, by taking a subsequence if necessary, we have $\mu_n \to \infty$. Note that, for $t \in I$, Equation (3·17) can be written as

$$Ly = (\lambda + g(y)/y)w(t)y \tag{3·21}$$

with $g(y)/y = f(y)/y - f_0$ bounded below, and every solution of Equation (2·1) with sufficiently large λ has at least two zeros in I. By the Sturm Comparison theorem, every solution of Equation (3·21), and hence of Equation (3·17), changes sign in I for $\lambda = \mu_n$ with sufficiently large $n \in \mathbb{N}$. This contradicts (3·19) and hence proves (3·20). Combining (3·18) and (3·20) we see that

$$\|y_n\|_1 \longrightarrow \infty \quad \text{as } n \longrightarrow \infty. \tag{3·22}$$

(a) We assume $f_\infty < \infty$. We observe that the function g in (3·17) can be written as

$$g(y) = f(y) - f_0 y = (f_\infty - f_0)y + h(y),$$

where

$$h(y) = f(y) - f_\infty y = o(|y|) \text{ as } |y| \longrightarrow \infty.$$

Thus, for $n \in \mathbb{N}$, $y_n(t)$ is a solution of the BVP consisting of the equation

$$Ly = (\mu_n + f_\infty - f_0)w(t)y + w(t)h(y)$$

and BC (1·2). With the same reasoning as in the proof of Lemma 3·2 we may assume that $\lambda = 0$ is not an eigenvalue of SLP (2·1), (1·2). Therefore, there is a Green's function $H(t, s)$ for the BVP consisting of the equation $Ly = 0$ and BC (1·2). Then,

$$y_n(t) = \int_a^b H(t, s)\,[(\mu_n + f_\infty - f_0)w(s)y_n(s) + w(s)h(y_n(s))]\,ds$$

and hence

$$\frac{y_n(t)}{\|y_n\|_1} = \int_a^b H(t, s)\left[(\mu_n + f_\infty - f_0)w(s)\frac{y_n(s)}{\|y_n\|_1} + w(s)\frac{h(y_n(s))}{\|y_n\|_1}\right]ds. \tag{3·23}$$

It is easy to see that the sequence $\{y_n(t)/\|y_n\|_1\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous on $[a, b]$. By the Arzelà–Ascoli theorem, we may assume, after taking a subsequence if necessary, that

$$\lim_{n\to\infty} \frac{y_n(t)}{\|y_n\|_1} = y_*(t) \quad \text{for some } y_* \in C[a, b] \text{ with } \|y_*\|_1 = 1.$$

Note that $\{\mu_n\}_{n=1}^{\infty}$ is bounded, we may also assume that $\lim_{n\to\infty} \mu_n = \mu_* \in \mathbb{R}$. Denote $h^*(y) = \max_{-y\leqslant z\leqslant y} |h(z)|$. Then from $h(y) = o(|y|)$ as $|y| \to \infty$, we have $h^*(y) = o(|y|)$ as $|y| \to \infty$. Hence for $t \in (a, b)$

$$\frac{|h(y_n(t))|}{\|y_n\|_1} \leqslant \frac{h^*(\|y_n\|)}{\|y_n\|_1} \leqslant \frac{h^*(\|y_n\|_1)}{\|y_n\|_1}.$$

This, together with (3·22), shows that for $t \in (a, b)$ $\lim_{n\to\infty} |h(y_n(t))|/\|y_n\|_1 = 0$. Taking limits as $n \to \infty$ in (3·23) and using the Lebesgue dominated convergence theorem, we

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obtain

$$y_*(t) = (\mu_* + f_\infty - f_0) \int_a^b H(t,s)w(s)y_*(s)\,ds,$$

i.e., y_* is a nontrivial solution of the equation

$$Ly = (\mu_* + f_\infty - f_0)w(t)y$$

satisfying BC (1·2). In other words, y_* is an eigenfunction of SLP (2·1), (1·2) associated with the eigenvalue $\mu_* + f_\infty - f_0$. Since y_n, $n = 1, 2, \ldots$, has k simple zeros in (a, b) and y_* has only simple zeros in $[a, b]$, we conclude that y_* has exactly k simple zeros in (a, b). As a result, $\mu_* + f_\infty - f_0 = \lambda_k$, or $\mu_* = \lambda_k - f_\infty + f_0 < f_0$. This contradicts the assumption that $\mu_n > f_0$ for all $n \in \mathbb{N}$ and thus completes the proof.

