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Mixing in High-Dimensional Expanders

ORI PARZANCHEVSKI

School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA
(e-mail: parzan@ias.edu)

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We establish a generalization of the Expander Mixing Lemma for arbitrary (finite) simplicial complexes. The original lemma states that concentration of the Laplace spectrum of a graph implies combinatorial expansion (which is also referred to as *mixing*, or *pseudo-randomness*). Recently, an analogue of this lemma was proved for simplicial complexes of arbitrary dimension, provided that the skeleton of the complex is complete. More precisely, it was shown that a concentrated spectrum of the simplicial Hodge Laplacian implies a similar type of pseudo-randomness as in graphs. In this paper we remove the assumption of a complete skeleton, showing that simultaneous concentration of the Laplace spectra in all dimensions implies pseudo-randomness in any complex. We discuss various applications and present some open questions.

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1. Introduction

The *spectral gap* of a finite graph $G = (V, E)$ is the smallest non-trivial eigenvalue of its Laplace operator. The so-called *discrete Cheeger inequalities* [2, 4, 10, 41] relate the spectral gap to expansion in the graph: if the spectral gap is large, then for any partition $V = A \cup B$ there is a large number of edges connecting a vertex in A with a vertex in B. Nevertheless, a large spectral gap does not suffice to control the number of edges between any two sets of vertices. For example, there exist 'bipartite expanders' (see *e.g.* [29, 31]): regular graphs with a large spectral gap, which are bipartite, so that they contain $A, B \subseteq V$ of size $|V|/4$ with no edges between them. The *Expander Mixing Lemma* remedies this inconvenience, using not only the spectral gap but also the maximal eigenvalue of the Laplacian.

Theorem 1.1 (Expander Mixing Lemma [3, 5, 16]). *Let $G = (V, E)$ be a graph on n vertices. If the non-trivial spectrum of the Laplacian is contained within $[k(1-\varepsilon), k(1+\varepsilon)]$,*

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then for any two sets of vertices A, B one has

$$\left| |E(A,B)| - \frac{k}{n}|A||B| \right| \leqslant \varepsilon k \sqrt{|A||B|},$$

where $E(A,B)$ are the edges with one endpoint in A and the other in B.

If k is the average degree of a vertex in G, then $(k/n)|A||B|$ is about the expected size of $|E(A,B)|$ (the exact value is $(k/(n-1))|A||B|$). Thus, the lemma states that a concentrated spectrum indicates pseudo-random behaviour. In light of the Expander Mixing Lemma, we call a graph whose non-trivial Laplace spectrum is contained in $[k(1-\varepsilon), k(1+\varepsilon)]$ a (k,ε)-*expander*.†

In [37] a generalization of the Expander Mixing Lemma is established for finite simplicial complexes of arbitrary dimension, assuming that they have a complete skeleton, namely, that every possible cell of dimension smaller than the maximal one is in the complex. The Laplace operator studied in [37] and in the current paper originates in Eckmann's work [12]. It is a natural analogue of the Hodge Laplace operator in Riemannian geometry, and it has been studied in several prominent works [1, 9, 11, 14, 17, 25, 26, 43], sometimes under the name *combinatorial Laplacian*. For a complex of dimension d, Eckmann defines d Laplace operators (see Section 2), with the jth one acting on the cells of dimension j. We say that X is a (j,k,ε)-*expander* if the non-trivial spectrum of the jth Laplacian is contained within $[k(1-\varepsilon), k(1+\varepsilon)]$ (see Section 2.1 for the precise definition).

Theorem 1.2 ([37]). *Let X be a d-dimensional complex on n vertices with a complete skeleton, which is a $(d-1,k,\varepsilon)$-expander. For any pairwise disjoint $A_0, \ldots, A_d \subseteq V$,*

$$\left| |F(A_0,\ldots,A_d)| - \frac{k}{n}|A_0| \cdot \ldots \cdot |A_d| \right| \leqslant \varepsilon k (|A_0| \cdot \ldots \cdot |A_d|)^{d/(d+1)},$$

where $F(A_0,\ldots,A_d)$ is the set of d-cells with one vertex in each A_i.

In this paper we prove a mixing lemma for arbitrary (finite) complexes. Our main result is the following theorem.

Theorem 1.3. *If a d-dimensional complex X is a (j,k_j,ε_j)-expander for $0 \leqslant j \leqslant d-1$, and $A_0, \ldots, A_d$ are pairwise disjoint sets of vertices in X, then*

$$\left| |F(A_0,\ldots,A_d)| - \frac{k_0 \cdots k_{d-1}}{n^d}|A_0| \cdot \ldots \cdot |A_d| \right| \leqslant c_d k_0 \cdots k_{d-1}(\varepsilon_0 + \cdots + \varepsilon_{d-1}) \max |A_i|,$$

where c_d depends only on d.

In order to understand $F(A_0,\ldots,A_d)$ in the case of general complexes, we study a more general counting problem.

† In [42] this is referred to as a 'two-sided (k,ε)-expander', as the spectrum is bounded on both sides.

