Nonlinear transient analysis of moderately thick rectangular composite plates

H. Tanrıöver
E. Şenocak
senocak@itu.edu.tr.
Istanbul Technical University
Department of Mechanical Engineering
Istanbul
Turkey

ABSTRACT

This paper presents an analytical-numerical methodology for the geometrically nonlinear analysis of laminated composite plates under dynamic loading. The methodology employs Galerkin technique, in which suitable polynomials are chosen as trial functions. In the solution process, Newmark's scheme for time integration, and modified Newton-Raphson method for the solution of resulting nonlinear equations are used. In the formulation, first order shear deformation theory based on Mindlin's hypothesis and von Kármán type geometric nonlinearity are considered. The results are compared to that of finite strips, and Chebyshev series published elsewhere. The method is found to determine closely both the displacements and the stresses. A finite element analysis has also been carried out for the validation of the results. The present method can be efficiently and easily applied for the nonlinear transient analysis of laminated composite plates with various boundary conditions.

1.0 INTRODUCTION

The transient response of laminated composite plates to dynamic loading is a topic of considerable engineering importance. High performance of such laminates in constructions can only be achieved by a proper design based on effective and accurate dynamic analysis.

Various numerical techniques have been utilised to investigate the geometrically nonlinear (in the von Kármán sense) dynamic response of plates. A mixed element formulation for finite element method (FEM) was utilised for large deflection dynamic response of isotropic plates by Akay[1]. Reddy investigated forced motions of laminated composite plates using a finite element that accounts for shear deformation, rotary inertia and geometric nonlinearity[2]. Ali and Al-Noury[3] solved the nonlinear transient response of rectangular plates using an implicit finite difference method. Carrera and Krause investigated nonlinear dynamic analysis of laminates concerning interlaminar continuity effects with the help of FEM[4]. Chen et al. developed a semi-analytical finite strip method (FSM)[5] and Nath and Shukla[6] used Chebyshev series (CS) technique for the nonlinear transient analysis of laminates. Chebyshev-collocation method with Newmark scheme is applied to the von Kármán plate theory by Kirby and Yosibash[7]. A comprehensive list of papers published in the area is given in references[8,9].

The Galerkin method (GM) is a powerful numerical solution technique to differential equations. This method involves the direct use of differential equations. Trial functions do not need to satisfy all the boundary conditions, and this gives flexibility to the method, which can be applied to a broad class of problems. The large amplitude dynamic response of rectangular plates under transient loads using Galerkin technique with a multi-mode approach was analyzed by Quaisi[10]. Making use of the GM, the dynamic analysis of quadrilateral plates having intermediate line and point supports is accomplished by Saadatpour *et al*[11]. Shi *et al* performed small amplitude free vibration analysis of laminated plates by using GM[12]. Tanrıöver and Şenocak used the Galerkin technique to study static and dynamic large deflection responses of laminated composite plates[13,14]. To the best knowledge of the authors, a few published

Paper No. 3376. Manuscript received 17 November 2008, revised version received 22 September 2009 accepted 21 January 2010.

https://doi.org/10.1017/S0001924000003912 Published online by Cambridge University Press

research works consider this technique to solve the nonlinear transient response of plates.

In the present paper, geometrically nonlinear transient analysis of moderately thick laminated composite plates is presented using the GM. In the formulation, first order shear deformation theory (FSDT) based on Mindlin's hypothesis with von Kármán nonlinearity is utilised taking into account rotary and in-plane inertia effects.

Polynomials satisfying essential boundary conditions are chosen (as trial functions) to approximate the plate displacemen ts. The trial functions do not have to satisfy natural boundary conditions since they are handled in the formulation of the GM. A detailed explanation on how to choose proper polynomials depending on boundary conditions is given in Section solution procedure. The analysis is performed by using Newmark's scheme for time integration, and modified Newton-Raphson method for the solution of resulting nonlinear equations. In the solution process, computations have been carried out analytically wherever it is possible and analytical-numerical type approach has been made for all cases. The stresses and displacements are computed and the results are compared to that of finite strips, Chebyshev series and commercial FEM code ABAQUS[15]. The transient responses of rectangular orthotropic, cross-ply and angle-ply laminates are presented. Displacements and stresses at the centre of the laminates are provided for all problems.

2.0 GOVERNING EQUATIONS

Consider a rectangular laminated plate with dimensions a, b and uniform thickness h. The origin of the coordinate system is chosen to coincide with the centre of the midplane of the undeformed plate (see Fig. 1). The plate is assumed to be subjected uniform transverse pressure q_0, and it is constructed of finite homogenous orthotropic layers perfectly bonded together.

Under the assumptions of first order shear deformation theory based on Mindlin's hypothesis; let u, v, w denote the displacements at an arbitrary point of the plate in the x, y, z directions and $u^0(x,y)$, $v^0(x,y)$, $w^0(x,y)$ are the displacements at a corresponding point of the midplane of the plate in the x, y and z directions respectively. Then the displacement field of the first order theory is of the form[16]:

$$U(x,y,z,t) - u^o(x,y,t) + \phi_x(x,y,t)z,$$
$$v(x,y,z,t) = v^o(x,y,z,t) + \phi_y(x,y,t)z, \quad \ldots (1)$$
$$w(x,y,z,t) - w^o(x,y,t)$$

where ϕ_x and ϕ_x are the rotations of a transverse normal about the y and x axes respectively. The corresponding total strains could be expressed as follows:

$$\varepsilon_x = \varepsilon_x^0 + \kappa_x z,\ \varepsilon_y = \varepsilon_y^0 + \kappa_y z,\ \gamma_{xy} = \gamma_{xy}^0 + \kappa_{xy} z,\ \gamma_{xz} = \gamma_{xz}^0,\ \gamma_{yz} = \gamma_{yz}^0 \quad \ldots (2)$$

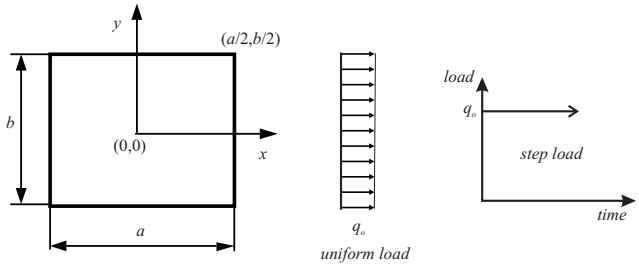


Figure 1. Plate geometry and loading.

Considering von Kármán type geometric nonlinearity[17], the strain displacement relations can be written as;

$$\varepsilon_x^0 = u_{,x}^0 + 1/2\, w_{,x}^2,\ \varepsilon_y^0 = v_{,x}^0 + 1/2\, w_{,y}^2,\ \gamma_{xy}^0 = u_{,y}^0 + v_{,x}^0 + w_{,x} w_{,y},$$
$$\gamma_{xz}^0 = w_{,x} + \phi_x,\ \gamma_{yz}^0 = w_{,y} + \phi_y,$$

where differentiations are denoted by comma. Midplane curvatures and twist of the plate are the following;

$$\kappa_x = \phi_{x,x},\ \kappa_y = \phi_{y,y},\ \kappa_{x,y} = \phi_{x,y} + \phi_{y,x} \quad \ldots (3)$$

For a plate with an arbitrary number of layers, the constitutive relations are

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}, \quad \ldots (4)$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}, \quad \ldots (5)$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix}, \quad \ldots (6)$$

where N_x, N_y and N_{xy} are the resultant in-plane, Q_x and Q_y are transverse forces and M_x, M_y and M_{xy} are moments conjugate to in-plane strains (ε_x^0, ε_y^0, ε_{xy}^0) transverse shear strains ($\gamma_{yz}^0, \gamma_{xz}^0$) and curvatures ($\kappa_x, \kappa_y, \kappa_{xy}$) respectively. The parameter K introduced in the above equation is shear correction factor[18] that is taken as 5/6 in the present work. A_{ij}, B_{ij} and D_{ij} are symmetric matrices defined as follows:

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} (1, z, z^2) Q_{ij}\, dz, \quad \ldots (7)$$

where Q_{ij} is the corresponding reduced stiffness coefficients.
Five governing equations of motion for the plate can be written as follows in the general form[8]:

$$\begin{aligned} R_1 &= N_{x,x} + N_{xy,y} - I_0 u_{,tt}^0 - I_1 \phi_{x,tt} = 0, \\ R_2 &= N_{xy,x} + N_{y,y} - I_0 v_{,tt}^0 - I_1 \phi_{y,tt} = 0, \\ R_3 &= Q_{x,x} + Q_{y,y} + \left(w_{,x} N_x + w_{,y} N_{xy}\right)_{,x} \\ &\quad + \left(w_{,x} N_{xy} + w_{,y} N_y\right)_{,y} + q_0 - I_0 w_{,tt} = 0, \\ R_4 &= M_{x,x} + M_{xy,y} - Q_x - I_2 \phi_{x,tt} - I_1 u_{,tt}^0 = 0, \\ R_5 &= M_{xy,x} + M_{y,y} - Q_y - I_2 \phi_{y,tt} - I_1 v_{,tt}^0 = 0, \end{aligned} \quad \ldots (8)$$

where I_0, I_1 and I_2 are mass moments of inertia defined as

$$(I_0, I_1, I_2) = \int_{-h/2}^{h/2} \rho(1, z, z^2)\, dz = \sum_i \int_{z_i}^{z_{i+1}} \rho^{(i)}(1, z, z^2)\, dz, \quad \ldots (9)$$

being the material density of the ith layer. The equations of motion[8] in terms of displacements is obtained by making use of constitutive[4,6], strain-displacement and curvature-displacement[3] relations. The normalised equations are given in Appendix A.