(b) We assume $f_\infty = \infty$. We show that in this case, by taking a subsequence if necessary, for the closed interval I defined earlier, we have

$$\max_{t\in I} |y_n(t)| \longrightarrow \infty \quad \text{as } n \longrightarrow \infty. \tag{3·24}$$

Assume for the contrary, there exists $M_1 > 0$ such that $|y_n(t)| \leqslant M_1$ for $t \in I$ and $n \in \mathbb{N}$. Then we claim that for $n \in \mathbb{N}$, there exists $t_n \in I$ such that $|(py_n')(t_n)| \leqslant M_2 := M_1(\int_I 1/p(t)dt)^{-1}$. For otherwise, $|(py')(t)| > M_2$ for all $t \in I$ and hence $y_n(t)$ and $(py_n')(t)$ do not change sign on I. Let $I = [c_0, d_0]$. Then

$$\max\{|y_n(c_0)|, |y_n(d_0)|\} > |y_n(d_0) - y_n(c_0)| = \int_I \frac{1}{p(t)}|(py')(t)|dt \geqslant M_1,$$

contradicting the boundedness assumption of y_n.

For $t \in [a, b]$, by integrating (3·17) with $\lambda = \mu_n$ and $y = y_n(t)$ from t_n to t we obtain

$$(py_n')(t) = (py_n')(t_n) + \int_{t_n}^t \{[q(\tau) - (\mu_n - f_0)w(\tau)]y_n(\tau) - w(\tau)f(y_n(\tau))\}\,d\tau \tag{3·25}$$

and

$$\begin{aligned} y_n(t) = y_n(t_n) &+ (py_n')(t_n)\int_{t_n}^t \frac{1}{p(s)}ds \\ &+ \int_{t_n}^t \left(\int_\tau^t \frac{1}{p(s)}ds\right)\{[q(\tau) - (\mu_n - f_0)w(\tau)]y_n(\tau) - w(\tau)f(y_n(\tau))\}\,d\tau. \end{aligned} \tag{3·26}$$

Recall that for sufficiently large $n \in \mathbb{N}$, $I \subset (\tau(i_0, n), \tau(i_0 + 1, n))$ and $f(y_n(t))$ has the same sign as $y_n(t)$. Then, from (3·26), for $t \in [t_n, \tau(i_0 + 1, n)]$

$$\begin{aligned} |y_n(t)| &= |y_n(t_n)| + (py_n')(t_n)\mathrm{sgn}(y_n(t))\int_{t_n}^t \frac{1}{p(s)}ds \\ &\quad + \int_{t_n}^t \left(\int_\tau^t \frac{1}{p(s)}ds\right)\{[q(\tau) - (\mu_n - f_0)w(\tau)]|y_n(\tau)| - w(\tau)|f(y_n(\tau))|\}\,d\tau \\ &\leqslant |y_n(t_n)| + |(py_n')(t_n)|\int_a^b \frac{1}{p(s)}ds \\ &\quad + \int_{t_n}^t \left(\int_a^b \frac{1}{p(s)}ds\right)\{[|q(\tau)| + (\mu_n - f_0)w(\tau)]|y_n(\tau)|\}\,d\tau \\ &\leqslant M_1 + M_2\int_a^b \frac{1}{p(s)}ds + \int_a^b \frac{1}{p(s)}ds\int_{t_n}^t \{[|q(\tau)| + (K - f_0)w(\tau)]|y_n(\tau)|\}\,d\tau, \end{aligned}$$