For the Galerkin approach, the normalised displacements of the plate are approximated in the form shown below:

https://doi.org/10.1017/S0001924000003912 Published online by Cambridge University Press

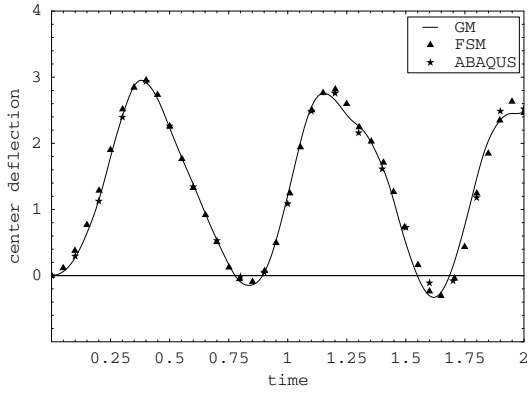


Figure 2. Dimensionless centre deflection ($w = \bar{w}/h$) for the orthotropic plate in time (millisec).

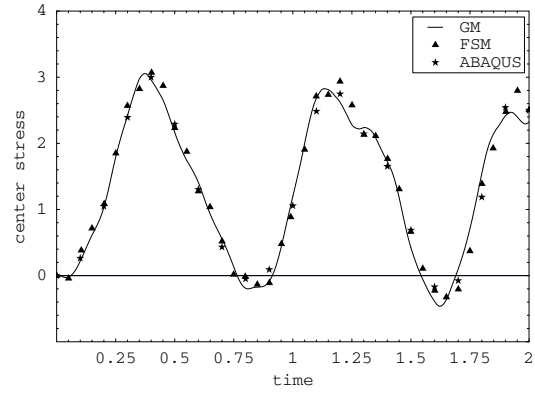


Figure 3. Dimensionless centre stress in the x direction ($\sigma_{xc} = 525\, s_{xc}/E_1$) for the orthotropic plate in time (millisec).

$$\begin{aligned}
\bar{u} &= \sum_{m=0}^{M}\sum_{n=0}^{N} a_{mn}(\bar{t})U_{mn}(\bar{x},\bar{y}) = \sum_i a_i U_i,\\
\bar{v} &= \sum_{m=0}^{M}\sum_{n=0}^{N} b_{mn}(\bar{t})V_{mn}(\bar{x},\bar{y}) = \sum_i b_i V_i,\\
\bar{w} &= \sum_{m=0}^{M}\sum_{n=0}^{N} c_{mn}(\bar{t})W_{mn}(\bar{x},\bar{y}) = \sum_i c_i W_i,\\
\bar{\phi}_x &= \sum_{m=0}^{M}\sum_{n=0}^{N} d_{mn}(\bar{t})S_{mn}(\bar{x},\bar{y}) = \sum_i d_i S_i,\\
\bar{\phi}_y &= \sum_{m=0}^{M}\sum_{n=0}^{N} e_{mn}(\bar{t})T_{mn}(\bar{x},\bar{y}) = \sum_i e_i T_i,\\
&(\text{i.e., } a_1 = a_{00}(\bar{t}), a_2 = a_{01}(\bar{t}), \ldots),
\end{aligned} \quad \ldots (10)$$

where a_{mn}, b_{mn}, c_{mn}, d_{mn} and e_{mn} are unknown functions of time, U_{mn}, V_{mn}, W_{mn}, S_{mn} and T_{mn} are the trial functions, and M and N are the number of terms in $\bar{x}$ and $\bar{y}$ directions respectively. Herein, polynomials are used as trial functions, which are chosen to satisfy the geometric boundary conditions, where as natural boundary conditions are not satisfied. In this case, simultaneous approximation is made to the solutions of differential equations and to the boundary conditions.

Substituting Equation (10) into nonlinear plate equilibrium equations and the boundary terms yields the residuals in the domain of the plate and at the boundaries of the plate. Forcing these residuals to be orthogonal to each member of a set of trial functions yields the following (nondimensionalised) Galerkin equations.

$$\begin{aligned}
&\int_{-1}^{+1}\int_{-1}^{+1} U_{mn}\bar{R}_1 \mathrm{d}\bar{x}\mathrm{d}\bar{y} - \int_{-1}^{+1} U_{mn}\bar{N}_x|_{\bar{x}=\pm1}\mathrm{d}\bar{y} - \int_{-1}^{+1} U_{mn}\bar{N}^u_{xy}|_{\bar{y}=\pm1}\mathrm{d}\bar{x} = 0,\\
&\int_{-1}^{+1}\int_{-1}^{+1} V_{mn}\bar{R}_2 \mathrm{d}\bar{x}\mathrm{d}\bar{y} - \int_{-1}^{+1} V_{mn}\bar{N}_y|_{\bar{y}=\pm1}\mathrm{d}\bar{x} - \int_{-1}^{+1} V_{mn}\bar{N}^v_{xy}|_{\bar{x}=\pm1}\mathrm{d}\bar{y} = 0,\\
&\int_{-1}^{+1}\int_{-1}^{+1} W_{mn}\bar{R}_3 \mathrm{d}\bar{x}\mathrm{d}\bar{y} = 0,\\
&\int_{-1}^{+1}\int_{-1}^{+1} S_{mn}\bar{R}_4 \mathrm{d}\bar{x}\mathrm{d}\bar{y} - \int_{-1}^{+1} S_{mn}\bar{M}_x|_{\bar{x}=\pm1}\mathrm{d}\bar{y} = 0,\\
&\int_{-1}^{+1}\int_{-1}^{+1} T_{mn}\bar{R}_5 \mathrm{d}\bar{x}\mathrm{d}\bar{y} - \int_{-1}^{+1} T_{mn}\bar{M}_y|_{\bar{y}=\pm1}\mathrm{d}\bar{x} = 0,
\end{aligned} \quad \ldots (11)$$

where $\bar{R}_i$ (i = 1, ...5) in the domain, $\bar{N}_x$, N_{xy}, N_y, $\bar{M}_x$, M_{xy} and M_y at the boundaries are given in Appendices A and B.

3.0 SOLUTION PROCEDURE

In the application of the Galerkin method the geometrical boundary conditions are satisfied by choosing appropriate trial functions. The trial functions are weighted polynomials given as follows:

$$\begin{aligned}
U_{mn} &= \Phi_1 \bar{x}^m \bar{y}^n,\\
V_{mn} &= \Phi_2 \bar{x}^m \bar{y}^n,\\
W_{mn} &= \Phi_3 \bar{x}^m \bar{y}^n,\\
S_{mn} &= \Phi_4 \bar{x}^m \bar{y}^n,\\
T_{mn} &= \Phi_5 \bar{x}^m \bar{y}^n,
\end{aligned} \quad \ldots (12)$$

where Φ_i (i = 1, ...5) denote the weight functions. Various types of boundary conditions can be implemented by choosing proper weight functions. In this work, three different boundary conditions are considered and presented in Table 1.

For the orthotropic and cross-ply laminates in the present paper, under SS-1 and SS-2 type boundary conditions, trial functions can be constructed taking into account the symmetry of the problem in such a manner that,

$$\left.\begin{matrix} m = odd \\ n = even \end{matrix}\right\} U_{mn}, S_{mn} \quad \left.\begin{matrix} m = even \\ n = odd \end{matrix}\right\} V_{mn}, T_{mn} \quad \left.\begin{matrix} m = even \\ n = even \end{matrix}\right\} W_{mn} \quad \ldots (13)$$

An angle-ply laminate under CS-1 type boundary condition is also taken as an example here, and for this case both even and odd powers of polynomial terms in the trial function representation must

Table 1
Boundary conditions and corresponding weight functions

SS-1	$\bar{u} = \bar{v} = \bar{w} = \bar{M}_x = \bar{\phi}_y = 0$ at $\bar{x} = \pm1$, $\bar{u} = \bar{v} = \bar{w} = \bar{M}_y = \bar{\phi}_x = 0$ at $\bar{y} = \pm1$. $\Phi_i = (\bar{x}^2 - 1)(\bar{y}^2 - 1)$ $(i = 1, \ldots 3)$, $\Phi_4 = (\bar{y}^2 - 1)$, $\Phi_5 = (\bar{x}^2 - 1)$.
SS-2	$\bar{N}_x = \bar{v} = \bar{w} = \bar{M}_x = \bar{\phi}_y = 0$ at $\bar{x} = \pm1$, $N_y = \bar{u} = \bar{w} = \bar{M}_y = \bar{\phi}_x = 0$ at $\bar{y} = \pm1$. $\Phi_3 = (\bar{x}^2 - 1)(\bar{y}^2 - 1), \Phi_i = (\bar{y}^2 - 1)$ $(i = 1, 4), \Phi_i = (\bar{x}^2 - 1)$ $(i = 2, 5)$.
CS-1	$\bar{u} = \bar{v} = \bar{w} = 0$ at $\bar{x} = \pm1$ and $\bar{y} = \pm1$, $\bar{\phi}_x = 0$ at $\bar{x} = -1$ and $\bar{y} = \pm1, \bar{M}_x = 0$ at $\bar{x} = 1$, $M_y = 0$ at $\bar{y} = -1$, $\bar{\phi}_y = 0$ at $\bar{x} = \pm1$ and $\bar{y} = 1$. $\Phi_i = (\bar{x}^2 - 1)(\bar{y}^2 - 1)$ $(i = 1, \ldots 3)$, $\Phi_4 = (\bar{x} + 1)(\bar{y}^2 - 1)$, $\Phi_5 = (\bar{x}^2 - 1)(\bar{y} - 1)$.

https://doi.org/10.1017/S0001924000003912 Published online by Cambridge University Press

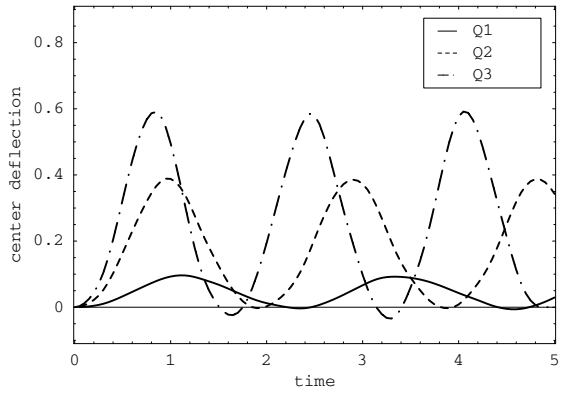


Figure 4. Dimensionless centre deflections ($\bar{w}$) of the orthotropic plate for different load values in time ($\bar{t}$).

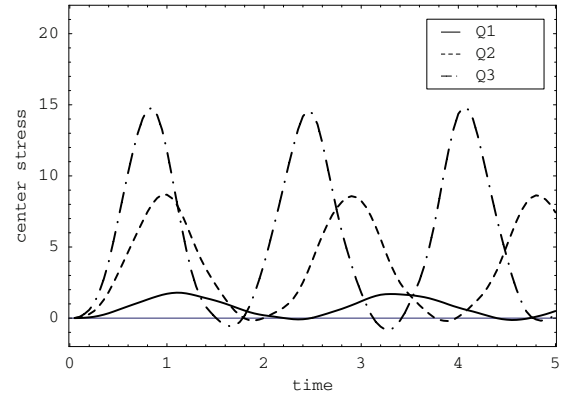


Figure 5. Dimensionless centre stresses ($\bar{\sigma}_{xc} = 525\,\sigma_{xc}/E_1$) of the orthotropic plate for different load values in time ($\bar{t}$).

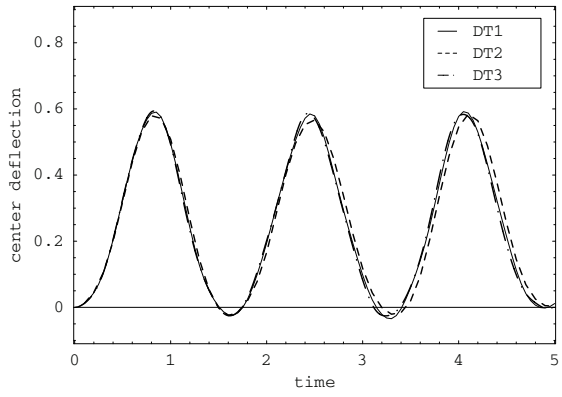


Figure 6. Dimensionless centre deflection ($\bar{w}$) of the orthotropic plate in time ($\bar{t}$) for different time steps.

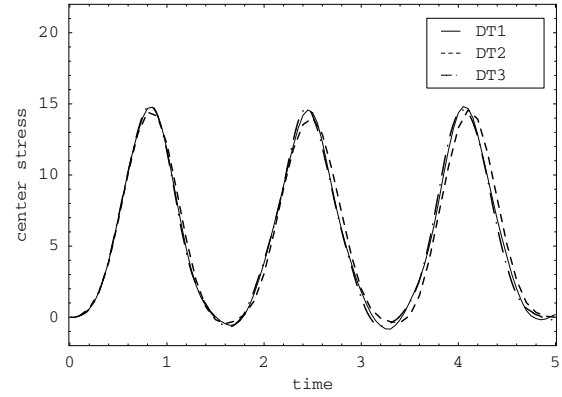


Figure 7. Dimensionless centre stress ($\bar{\sigma}_{xc} = 525\,\sigma_{xc}/E_1$) of the orthotropic plate in time for different time steps ($\bar{t}$).

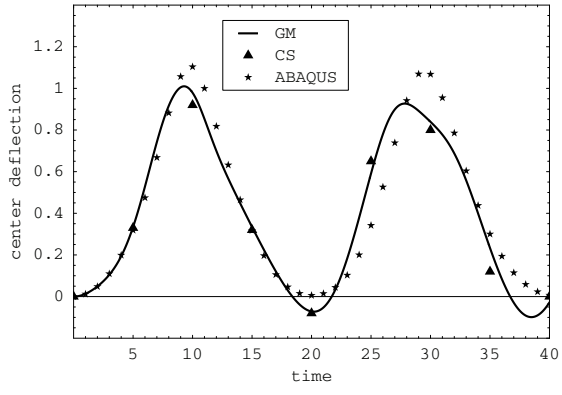


Figure 8. Dimensionless centre deflection ($\bar{w}$) of the cross-ply (0°/90°/0°/90°) laminate in time ($\bar{\tau}$). $\Delta\tau$ – 0·1.

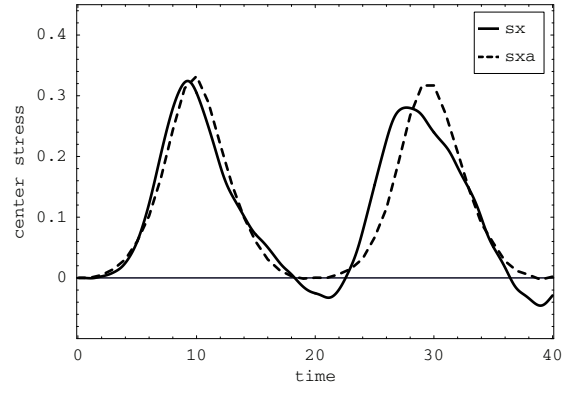


Figure 9. Dimensionless centre stress ($\sigma_{xc}\bar{\xi}^2/E_1$) of the cross-ply (0°/90°/0°/90°) laminate in time (τ). sx: GM results. sxa: ABAQUS results.

be taken into account.

Galerkin equations (11) can be written in brief form as:

$$\mathbf{F}^{(1)} - \mathbf{M}^{(11)}\ddot{\mathbf{a}} - \mathbf{M}^{(14)}\ddot{\mathbf{d}} = \mathbf{0}$$

$$\mathbf{F}^{(2)} - \mathbf{M}^{(22)}\ddot{\mathbf{a}} - \mathbf{M}^{(25)}\ddot{\mathbf{e}} = \mathbf{0}$$

$$\mathbf{F}^{(3)} - \mathbf{M}^{(33)}\ddot{\mathbf{c}} = \mathbf{0} \qquad \ldots (14)$$

$$\mathbf{F}^{(4)} - \mathbf{M}^{(41)}\ddot{\mathbf{a}} - \mathbf{M}^{(44)}\ddot{\mathbf{d}} = \mathbf{0}$$

$$\mathbf{F}^{(5)} - \mathbf{M}^{(25)}\ddot{\mathbf{b}} - \mathbf{M}^{(55)}\ddot{\mathbf{e}} = \mathbf{0}$$

or,

$$\mathbf{F} - \mathbf{M}\ddot{\mathbf{U}} = \mathbf{0} \qquad \ldots (15)$$

where $\mathbf{F}$ is the force vector, $\mathbf{M}$ is the mass matrix and $\mathbf{U}$ is the generalised displacement vector. Note that, the number of unknown coefficients in Equation (15) is $5((M + 1)/2)$ $((N + 1)/2)$ for the orthotropic plates and the cross-ply laminate, $5(M + 1)$ $(N + 1)$ for the angle-ply laminate. The detailed expressions for this equation can be found in Appendix C.