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where K is defined by (3·20). By the Gronwall inequality we have that for $t \in [t_n, \tau(i_0 + 1, n)]$

$$|y_n(t)| \leqslant \left(M_1 + M_2 \int_a^b \frac{1}{p(s)} ds\right) \exp\left\{\int_a^b \frac{1}{p(s)} ds \int_a^b [|q(\tau)| + (K - f_0)w(\tau)]\, d\tau\right\}.$$

This means that $\{y_n(t)\}_{n=1}^{\infty}$ is uniformly bounded on $[t_n, \tau(i_0 + 1, n)]$. Then by (3·25), $\{(py_n')(t)\}_{n=1}^{\infty}$ is uniformly bounded on $[t_n, \tau(i_0 + 1, n)]$. In particular, $\{y_n(\tau(i_0 + 1, n))\}_{n=1}^{\infty}$ and $\{(py_n')(\tau(i_0 + 1, n))\}_{n=1}^{\infty}$ are bounded. By the same process, we see that $\{y_n(t)\}_{n=1}^{\infty}$ and $\{(py_n')(t)\}_{n=1}^{\infty}$ are uniformly bounded on $[\tau(i_0 + 1, n), \tau(i_0 + 2, n)]$ and thereafter on intervals $[\tau(i, n), \tau(i + 1, n)]$ for all $i = i_0 + 1, \ldots, k$. With the backward Gronwall inequality we also see that $\{y_n(t)\}_{n=1}^{\infty}$ and $\{(py_n')(t)\}_{n=1}^{\infty}$ are uniformly bounded on $[\tau(i, n), \tau(i+1, n)]$ for all $i = 0, \ldots, i_0 - 1$. This contradicts (3·22) and thus proves (3·24).

We observe that for all $n \in \mathbb{N}$, $y_n(t)$ satisfy the same BC (3·3) with $c_n = \tau(i_0, n)$, $d_n = \tau(i_0 + 1, n)$,

$$\bar{\alpha} = \begin{cases} \alpha & \text{if} \quad i_0 = 0, \\ 0 & \text{if} \quad i_0 > 0, \end{cases} \quad \text{and} \quad \bar{\beta} = \begin{cases} \beta & \text{if} \quad i_0 = k, \\ \pi & \text{if} \quad i_0 < k. \end{cases}$$

By Lemma 3·2 with $q_n(t) = q(t) - \mu_n w(t)$, there exists $r > 0$ such that (3·4) holds, and hence $|y_n(t)| \to \infty$ uniformly on I as $n \to \infty$. Since

$$\frac{g(y)}{y} = \frac{f(y)}{y} - f_0 \longrightarrow \infty \text{ as } |y| \longrightarrow \infty,$$

it follows from (3·17) and (3·20) that for any $\lambda^* > 0$ there exists $N \in \mathbb{N}$ such that for $n \geqslant N$

$$Ly_n(t) = \left(\mu_n + \frac{g(y_n(t))}{y_n(t)}\right) w(t) y_n(t) \geqslant \lambda^* w(t) y_n(t).$$

Note that every solution of Equation (2·1) with $\lambda = \lambda^*$ sufficiently large has at least two zeros in I. By the Sturm comparison theorem, every solution of Equation (3·17) changes sign in I for sufficiently large $n \in \mathbb{N}$. This contradicts the definition of the interval I and thus completes the proof.

(II) Finally, we prove the case where $0 \leqslant f_\infty < \lambda_k < f_0 \leqslant \infty$. Let $g(y) := f(y) - f_\infty y$. Then Equation (1·1) becomes

$$Ly = f_\infty w(t) y + w(t) g(y), \tag{3·27}$$

and $g(y) = o(y)$ as $|y| \to \infty$. We consider the equation

$$Ly = \lambda w(t) y + w(t) g(y). \tag{3·28}$$

By the standard transformation $z = y/\|y\|^2$ for $y \neq 0$ (see for example [**1**, **14**]), Equation (3·28) becomes

$$Lz = \lambda w(t) z + w(t) G(z), \tag{3·29}$$

where $G: C[a, b] \to C[a, b]$ is defined by

$$G(z) = \begin{cases} \|z\|^2 g\left(\dfrac{z}{\|z\|^2}\right), & z \neq 0, \\ 0, & z = 0. \end{cases}$$

It is easy to verify that G is continuous at $z = 0$, $z \neq 0$ is a solution of BVP (3·29), (1·2) if and only if $y = z/\|z\|^2$ is a solution of BVP (3·28), (1·2), and $\|z\| = 1/\|y\|$.