3.1 Time integration

To integrate Equation (15), Newmark's direct integration scheme[19] is employed. In the Newmark scheme the first time derivative of the displacement field $\dot{\mathbf{U}}$ and the solution $\mathbf{U}$ are approximated at $(n + 1)$ time step (i.e., at time $t = t_{n+1} = (n + 1)\Delta t$) by the following expressions:

https://doi.org/10.1017/S0001924000003912 Published online by Cambridge University Press

$$\dot{\mathbf{U}}_{n+1} = \dot{\mathbf{U}}_n + \Delta t[(1-\gamma)\ddot{\mathbf{U}}_n + \gamma\ddot{\mathbf{U}}_{n+1}],$$
$$\mathbf{U}_{n+1} = \mathbf{U}_n + \dot{\mathbf{U}}_n\Delta t + \tfrac{\Delta t^2}{2}[(1-2\beta)\ddot{\mathbf{U}}_n + 2\beta\ddot{\mathbf{U}}_{n+1}], \quad \ldots (16)$$

where γ and β parameters are chosen as ½ and ¼ respectively in all of the present analyses.

Initial displacement and velocity fields are taken as zero in the analyses. Initial acceleration is obtained by substituting initial displacement field into Equations (11) and solving the yielding linear equation system. In each time step, equations of motion (8) is utilised with Newmark's scheme by substituting acceleration at $t_n + 1$ from Equations (8) into the second equation of Newmark scheme. GM is applied to the resulting nonlinear differential equations in terms of displacements at $t_n + 1$. The method gives nonlinear equations in terms of unknown coefficients a_i, b_i, c_i, d_i and e_i. These equations are solved by employing the modified Newton-Raphson methodology. Note that the evaluations of integrals and derivatives are symbolically computed by using a commercial computer math code Mathematica™[20].

4.0 NUMERICAL EXAMPLES

Geometrically nonlinear transient analyses of composite laminates subjected to step loading are studied. Orthotropic, cross-ply and angle-ply type laminates, whose solutions are available in the literature, are considered to check the results. Comparisons of the results with other numerical techniques such as Chebyshev series and finite strip method are presented. A commercial FEM code ABAQUS is also used to check the results. In the ABAQUS analyses 10 × 10 mesh with S4 type general shell elements is used.

A convergence study of the proposed solution technique is carried out, and it has been determined that taking M and N in Equation (10) as five is appropriate for the analyses[21]. Hence, for all the GM applications given here, M and N are taken as five and computations are performed on a dual processor SUN Blade 2,500 workstation with 8Gb random access memory.

4.1 Nonlinear transient analyses of orthotropic plates

The orthotropic plates under uniform step loading are considered. The material data for the plates are taken from Ref (5) and given below.

$E_1 = 525{,}000\text{N/mm}^2$, $E_2 = 21{,}000\text{N/mm}^2$

$G_{12} = G_{13} = G_{23} = 104{,}500\text{N/mm}^2$

$v_{12} = 0{\cdot}25$, $p = 800\text{kg/mm}^3$, $a = b = 250\text{mm}$

For the validation of the present technique an orthotropic plate under SS-2 type boundary condition is considered and the results are compared to that of FSM[5] and ABAQUS. Here the plate is relatively thin, whereas a/h – 50 and under a lateral load of $q_0 = 1$ N/mm^2. In the analyses the time step Δt is taken as 10μsec which is the same value chosen in Ref 5. Comparisons are given for the centre deflection and centre stress at the surface of the plate and can be seen in Figs 2 and 3.

An orthotropic plate of moderate thickness where a/h – 10 under SS-1 type boundary condition is also analysed. In Fig. 4 and 5, centre deflections and stresses *versus* time for different load values such as Q1 = 10N/mm^2, Q2 = 50N/mm^2, Q3 = 100N/mm^2 are presented. Here, time step is taken as $\Delta t = 0{\cdot}05$ (see Equation (17) for normalisation in time).

Effect of the time step size to the accuracy of the solutions is conducted and shown in Figs 6 and 7 for the same plate under SS-1 type boundary condition and load of 100N/mm^2. Here, three different time steps $\Delta t = 0{\cdot}05$ (DT1), $\Delta t = 0{\cdot}1$ (DT2) and $\Delta t = 0{\cdot}01$ (DT3) are considered. It is found to be adequate to take the time step as $\Delta t = 0{\cdot}05$ for this plate.

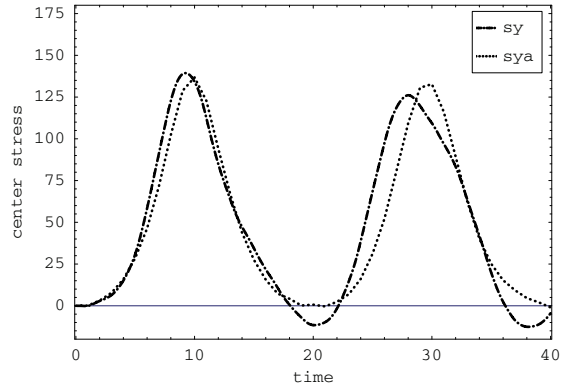


Figure 10. Dimensionless centre stress of the cross-ply (0°/90°/0°/90°) laminate in time (τ). sy: GM results. sya: ABAQUS results.

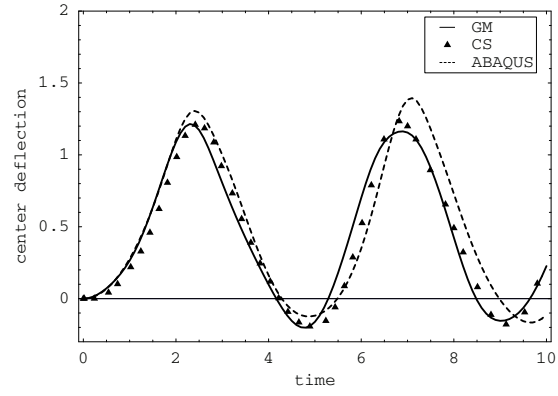


Figure 11. Dimensionless centre deflection ($\bar{w}$) of the angle-ply (15°/–15°/15°/–15°) laminate in time (τ). $\Delta\tau$ – 0·1 CS results are taken from Ref (6).

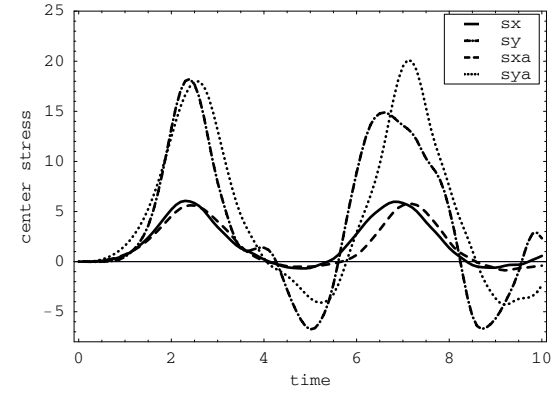


Figure 12: Dimensionless center stress in the x ($\sigma_{xc}\xi^2/E_1$) and y ($\sigma_{yc}\xi^2/E_2$) directions of the angle-ply (15°/–15°/15°/–15°) laminate in time (τ). sx and sy: GM results. sxa and sya: ABAQUS results.

4.2 Nonlinear transient analysis of an unsymmetric cross-ply and an angle-ply laminate

For the analyses of unsymmetric laminates the following material properties, geometric data and related normalised variables are used which are taken from Ref. 6.

https://doi.org/10.1017/S0001924000003912 Published online by Cambridge University Press

$$E_1/E_2 = 25,\ G_{12} = G_{13} = E_2/2,\ G_{23} = E_2/5,$$
$$\nu_{12} = 0.25,\ a = b,\ \xi = a/h = 10,$$
$$q' = \frac{q_o a^4}{E_2 h^4},\ \tau = t\sqrt{\frac{4A_{22}}{I_0 h^2 \xi^2}},\ \Delta\tau = 0.1. \quad \ldots(16)$$

Nonlinear transient analysis of an unsymmetric cross-ply (0°/90°/0°/90°) laminate under SS-1 type boundary condition and a lateral load of $q' = 125$ is performed. Centre deflection versus time plot of the laminate is shown in Fig. 8 as a comparison of the results from the present technique, CS technique[6] and the commercial FEM program ABAQUS. Centre stress values of the cross-ply laminate in the x and y directions are given in Figs 9 and 10, respectively.

In Fig. 11, centre deflection of an angle-ply laminate (15°/–15°/15°/–15°) under CS-1 type boundary condition and a lateral load of $q' = 125$ is presented. Centre stresses of the angle-ply laminate are shown in Fig. 12.

5.0 CONCLUSIONS

An analytical-numerical method is presented to study the nonlinear transient behavior of moderately thick composite plates. Different boundary conditions are considered and the details on how to choose the trial functions are provided. The use of polynomials as trial functions is found to be effective and easy to implement. The results are compared to that of known other approximating methods (finite strips, Chebyshev series and finite elements). A good agreement is seen. It is observed that Galerkin method performs well for the nonlinear transient solution of composite plates. The displacements and stresses can be obtained accurately with a few number of terms. The present solution methodology may be used to solve nonlinear transient response of the moderately thick laminates in an easy and effective way with the help of a symbolic math package.

ACKNOWLEDGEMENTS

This work is partly supported by Istanbul Technical University.

REFERENCES

1. AKAY, H.U. Dynamic large deflection analysis of plates using mixed finite elements, *Computers and Structures*, 1980, **11**, pp 1-11.
2. REDDY, J.N. Geometrically nonlinear transient analysis of laminated composite plates, *AIAA J*, 1983, **21**, (4), pp 621-629.
3. ALI, S.A. and AL-NOURY, S. Nonlinear dynamic response of rectangular plates, *Computers and Structures*, 1986, **22**, (3), pp 433-437.
4. CARRERA, E. and KRAUSE, H. Investigation of non-linear dynamics of multilayered plates accounting for requirements, *Computers and Structures*, 1998, **69**, pp 473-486.
5. CHEN, J., DAWE, D.J. and WANG. S. Nonlinear transient analysis of rectangular composite laminated plates, *Composite Structures*, 2000, **49**, pp 129-139.
6. SHUKLA K.K. and NATH, Y. Non-linear transient analysis of moderately thick laminated composite plates, *J Sound and Vibration*, 2001, **247**, (3), pp 509-526.
7. KIRBY, R.M. and YOSIBASH, Z. Solution of von Kármán dynamic non-linear plate equations using a pseudo-spectral method, *Computer Methods in Applied Mechanics and Engineering*, 2004, **193**, pp 575-599.
8. REDDY, J.N. *Mechanics of Laminated Composite Plates*, CRC Press, New York, USA, 1997.
9. SATHYAMOORTHY, M. *Nonlinear Analysis of Structures*, CRC Press, New York, USA, 1998.
10. QUASI, M.I. Large amplitude response of rectangular plates subjected to transient loads, *Applied Acoustics*, 1994, **42**, pp 267-276.
11. SAADATPOUR, M.M., AZHARI, M. and BRADFORD, M.A. Vibration analysis of simply supported plates of general shape with internal point and line supports using the Galerkin method, *Engineering Structures*, 2000, **22**, pp 1180-1188.
12. NAKATANI, A., SHI, J.W. and KITAGAWA, H. Vibration analysis of fully clamped arbitrarily laminated plate, *Composite Structures*, 2004, **63**, pp 115-122.
13. TANRIÖVER, H. and ŞENOCAK, E. Large deflection analysis of unsymmetrically laminated composite plates: analytical-numerical type approach, *Int J Non-Linear Mechanics*, 2004, **39**, (8), pp 1385-1392.
14. TANRIÖVER, H. and ŞENOCAK, E. Dynamic nonlinear behavior of composite plates, In MALLA, R.B. and MAJI, A. (Eds), Proceedings of ASCE Aerospace Division International Conference on Engineering, Construction and Operations in Challenging Environments, pp 519-525, Houston, TX, USA, 2004.
15. ABAQUS User's Examples and Theory Manual, Version 6.4, 2003.
16. MINDLIN, R.D. Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates, *J Applied Mechanics*, 1951, **18**, pp 31-38.
17. NOVOZHILOV, V.V. *Foundations of The Nonlinear Theory of Elasticity*, Graylock Press, Rochester, New York, USA, 1953.
18. WHITNEY, J.M. *Structural Analysis of Laminated Anizotropic Plates*, Technomic Publ. Co, PA, USA, 1987.
19. NEWMARK, N.M. A method of computation for structural dynamics, *J Engineering Mechanics Division*, 1959, **8**, pp 67-94.
20. WOLFRAM, S. *Mathematica: A System for Doing Mathematics by Computer*, Redwood City, CA, USA, 1988.
21. TANRIÖVER, H. Dynamic Nonlinear Behavior of Composite Plates, Phd Thesis, Istanbul Technical University, Istanbul, Turkey, 2005.

A. Equilibrium Equations

$$\psi = \frac{a}{b},\ \bar{x} = \frac{2x}{a},\ \bar{y} = \frac{2y}{b},\ \bar{q} = \frac{q_o a^4}{hD_{11}},\ \bar{u} = \frac{au^0}{h^2},\ \bar{v} = \frac{bv^0}{h^2},\ \bar{w} = \frac{w}{h},$$
$$\xi = \frac{a}{h},\ \eta = \frac{b}{h},\ \bar{\phi}_x = \phi_x\frac{a}{2h},\ \bar{\phi}_y = \phi_y\frac{b}{2h},$$
$$\bar{B}_{ij} = \frac{B_{ij}h}{D_{11}},\ \bar{D}_{ij} = \frac{D_{ij}}{D_{11}}\quad (i,j = 1,2,6) \quad \ldots(17)$$
$$\bar{A}_{ij} = \frac{A_{ij}h^2}{D_{11}}\quad (i,j = 1,2,6; i,j = 4,5),$$
$$\bar{t} = \frac{t}{t_0}\quad (t_0 = \sqrt{\frac{a^2b^2h\rho^{(1)}}{4D_{11}}}), \bar{I}_0 = \frac{I_0}{\rho^{(1)}h}, \bar{I}_1 = \frac{I_1}{\rho^{(1)}h^2}, \bar{I}_2 = \frac{I_2}{\rho^{(1)}h^3}.$$

$$\bar{R}_1(\bar{u},\bar{v},\bar{w}) = \bar{R}_1^{(s)} + \bar{R}_1^{(d)} = 0,$$
$$\bar{R}_1^{(s)} = \bar{A}_{11}(\bar{u}_{,\bar{x}\bar{x}} + 2\bar{w}_{,\bar{x}}\bar{w}_{,\bar{x}\bar{x}}) + \bar{A}_{12}\psi^2(\bar{v}_{,\bar{x}\bar{y}} + 2\bar{w}_{,\bar{y}}\bar{w}_{,\bar{x}\bar{y}})$$
$$+\bar{A}_{16}\psi(2\bar{u}_{,\bar{x}\bar{y}} + \bar{v}_{,\bar{x}\bar{x}} + 4\bar{w}_{,\bar{x}}\bar{w}_{,\bar{x}\bar{y}} + 2\bar{w}_{,\bar{y}}\bar{w}_{,\bar{x}\bar{x}})$$
$$+\bar{A}_{26}\psi^3(\bar{v}_{,\bar{y}\bar{y}} + 2\bar{w}_{,\bar{y}}\bar{w}_{,\bar{y}\bar{y}}) + \bar{A}_{66}\psi^2(\bar{u}_{,\bar{y}\bar{y}} + \bar{v}_{,\bar{x}\bar{y}} + 2(\bar{w}_{,\bar{y}}\bar{w}_{,\bar{x}\bar{y}} + \bar{w}_{,\bar{x}}\bar{w}_{,\bar{y}\bar{y}}))$$
$$+2\bar{B}_{11}\bar{\phi}_{x,\bar{x}\bar{x}} + 2\bar{B}_{12}\psi^2\bar{\phi}_{y,\bar{x}\bar{y}} + 2\bar{B}_{16}\psi(2\bar{\phi}_{x,\bar{x}\bar{y}} + \bar{\phi}_{y,\bar{x}\bar{x}})$$
$$+2\bar{B}_{26}\psi^3\bar{\phi}_{y,\bar{y}\bar{y}} + 2\bar{B}_{66}\psi^2(\bar{\phi}_{x,\bar{y}\bar{y}} + \bar{\phi}_{y,\bar{x}\bar{y}}),$$
$$\bar{R}_1^{(d)} = -\frac{1}{\eta^2}(\bar{I}_0\bar{u}_{,\bar{t}\bar{t}} + 2\bar{I}_1\bar{\phi}_{x,\bar{t}\bar{t}}). \quad \ldots(18)$$

$$\bar{R}_2(\bar{u},\bar{v},\bar{w}) = \bar{R}_2^{(s)} + \bar{R}_2^{(d)} = 0,$$
$$\bar{R}_2^{(s)} = \bar{A}_{22}(\bar{v}_{,\bar{y}\bar{y}} + 2\bar{w}_{,\bar{y}}\bar{w}_{,\bar{y}\bar{y}}) + \frac{\bar{A}_{12}}{\psi^2}(\bar{u}_{,\bar{x}\bar{y}} + 2\bar{w}_{,\bar{x}}\bar{w}_{,\bar{x}\bar{y}})$$
$$+\frac{\bar{A}_{16}}{\psi^3}(\bar{u}_{,\bar{x}\bar{x}} + 2\bar{w}_{,\bar{x}}\bar{w}_{,\bar{x}\bar{x}}) + \frac{\bar{A}_{26}}{\psi}(\bar{u}_{,\bar{y}\bar{y}} + 2\bar{v}_{,\bar{x}\bar{y}} + 4\bar{w}_{,\bar{y}}\bar{w}_{,\bar{x}\bar{y}} + 2\bar{w}_{,\bar{x}}\bar{w}_{,\bar{y}\bar{y}})$$
$$+\frac{\bar{A}_{66}}{\psi^2}(\bar{v}_{,\bar{x}\bar{x}} + \bar{u}_{,\bar{x}\bar{y}} + 2(\bar{w}_{,\bar{y}}\bar{w}_{,\bar{x}\bar{x}} + \bar{w}_{,\bar{x}}\bar{w}_{,\bar{x}\bar{y}}))$$
$$+2\bar{B}_{22}\bar{\phi}_{y,\bar{y}\bar{y}} + \frac{2\bar{B}_{12}}{\psi^2}\bar{\phi}_{x,\bar{x}\bar{y}} + \frac{2\bar{B}_{16}}{\psi^3}\bar{\phi}_{x,\bar{x}\bar{x}}$$
$$+\frac{2\bar{B}_{26}}{\psi}(\bar{\phi}_{x,\bar{y}\bar{y}} + 2\bar{\phi}_{y,\bar{x}\bar{y}}) + \frac{2\bar{B}_{66}}{\psi^2}(\bar{\phi}_{x,\bar{x}\bar{y}} + \bar{\phi}_{y,\bar{x}\bar{x}}),$$
$$\bar{R}_2^{(d)} = -\frac{1}{\xi^2}(\bar{I}_0\bar{v}_{,\bar{t}\bar{t}} + 2\bar{I}_1\bar{\phi}_{y,\bar{t}\bar{t}}). \quad \ldots(19)$$