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Now we show that

$$\|G(z)\| = o(\|z\|) \quad \text{as } \|z\| \longrightarrow 0. \tag{3·30}$$

To do this, let $g^*: [0, \infty) \to [0, \infty)$ be given by

$$g^*(y) = \max_{x \in [-y, y]} |g(x)|.$$

Then, g^* is nondecreasing and from the condition $g(y) = o(y)$ we see that

$$\lim_{y \to \infty} \frac{g^*(y)}{y} = 0. \tag{3·31}$$

Thus for $z \neq 0$ and $t \in [a, b]$

$$\begin{aligned} |G(z(t))| &= \|z\|^2 g\left(\frac{z(t)}{\|z\|^2}\right) \leqslant \|z\|^2 g^*\left(\frac{|z(t)|}{\|z\|^2}\right) \\ &\leqslant \|z\|^2 g^*\left(\frac{\|z\|}{\|z\|^2}\right) \leqslant \|z\| \frac{g^*(1/\|z\|)}{1/\|z\|}. \end{aligned}$$

This, together with (3·31), implies (3·30).

We then show that the only solution of Equation (3·29) with a double zero in $[a, b]$ is the trivial solution. In fact, if $t_0 \in [a, b]$ is a double zero of a solution $z(t)$, then $z(t_0) = (pz')(t_0) = 0$. Without loss of generality we only show that $z(t) \equiv 0$ on $[t_0, b]$. For any $t \in [t_0, b]$ and $s \in [a, b]$ let

$$z(t; s) = \begin{cases} z(t), & t < s \leqslant b \\ z(s), & t_0 \leqslant s \leqslant t \\ 0, & a \leqslant s \leqslant t_0. \end{cases}$$

Then $z(t; \cdot) \in C[a, b]$. Since $z(t; s) \to 0$ as $t \to t_0+$ uniformly for $s \in [a, b]$, by (3·30)

$$\|G(z(t; \cdot)\| = o(\|z(t; \cdot)\|) \quad \text{as } t \longrightarrow t_0.$$

This implies that there exists $M > 0$ such that for $t \geqslant t_0$ sufficiently close to t_0 and $0 \leqslant s \leqslant t$ we have

$$|G(z(t; s))| \leqslant M \|z(t; \cdot)\|.$$

Let $z^*(t) = \|z(t; \cdot)\| = \max_{s \in [a,b]} |z(t; s)|$. Then there exists $t^* \in [0, t]$ such that $z^*(t) = |z(t^*)|$. For $t \geqslant t_0$ sufficiently close to t_0, by integrating Equation (3·29) twice from t_0 to t^* we obtain

$$\begin{aligned} z^*(t) = |z(t^*)| &= \left| \int_{t_0}^{t^*} \left(\int_{\tau}^{t^*} 1/p(s) ds \right) [(q(\tau) - \lambda w(\tau)) z(\tau) - w(\tau) G(z(\tau))] \, d\tau \right| \\ &\leqslant \left(\int_a^b 1/p(s) ds \right) \int_{t_0}^{t} [|q(\tau) - \lambda w(\tau)\| z(\tau; \tau)| + w(\tau) |G(z(\tau; \tau))|] d\tau \\ &\leqslant \left(\int_a^b 1/p(s) ds \right) \int_{t_0}^{t} [|q(\tau)| + (\lambda + M) w(\tau)] \|z(\tau; \cdot)\| d\tau \\ &\leqslant \left(\int_a^b 1/p(s) ds \right) \int_{t_0}^{t} [|q(\tau)| + (\lambda + M) w(\tau)] z^*(\tau) d\tau. \end{aligned}$$

Hence the Gronwall inequality implies that $z^*(t) \equiv 0$ and hence $z(t) \equiv 0$ for $t \geqslant t_0$ sufficiently close to t_0. The conclusion on the whole interval $[t_0, b]$ is then obtained by extension.