https://doi.org/10.1017/S0001924000003912 Published online by Cambridge University Press

$$\bar{R}_3(\bar{u},\bar{v},\bar{w}) = \bar{R}_3^{(s)} + \bar{R}_3^{(d)} = 0,$$
$$\bar{R}_3^{(s)} = \frac{\bar{A}_{11}}{2\psi^2}(\bar{u}_{,\bar{x}}\bar{w}_{,\bar{x}\bar{x}} + \bar{u}_{,\bar{x}\bar{x}}\bar{w}_{,\bar{x}} + 3\bar{w}_{,\bar{x}\bar{x}}\bar{w}_{,\bar{x}}^2)$$
$$+\frac{\bar{A}_{12}}{2}(\bar{u}_{,\bar{x}}\bar{w}_{,\bar{y}\bar{y}} + \bar{w}_{,\bar{y}\bar{y}}\bar{w}_{,\bar{x}}^2 + \bar{u}_{,\bar{x}\bar{y}}\bar{w}_{,\bar{y}} + \bar{v}_{,\bar{x}\bar{y}}\bar{w}_{,\bar{x}} + 4\bar{w}_{,\bar{x}}\bar{w}_{,\bar{y}}\bar{w}_{,\bar{x}\bar{y}}$$
$$+\bar{v}_{,\bar{y}}\bar{w}_{,\bar{x}\bar{x}} + \bar{w}_{,\bar{x}\bar{x}}\bar{w}_{,\bar{y}}^2) + \frac{\bar{A}_{22}\psi^2}{2}(\bar{v}_{,\bar{y}\bar{y}}\bar{w}_{,\bar{y}} + \bar{v}_{,\bar{y}}\bar{w}_{,\bar{y}\bar{y}} + 3\bar{w}_{,\bar{y}\bar{y}}\bar{w}_{,\bar{y}}^2)$$
$$+\frac{\bar{A}_{16}}{\psi}(\bar{u}_{,\bar{x}\bar{y}}\bar{w}_{,\bar{x}} + \bar{u}_{,\bar{x}}\bar{w}_{,\bar{x}\bar{y}} + 3\bar{w}_{,\bar{x}}(\bar{w}_{,\bar{x}}\bar{w}_{,\bar{x}\bar{y}} + \bar{w}_{,\bar{y}}\bar{w}_{,\bar{x}\bar{x}})$$
$$+\frac{1}{2}(\bar{w}_{,\bar{y}}\bar{u}_{,\bar{x}\bar{x}} + \bar{v}_{,\bar{x}\bar{x}}\bar{w}_{,\bar{x}} + \bar{u}_{,\bar{y}}\bar{w}_{,\bar{x}\bar{x}} + \bar{v}_{,\bar{x}}\bar{w}_{,\bar{x}\bar{x}}))$$
$$+\bar{A}_{26}\psi(\bar{v}_{,\bar{x}\bar{y}}\bar{w}_{,\bar{y}} + \bar{v}_{,\bar{y}}\bar{w}_{,\bar{x}\bar{y}} + 3\bar{w}_{,\bar{y}}(\bar{w}_{,\bar{x}}\bar{w}_{,\bar{y}\bar{y}} + \bar{w}_{,\bar{y}}\bar{w}_{,\bar{x}\bar{y}})$$
$$+\frac{1}{2}(\bar{u}_{,\bar{y}\bar{y}}\bar{w}_{,\bar{y}} + \bar{u}_{,\bar{y}}\bar{w}_{,\bar{y}\bar{y}} + \bar{v}_{,\bar{x}}\bar{w}_{,\bar{y}\bar{y}} + \bar{v}_{,\bar{y}\bar{y}}\bar{w}_{,\bar{x}}))$$
$$+\bar{A}_{66}(\frac{1}{2}\bar{u}_{,\bar{y}\bar{y}}\bar{w}_{,\bar{x}} + \bar{w}_{,\bar{y}\bar{y}}\bar{w}_{,\bar{x}}^2 + \frac{1}{2}\bar{u}_{,\bar{x}\bar{y}}\bar{w}_{,\bar{y}} + \frac{1}{2}\bar{v}_{,\bar{x}\bar{y}}\bar{w}_{,\bar{x}}$$
$$+\bar{u}_{,\bar{y}}\bar{w}_{,\bar{x}\bar{y}} + \bar{v}_{,\bar{x}}\bar{w}_{,\bar{x}\bar{y}} + 4\bar{w}_{,\bar{y}}\bar{w}_{,\bar{x}}\bar{w}_{,\bar{x}\bar{y}} + \frac{1}{2}\bar{v}_{,\bar{x}\bar{x}}\bar{w}_{,\bar{y}} + \bar{w}_{,\bar{x}\bar{x}}\bar{w}_{,\bar{y}}^2)$$
$$+\bar{A}_{44}\frac{K\xi^2}{4}(\bar{\phi}_{y,\bar{y}} + \bar{w}_{,\bar{y}\bar{y}}) + \bar{A}_{45}\frac{K\xi\eta}{4}(\bar{\phi}_{x,\bar{y}} + \bar{\phi}_{y,\bar{x}} + 2\bar{w}_{,\bar{x}\bar{y}})$$
$$+\bar{A}_{55}\frac{K\eta^2}{4}(\bar{\phi}_{x,\bar{x}} + \bar{w}_{,\bar{x}\bar{x}}) + \frac{\bar{B}_{11}}{\psi^2}(\bar{\phi}_{x,\bar{x}}\bar{w}_{,\bar{x}\bar{x}} + \bar{\phi}_{x,\bar{x}\bar{x}}\bar{w}_{,\bar{x}})$$
$$+\bar{B}_{12}(\bar{\phi}_{x,\bar{x}}\bar{w}_{,\bar{y}\bar{y}} + \bar{\phi}_{x,\bar{x}\bar{y}}\bar{w}_{,\bar{y}} + \bar{\phi}_{x,\bar{x}\bar{y}}\bar{w}_{,\bar{x}} + \bar{\phi}_{y,\bar{y}}\bar{w}_{,\bar{x}\bar{x}})$$
$$+\bar{B}_{22}\psi^2(\bar{\phi}_{y,\bar{y}}\bar{w}_{,\bar{y}\bar{y}} + \bar{\phi}_{y,\bar{y}\bar{y}}\bar{w}_{,\bar{y}}) + \frac{\bar{B}_{16}}{\psi}(2\bar{\phi}_{x,\bar{x}}\bar{w}_{,\bar{x}\bar{y}} + 2\bar{\phi}_{x,\bar{x}\bar{y}}\bar{w}_{,\bar{x}}$$
$$+\bar{\phi}_{x,\bar{y}}\bar{w}_{,\bar{x}\bar{x}} + \bar{\phi}_{y,\bar{x}}\bar{w}_{,\bar{x}\bar{x}} + \bar{\phi}_{x,\bar{x}\bar{x}}\bar{w}_{,\bar{y}} + \bar{\phi}_{y,\bar{x}\bar{x}}\bar{w}_{,\bar{x}})$$
$$+\bar{B}_{26}\psi(\bar{\phi}_{x,\bar{y}}\bar{w}_{,\bar{y}\bar{y}} + \bar{\phi}_{x,\bar{y}\bar{y}}\bar{w}_{,\bar{y}} + \bar{\phi}_{y,\bar{y}\bar{y}}\bar{w}_{,\bar{x}} + \bar{\phi}_{y,\bar{x}}\bar{w}_{,\bar{y}\bar{y}}$$
$$+2\bar{\phi}_{y,\bar{y}}\bar{w}_{,\bar{x}\bar{y}} + 2\bar{\phi}_{y,\bar{x}\bar{y}}\bar{w}_{,\bar{y}}) + \bar{B}_{66}(\bar{\phi}_{x,\bar{y}\bar{y}}\bar{w}_{,\bar{x}} + 2\bar{\phi}_{x,\bar{y}}\bar{w}_{,\bar{x}\bar{y}} + 2\bar{\phi}_{y,\bar{x}}\bar{w}_{,\bar{x}\bar{y}}$$
$$+\bar{\phi}_{x,\bar{x}\bar{y}}\bar{w}_{,\bar{y}} + \bar{\phi}_{y,\bar{x}\bar{y}}\bar{w}_{,\bar{x}} + \bar{\phi}_{y,\bar{x}\bar{x}}\bar{w}_{,\bar{y}}) + \frac{\bar{q}}{16\psi^2},$$
$$\bar{R}_3^{(d)} = -\frac{\bar{I}_0}{4}\bar{w}_{,\bar{t}\bar{t}}.$$