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Now, all the conditions of Lemma 3·1 and Remark 3·1 are satisfied. Therefore, for $\nu \in \{+, -\}$, there exists a continuum C_k^ν of solutions of (3·29) in $\Phi_k^\nu \cup \{(\tilde{\lambda}_k, 0)\}$ that meets $(\lambda_k, 0)$ and ∞ in $\mathcal{P}$. It is clear that every solution of Equation (3·29) of the form (f_∞, z) in C_k^ν with $z \neq 0$ yields a solution $y = z/\|z\|$ of BVP (3·27), (1·2) with k simple zeros.

The rest of the proof is essentially in the same way as Part (I) except that f_0 and f_∞ interchange their roles. We omit the details.

Proof of Theorem 2·3. The proof for parts (i) and (ii) is based on the comparison theorem for Prüfer angles. Although the proof of [**8**, theorem 2·1, (i) and (ii)] is for the special case with $p \equiv 1$ and other stronger assumptions, it works here for the general problem. We omit the details.

To prove part (iii), we only need to show that under our assumptions, no nontrivial solution $y(t)$ of BVP (1·1), (1·2) can have an infinite number of zeros, and hence the conclusion follows from parts (i) and (ii). For otherwise, there exist a nontrivial solution $y(t)$ and $t_* \in [a, b]$ such that $y(t_*) = (py')(t_*) = 0$.

Assume $f_0 < \infty$. Note that $y(t)$ is a solution of Equation (3·16). It is easy to see that $y(t) \equiv 0$ on $[a, b]$.

Assume $f_\infty < \infty$. If $y(t) \not\equiv 0$ on $[a, b]$, let $z = y/\|y\|^2$. Then $z(t)$ is a nontrivial solution of Equation (3·29) satisfying $z(t_0) = (pz')(t_0) = 0$. The by Part (II) of the proof of Theorem 2·1 we have $z(t) \equiv 0$ on $[a, b]$. We have reached a contradiction and hence completes the proof.

Proof of Theorem 2·5. This theorem is a generalization of [**8**, theorem 2·4] to our case with weaker assumptions. The proof is essentially the same and hence is omitted.

Proof of Theorem 2·6. This theorem is a generalization of [**8**, theorem 2·6] to our case with weaker assumptions. The proof is essentially the same and hence is omitted.

Proof of Theorem 2·7. For $s \in \mathbb{R}$ and $i \in \mathbb{N}_0$, we denote by $\lambda_i(s)$ the ith eigenvalue of the SLP consisting of the equation

$$-(p(t)y')' + [q(t) + sh(t)]y = \lambda w(t)y \quad \text{on } [a, b]$$

and BC (1·2). Let $h_* = \min\{h(t)/w(t) : t \in [a, b]\}$, and denote by $\mu_i(s)$ the ith eigenvalue of the SLP consisting of the equation

$$-(p(t)y')' + [q(t) + sh_*w(t)]y = \mu w(t)y \quad \text{on } [a, b] \tag{3·32}$$

and BC (1·2).