. . . (20)

$$\bar{R}_4(\bar{u},\bar{v},\bar{w}) = \bar{R}_4^{(s)} + \bar{R}_4^{(d)} = 0,$$
$$\bar{R}_4^{(s)} = -\bar{A}_{45}\frac{K\psi\xi^2}{2}(\bar{\phi}_y + \bar{w}_{,\bar{y}}) - \bar{A}_{55}\frac{K\xi^2}{2}(\bar{\phi}_x + \bar{w}_{,\bar{x}})$$
$$+\bar{B}_{11}(\bar{u}_{,\bar{x}\bar{x}} + 2\bar{w}_{,\bar{x}}\bar{w}_{,\bar{x}\bar{x}}) + \bar{B}_{12}\psi^2(\bar{v}_{,\bar{x}\bar{y}} + 2\bar{w}_{,\bar{y}}\bar{w}_{,\bar{x}\bar{y}})$$
$$+\bar{B}_{16}\psi(2\bar{u}_{,\bar{x}\bar{y}} + \bar{v}_{,\bar{x}\bar{x}} + 4\bar{w}_{,\bar{x}}\bar{w}_{,\bar{x}\bar{y}} + 2\bar{w}_{,\bar{y}}\bar{w}_{,\bar{x}\bar{x}}) + \bar{B}_{26}\psi^3(\bar{v}_{,\bar{y}\bar{y}} + 2\bar{w}_{,\bar{y}}\bar{w}_{,\bar{y}\bar{y}})$$
$$+\bar{B}_{66}\psi^2(\bar{u}_{,\bar{y}\bar{y}} + \bar{v}_{,\bar{x}\bar{y}} + 2\bar{w}_{,\bar{x}}\bar{w}_{,\bar{y}\bar{y}} + 2\bar{w}_{,\bar{y}}\bar{w}_{,\bar{x}\bar{y}})$$
$$+2\bar{D}_{11}\bar{\phi}_{x,\bar{x}\bar{x}} + 2\bar{D}_{12}\psi^2\bar{\phi}_{y,\bar{x}\bar{y}} + 2\bar{D}_{16}\psi(2\bar{\phi}_{x,\bar{x}\bar{y}} + \bar{\phi}_{y,\bar{x}\bar{x}})$$
$$+2\bar{D}_{26}\psi^3\bar{\phi}_{y,\bar{y}\bar{y}} + 2\bar{D}_{66}\psi^2(\bar{\phi}_{x,\bar{y}\bar{y}} + \bar{\phi}_{y,\bar{x}\bar{y}}),$$
$$\bar{R}_4^{(d)} = -\frac{\bar{I}_1}{\eta^2}\bar{u}_{,\bar{t}\bar{t}} - \frac{2\bar{I}_2}{\eta^2}\bar{\phi}_{x,\bar{t}\bar{t}}.$$

. . . (21)

$$\bar{R}_5(\bar{u},\bar{v},\bar{w}) = \bar{R}_5^{(s)} + \bar{R}_5^{(d)} = 0,$$
$$\bar{R}_5^{(s)} = -\bar{A}_{44}\frac{K\eta^2}{2}(\bar{\phi}_y + \bar{w}_{,\bar{y}}) - \bar{A}_{45}\frac{K\eta^2}{2\psi}(\bar{\phi}_x + \bar{w}_{,\bar{x}})$$
$$+\bar{B}_{22}(\bar{v}_{,\bar{y}\bar{y}} + 2\bar{w}_{,\bar{y}}\bar{w}_{,\bar{y}\bar{y}}) + \frac{\bar{B}_{12}}{\psi^2}(\bar{u}_{,\bar{x}\bar{y}} + 2\bar{w}_{,\bar{x}}\bar{w}_{,\bar{x}\bar{y}})$$
$$+\frac{\bar{B}_{16}}{\psi^3}(\bar{u}_{,\bar{x}\bar{x}} + 2\bar{w}_{,\bar{x}}\bar{w}_{,\bar{x}\bar{x}}) + \frac{\bar{B}_{26}}{\psi}(\bar{u}_{,\bar{y}\bar{y}} + 2\bar{v}_{,\bar{x}\bar{y}} + 2\bar{w}_{,\bar{x}}\bar{w}_{,\bar{y}\bar{y}} + 4\bar{w}_{,\bar{y}}\bar{w}_{,\bar{x}\bar{y}})$$
$$+\frac{\bar{B}_{66}}{\psi^2}(\bar{u}_{,\bar{x}\bar{y}} + \bar{v}_{,\bar{x}\bar{x}} + 2\bar{w}_{,\bar{x}}\bar{w}_{,\bar{x}\bar{y}} + 2\bar{w}_{,\bar{y}}\bar{w}_{,\bar{x}\bar{x}})$$
$$+2\bar{D}_{22}\bar{\phi}_{y,\bar{y}\bar{y}} + \frac{2\bar{D}_{12}}{\psi^2}\bar{\phi}_{x,\bar{x}\bar{y}} + \frac{2\bar{D}_{16}}{\psi^3}\bar{\phi}_{x,\bar{x}\bar{x}} + \frac{2\bar{D}_{26}}{\psi}(\bar{\phi}_{x,\bar{y}\bar{y}} + 2\bar{\phi}_{y,\bar{x}\bar{y}})$$
$$+\frac{2\bar{D}_{66}}{\psi^2}(\bar{\phi}_{x,\bar{x}\bar{y}} + \bar{\phi}_{y,\bar{x}\bar{x}}),$$
$$\bar{R}_5^{(d)} = -\frac{\bar{I}_1}{\xi^2}\bar{v}_{,\bar{t}\bar{t}} - \frac{2\bar{I}_2}{\xi^2}\bar{\phi}_{y,\bar{t}\bar{t}}.$$

. . . (22)

B. Boundary equations

$$\bar{N}_x = \bar{A}_{11}(\bar{u}_{,\bar{x}} + \bar{w}_{,\bar{x}}^2) + \bar{A}_{12}\psi^2(\bar{v}_{,\bar{y}} + \bar{w}_{,\bar{y}}^2) + \bar{A}_{16}\psi(\bar{u}_{,\bar{y}} + \bar{v}_{,\bar{x}} + 2\bar{w}_{,\bar{x}}\bar{w}_{,\bar{y}})$$
$$+2(\bar{B}_{11}\bar{\phi}_{x,\bar{x}} + \bar{B}_{12}\psi^2\bar{\phi}_{y,\bar{y}} + \bar{B}_{16}\psi(\bar{\phi}_{x,\bar{y}} + \bar{\phi}_{y,\bar{x}}))$$

. . . (23)

$$\bar{N}_{xy}^u = \bar{A}_{66}\psi^2(\bar{u}_{,\bar{y}} + \bar{v}_{,\bar{x}} + 2\bar{w}_{,\bar{x}}\bar{w}_{,\bar{y}}) + \bar{A}_{16}\psi(\bar{u}_{,\bar{x}} + \bar{w}_{,\bar{x}}^2) + \bar{A}_{26}\psi^3(\bar{v}_{,\bar{y}} + \bar{w}_{,\bar{y}}^2)$$
$$+2\psi(\bar{B}_{16}\bar{\phi}_{x,\bar{x}} + \bar{B}_{26}\psi^2\bar{\phi}_{y,\bar{y}} + \bar{B}_{66}\psi(\bar{\phi}_{x,\bar{y}} + \bar{\phi}_{y,\bar{x}}))$$

. . . (24)

$$\bar{N}_y = \bar{A}_{22}(\bar{v}_{,\bar{y}} + \bar{w}_{,\bar{y}}^2) + \frac{\bar{A}_{12}}{\psi^2}(\bar{u}_{,\bar{x}} + \bar{w}_{,\bar{x}}^2) + \frac{\bar{A}_{26}}{\psi}(\bar{u}_{,\bar{y}} + \bar{v}_{,\bar{x}} + 2\bar{w}_{,\bar{x}}\bar{w}_{,\bar{y}})$$
$$+2(\bar{B}_{22}\bar{\phi}_{y,\bar{y}} + \frac{\bar{B}_{12}}{\psi^2}\bar{\phi}_{x,\bar{x}} + \frac{\bar{B}_{26}}{\psi}(\bar{\phi}_{x,\bar{y}} + \bar{\phi}_{y,\bar{x}}))$$