(i) Since for $s \leqslant 0$

$$q(t) + sh(t) \leqslant q(t) + sh_*w(t),$$

by [**9**, theorem 4·2 (6)], $\lambda_i(s) \leqslant \mu_i(s)$ for all $s \leqslant 0$ and $i \geqslant 0$. Note that Equation (3·32) is the same as the equation

$$-(p(t)y')' + q(t)y = (\mu - sh_*)w(t)y \quad \text{on } [a, b].$$

Thus, for $s \leqslant 0$ and $i \geqslant 0$, $\mu_i(s) - sh_* = \mu_i(0)$ which implies that

$$\mu_i(s) = \mu_i(0) + sh_* \longrightarrow -\infty \text{ as } s \longrightarrow -\infty,$$

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and hence

$$\lambda_i(s) \longrightarrow -\infty \text{ as } s \longrightarrow -\infty,\ i \geqslant 0.$$

Then, for any $n \in \mathbb{N}$ there exists $s_n \leqslant 0$ such that $\lambda_i < 0$ for all $i \leqslant n$ and $s < s_n$. Therefore, the conclusion follows from Theorem 2·3.

(ii) Without loss of generality assume $f_0 < \infty$ and $f_\infty = \infty$. Similar to the argument in (i) we have

$$\lambda_i(s) \longrightarrow \infty \text{ as } s \longrightarrow \infty,\ i \geqslant 0. \tag{3·33}$$

Then for any $n \in \mathbb{N}_0$ there exists $s_n \geqslant 0$ such that for any $s > s_n$ we have $f_0 < \lambda_n(s) < f_\infty$ and hence $f_0 < \lambda_i(s) < f_\infty$ for all $i \geqslant n$. Therefore, the conclusion follows from Theorem 2·1.

(iii) From (3·33), there exists $s_* \geqslant 0$ such that for any $s > s_*$ we have that $\lambda_n(s) > f^* := \sup\{f(y)/y: y \in (0, \infty)\}$. Therefore, the conclusion follows from Theorem 2·3.

Proof of Theorem 2·8. For $s \geqslant 0$ and $i \in \mathbb{N}_0$, we denote by $\lambda_i(s)$ the ith eigenvalue of the SLP consisting of the equation

$$-(p(t)y')' + q(t)y = \lambda[w(t) + sh(t)]y \quad \text{on } [a, b]$$

and BC (1·2). Let $h_* = \min\{h(t)/w(t): t \in [a, b]\}$, and denote by $\mu_i(s)$ the ith eigenvalue of the SLP consisting of the equation

$$-(p(t)y')' + q(t)y = \mu(1 + sh_*)w(t)y \quad \text{on } [a, b]$$

and BC (1·2). Since for $s \geqslant 0$

$$w(t) + sh(t) \geqslant (1 + sh_*)w(t),$$

by [**9**, theorem 4·2 (7)],

$$\lambda_i(s) \leqslant \mu_i(s) \text{ for all } s \geqslant 0 \text{ and } i \geqslant 0, \text{ whenever } \lambda_i(s) \geqslant 0. \tag{3·34}$$

Note that for $i \geqslant 0$, $\mu_i(s)(1 + sh_*) = \mu_i(0)$, we have

$$\mu_i(s) = \frac{\mu_i(0)}{1 + sh_*} \longrightarrow 0 \text{ as } s \longrightarrow \infty.$$

This together with (3·34) implies that $\lambda_i(s) < f_*$ as $s \to \infty$. Then, for any $n \in \mathbb{N}$ there exists $s_n \geqslant 0$ such that $\lambda_i < f_*$ for all $s > s_n$ and $i \leqslant n$. Therefore, the conclusion follows from Theorem 2·3.

Proof of Theorem 2·9. For $s \geqslant 0$ and $i \in \mathbb{N}_0$, we denote by $\lambda_i(s)$ the ith eigenvalue of the SLP consisting of the equation

$$-\left[\frac{1}{1/p(t) + sh(t)}y'\right]' + q(t)y = \lambda w(t)y$$

and BC (1·2) with an eigenfunction $u_i(t, s)$. Let $\theta_i(t, s)$ be the Prüfer angle of $u_i(t, s)$ satisfying $\theta_i(a, s) = \alpha$. Then

$$\theta_i'(t, s) = \left[\frac{1}{p(t)} + sh(t)\right]\cos^2\theta_i(t, s) + [\lambda_i w(t) - q(t)]\sin^2\theta_i(t, s). \tag{3·35}$$

By [**9**, theorem 4·2 (5)], $\lambda_i(s)$ is decreasing and hence

$$\lim_{s\to\infty} \lambda_i(s) = \lambda_i^* \in [-\infty, \infty).$$

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We show that $\lambda_i^* < f_*$ and then the conclusion follows from Theorem 2·3.