. . . (25)

$$\bar{N}_{xy}^v = \frac{\bar{A}_{66}}{\psi^2}(\bar{u}_{,\bar{y}} + \bar{v}_{,\bar{x}} + 2\bar{w}_{,\bar{x}}\bar{w}_{,\bar{y}}) + \frac{\bar{A}_{16}}{\psi^3}(\bar{u}_{,\bar{x}} + \bar{w}_{,\bar{x}}^2)$$
$$+\frac{\bar{A}_{26}}{\psi}(\bar{v}_{,\bar{y}} + \bar{w}_{,\bar{y}}^2) + 2(\frac{\bar{B}_{16}}{\psi^3}\bar{\phi}_{x,\bar{x}} + \frac{\bar{B}_{26}}{\psi}\bar{\phi}_{y,\bar{y}} + \frac{\bar{B}_{66}}{\psi^2}(\bar{\phi}_{x,\bar{y}} + \bar{\phi}_{y,\bar{x}}))$$

. . . (26)

$$\bar{M}_x = \bar{D}_{11}\bar{\phi}_{x,\bar{x}} + \bar{D}_{12}\psi^2\bar{\phi}_{y,\bar{y}} + \bar{D}_{16}\psi(\bar{\phi}_{x,\bar{y}} + \bar{\phi}_{y,\bar{x}}) + \frac{\bar{B}_{11}}{2}(\bar{u}_{,\bar{x}} + \bar{w}_{,\bar{x}}^2)$$
$$+\frac{\bar{B}_{12}\psi^2}{2}(\bar{v}_{,\bar{y}} + \bar{w}_{,\bar{y}}^2) + \frac{\bar{B}_{16}\psi}{2}(\bar{u}_{,\bar{y}} + \bar{v}_{,\bar{x}} + 2\bar{w}_{,\bar{x}}\bar{w}_{,\bar{y}})$$

. . . (27)

$$\bar{M}_y = \bar{D}_{22}\bar{\phi}_{y,\bar{y}} + \frac{\bar{D}_{12}}{\psi^2}\bar{\phi}_{x,\bar{x}} + \frac{\bar{D}_{26}}{\psi}(\bar{\phi}_{x,\bar{y}} + \bar{\phi}_{y,\bar{x}}) + \frac{\bar{B}_{22}}{2}(\bar{v}_{,\bar{y}} + \bar{w}_{,\bar{y}}^2)$$
$$+\frac{\bar{B}_{12}}{2\psi^2}(\bar{u}_{,\bar{x}} + \bar{w}_{,\bar{x}}^2) + \frac{\bar{B}_{26}}{2\psi}(\bar{u}_{,\bar{y}} + \bar{v}_{,\bar{x}} + 2\bar{w}_{,\bar{x}}\bar{w}_{,\bar{y}})$$

. . . (28)

C. Definitions

Definitions are introduced for the Equation (14).

$$\mathbf{F} = [\mathbf{F}^{(1)} \quad \mathbf{F}^{(2)} \quad \mathbf{F}^{(3)} \quad \mathbf{F}^{(4)} \quad \mathbf{F}^{(5)}]^{\mathrm{T}},$$
$$\mathbf{U} = [\mathbf{a} \quad \mathbf{b} \quad \mathbf{c} \quad \mathbf{d} \quad \mathbf{e}]^{\mathrm{T}},$$

. . . (29)

https://doi.org/10.1017/S0001924000003912 Published online by Cambridge University Press

$$\mathrm{M} = \begin{bmatrix} \mathbf{M^{(11)}} & [\mathbf{0}] & [\mathbf{0}] & \mathbf{M^{(14)}} & [\mathbf{0}] \\ [\mathbf{0}] & \mathbf{M^{(22)}} & [\mathbf{0}] & [\mathbf{0}] & \mathbf{M^{(25)}} \\ [\mathbf{0}] & [\mathbf{0}] & \mathbf{M^{(33)}} & [\mathbf{0}] & [\mathbf{0}] \\ \mathbf{M^{(41)}} & [\mathbf{0}] & [\mathbf{0}] & \mathbf{M^{(44)}} & [\mathbf{0}] \\ [\mathbf{0}] & \mathbf{M^{(52)}} & [\mathbf{0}] & [\mathbf{0}] & \mathbf{M^{(55)}} \end{bmatrix}, \qquad \ldots (30)$$

$$\begin{aligned}
\mathbf{F^{(1)}}_i &= \int_{-1}^{+1}\int_{-1}^{+1} \mathrm{U}_i\, \bar{R}_1^{(s)}\, \mathrm{d}\bar{x}\mathrm{d}\bar{y} \\
&\quad - \int_{-1}^{+1} \mathrm{U}_i\, \bar{N}_x|_{\bar{x}=\pm 1}\mathrm{d}\bar{y} - \int_{-1}^{+1} \mathrm{U}_i\, \bar{N}^u_{xy}|_{\bar{y}=\pm 1}\mathrm{d}\bar{x}, \\
\mathbf{F^{(2)}}_i &= \int_{-1}^{+1}\int_{-1}^{+1} \mathrm{V}_i\, \bar{R}_2^{(s)}\, \mathrm{d}\bar{x}\mathrm{d}\bar{y} \\
&\quad - \int_{-1}^{+1} \mathrm{V}_i\, \bar{N}_y|_{\bar{y}=\pm 1}\mathrm{d}\bar{x} - \int_{-1}^{+1} \mathrm{V}_i\, \bar{N}^v_{xy}|_{\bar{x}=\pm 1}\mathrm{d}\bar{y}, \\
\mathbf{F^{(3)}}_i &= \int_{-1}^{+1}\int_{-1}^{+1} \mathrm{W}_i\, \bar{R}_3^{(s)}\, \mathrm{d}\bar{x}\mathrm{d}\bar{y}, \\
\mathbf{F^{(4)}}_i &= \int_{-1}^{+1}\int_{-1}^{+1} \mathrm{S}_i\, \bar{R}_4^{(s)}\, \mathrm{d}\bar{x}\mathrm{d}\bar{y} - \int_{-1}^{+1} \mathrm{S}_i\, \bar{M}_x|_{\bar{x}=\pm 1}\mathrm{d}\bar{y}, \\
\mathbf{F^{(5)}}_i &= \int_{-1}^{+1}\int_{-1}^{+1} \mathrm{T}_i\, \bar{R}_5^{(s)}\, \mathrm{d}\bar{x}\mathrm{d}\bar{y} - \int_{-1}^{+1} \mathrm{T}_i\, \bar{M}_y|_{\bar{y}=\pm 1}\mathrm{d}\bar{x},
\end{aligned} \qquad \ldots (31)$$

$$\begin{aligned}
\mathbf{M^{(11)}}_{ij} &= \tfrac{1}{\eta^2}\int_{-1}^{+1}\int_{-1}^{+1} \bar{I}_0\, \mathrm{U}_i\, \mathrm{U}_j\, \mathrm{d}\bar{x}\mathrm{d}\bar{y}, \\
\mathbf{M^{(14)}}_{ij} &= \tfrac{2}{\eta^2}\int_{-1}^{+1}\int_{-1}^{+1} \bar{I}_1\, \mathrm{U}_i\, \mathrm{S}_j\, \mathrm{d}\bar{x}\mathrm{d}\bar{y}, \\
\mathbf{M^{(22)}}_{ij} &= \tfrac{1}{\xi^2}\int_{-1}^{+1}\int_{-1}^{+1} \bar{I}_0\, \mathrm{V}_i\, \mathrm{V}_j\, \mathrm{d}\bar{x}\mathrm{d}\bar{y}, \\
\mathbf{M^{(25)}}_{ij} &= \tfrac{2}{\xi^2}\int_{-1}^{+1}\int_{-1}^{+1} \bar{I}_1\, \mathrm{V}_i\, \mathrm{T}_j\, \mathrm{d}\bar{x}\mathrm{d}\bar{y}, \\
\mathbf{M^{(33)}}_{ij} &= \tfrac{1}{4}\int_{-1}^{+1}\int_{-1}^{+1} \bar{I}_0\, \mathrm{W}_i\, \mathrm{W}_j\, \mathrm{d}\bar{x}\mathrm{d}\bar{y}, \\
\mathbf{M^{(41)}}_{ij} &= \tfrac{1}{\eta^2}\int_{-1}^{+1}\int_{-1}^{+1} \bar{I}_1\, \mathrm{S}_i\, \mathrm{U}_j\, \mathrm{d}\bar{x}\mathrm{d}\bar{y}, \\
\mathbf{M^{(44)}}_{ij} &= \tfrac{2}{\eta^2}\int_{-1}^{+1}\int_{-1}^{+1} \bar{I}_2\, \mathrm{S}_i\, \mathrm{S}_j\, \mathrm{d}\bar{x}\mathrm{d}\bar{y}, \\
\mathbf{M^{(52)}}_{ij} &= \tfrac{1}{\xi^2}\int_{-1}^{+1}\int_{-1}^{+1} \bar{I}_1\, \mathrm{T}_i\, \mathrm{V}_j\, \mathrm{d}\bar{x}\mathrm{d}\bar{y}, \\
\mathbf{M^{(55)}}_{ij} &= \tfrac{2}{\xi^2}\int_{-1}^{+1}\int_{-1}^{+1} \bar{I}_2\, \mathrm{T}_i\, \mathrm{T}_j\, \mathrm{d}\bar{x}\mathrm{d}\bar{y}.
\end{aligned} \qquad \ldots (32)$$

https://doi.org/10.1017/S0001924000003912 Published online by Cambridge University Press