Assume the contrary, i.e., $\lambda_i^* \geqslant f_*$. Let $w_* = \min\{w(t): t \in [a, b]\}$. By (3·35)

$$\begin{aligned}\theta_i'(t, s) &\geqslant \left[\frac{1}{p(t)} + sh(t)\right]\cos^2\theta_i(t, s) + [\lambda_i^* w(t) - q(t)]\sin^2\theta_i(t, s)\\ &\geqslant \left[\frac{1}{p(t)} + sh(t)\right]\cos^2\theta_i(t, s) + (f_* - q^*)w_*\sin^2\theta_i(t, s).\end{aligned}$$

Let $\phi(t, s)$ be the solution of the equation

$$\phi'(t, s) = \left[\frac{1}{p(t)} + sh(t)\right]\cos^2\phi(t, s) + (f_* - q^*)w_*\sin^2\phi(t, s) \tag{3·36}$$

satisfying $\phi(a, s) = \alpha$. By the theory of differential inequalities we have $\phi(t, s) \leqslant \theta_i(t, s)$. In particular,

$$\phi(b, s) \leqslant \theta_i(b, s) = i\pi + \beta. \tag{3·37}$$

We observe from (3·36) and the Sturm comparison theorem that $\phi(t, s)$ is strictly increasing in t and s, and $0 < \phi(t, s) < (i+1)\pi$ for $t \in [a, b]$ and $s \geqslant 0$. Let $\phi^*(t) = \lim_{s\to\infty}\phi(t, s)$. Then $0 < \phi^*(t) \leqslant (i+1)\pi$ for $t \in [a, b]$. We claim that

$$\phi^*(t) \not\equiv k\pi + \frac{\pi}{2} \text{ on } (a, b], \tag{3·38}$$

for any $0 \leqslant k \leqslant i$. If not, for any $a_1 \in (a, b]$ and $\epsilon > 0$, there exists $s^* > 0$ such that for $s \geqslant s^*$

$$\phi(a_1, s) \in (k\pi + \pi/2 - \epsilon, k\pi + \pi/2),$$

which follows that

$$\phi(t, s) \in (k\pi + \pi/2 - \epsilon, k\pi + \pi/2) \text{ for } t \in [a_1, b].$$

This implies that

$$0 < \phi(b, s) - \phi(a_1, s) < \epsilon. \tag{3·39}$$

However, from (3·36) we see that for s^* sufficiently large,

$$\phi'(t, s) \geqslant \frac{1}{2}(f_* - q^*)w_* \text{ for } t \in [a_1, b].$$

This contradicts (3·39) and hence verifies (3·38).

It is easy to see that $\phi'(t, s) \to \phi^*(t)$ uniformly on $[a_1, b]$ as $s \to \infty$. Thus $\phi^*(t)$ is continuous on $[a_1, b]$. From (3·38), we can find nontrivial closed interval $[c, d] \subset [a_1, b]$ such that $\cos^2\phi^*(t) \geqslant k > 0$ for $t \in [c, d]$. Then from (3·36)

$$\phi'(t, s) \geqslant \left[\frac{1}{p(t)} + sh(t)\right]k \longrightarrow \infty \text{ uniformly for } t \in [c, d] \text{ as } s \longrightarrow \infty.$$

Therefore,

$$\begin{aligned}\phi(b, s) &\geqslant \phi(d, s) \geqslant \phi(c, s) + \int_c^d \left[\frac{1}{p(t)} + sh(t)\right]k\,ds\\ &\geqslant \int_c^d \left[\frac{1}{p(t)} + sh(t)\right]k\,ds \longrightarrow \infty \text{ as } s \longrightarrow \infty.\end{aligned}$$

This contradicts (3·37) and hence completes the proof.

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