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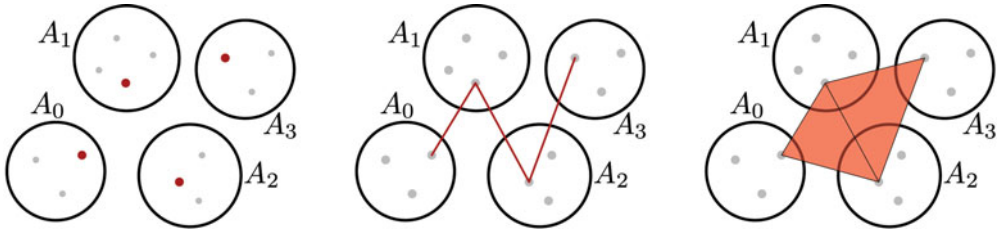


Figure 1. A 0-gallery, a 1-gallery and a 2-gallery in $A_0,\dots,A_3$.

Definition. Given $A_0,\dots,A_\ell \subseteq V$ and $j \leqslant \ell$, a *j-gallery in $A_0,\dots,A_\ell$* is a sequence of j-cells $\sigma_0,\dots,\sigma_{\ell-j} \in X^j$ such that σ_i is in $F(A_i,\dots,A_{i+j})$, and σ_i and σ_{i+1} intersect in a $(j-1)$-cell (which must lie in $F(A_{i+1},\dots,A_{i+j})$). We denote the set of j-galleries in $A_0,\dots,A_\ell$ by $F^j(A_0,\dots,A_\ell)$.

At the heart of our analysis is the next lemma, which estimates the size of $F^{j+1}(A_0,\dots,A_\ell)$ in terms of that of $F^j(A_0,\dots,A_\ell)$. Since $F(A_0,\dots,A_\ell)$ is $F^\ell(A_0,\dots,A_\ell)$ and $F^0(A_0,\dots,A_\ell) = A_0 \times \cdots \times A_\ell$, repeatedly applying the lemma allows us to estimate $|F(A_0,\dots,A_d)|$ in terms of $|F^0(A_0,\dots,A_d)| = |A_0| \cdot \ldots \cdot |A_d|$.

Lemma 1.4 (Descent Lemma). *Let $A_0,\dots,A_\ell$ be sets of vertices in X, such that each $j+1$ tuple $A_i, A_{i+1},\dots,A_{i+j+1}$ consists of pairwise disjoint sets. If X is an (i,k_i,ε_i)-expander for $i = j-1, i = j$, then*

$$\left| |F^{j+1}(A_0,\dots,A_\ell)| - \left(\frac{k_j}{k_{j-1}}\right)^{\ell-j} |F^j(A_0,\dots,A_\ell)| \right| \leqslant (\ell-j)k_j^{\ell-j}(\varepsilon_j + \varepsilon_{j-1})\sqrt{|F(A_0,\dots,A_j)|\,|F(A_{\ell-j},\dots,A_\ell)|}.$$

Building upon this we obtain a more general pseudo-randomness result for galleries, of which Theorem 1.3 is a special case.

Proposition 1.5. *For any $j < \ell$, there exists $c_{j,\ell}$ such that if X is an (i,k_i,ε_i)-expander for $0 \leqslant i \leqslant j$ and $A_0,\dots,A_\ell$ are pairwise disjoint sets of vertices in X, then*

$$\left| |F^{j+1}(A_0,\dots,A_\ell)| - \frac{k_0k_1\cdots k_{j-1}k_j^{\ell-j}}{n^\ell} \prod_{i=0}^{\ell} |A_i| \right| \leqslant c_{j,\ell}k_0k_1\cdots k_{j-1}k_j^{\ell-j}(\varepsilon_0 + \cdots + \varepsilon_j)\max|A_i|.$$

The proofs of the Descent Lemma and its corollaries appear in Section 3, after giving the required definitions in Section 2. In Section 4 we discuss applications of the mixing lemma for geometric expansion (in the sense of [13, 20]), chromatic bounds, isoperimetric constants and crossing numbers. We also present the idea of *ideal expanders* in this section, and list some open problems in Section 5.

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2. Simplicial Hodge theory

We describe here briefly the notions we shall need from simplicial Hodge theory. For a more detailed summary we refer the reader to [37, § 2].

Let X be a d-dimensional simplicial complex, with vertex set V of size n. For $-1 \leqslant j \leqslant d$ we denote by X^j the set of j-cells in X (cells of size $j+1$), and by $X^j_\pm$ the set of oriented j-cells, that is, ordered cells up to an even permutation. A *j-form* on X is an antisymmetric function on oriented j-cells,

$$\Omega^j = \Omega^j(X) = \{f : X^j_\pm \to \mathbb{R} \mid f(\overline{\sigma}) = -f(\sigma)\ \forall \sigma \in X^j_\pm\},$$

where $\overline{\sigma}$ is σ endowed with the opposite orientation. In dimensions 0 and -1 there is only one orientation, and so $\Omega^0 = \mathbb{R}^V$ and $\Omega^{-1} = \mathbb{R}^{\{\varnothing\}} \cong \mathbb{R}$. The *$j$th boundary operator* $\partial_j : \Omega^j \to \Omega^{j-1}$ is defined by

$$(\partial_j f)(\sigma) = \sum_{v \cup \sigma \in X^j} f(v\sigma).$$

The sequence $\Omega^{-1} \xleftarrow{\partial_0} \Omega^0 \xleftarrow{\partial_1} \cdots$ is a chain complex, that is,

$$B_j \stackrel{\text{def}}{=} \operatorname{im} \partial_{j+1} \subseteq \ker \partial_j \stackrel{\text{def}}{=} Z_j,$$

and $H^j = Z_j / B_j$ is the jth (real, reduced) homology group of X. We endow each Ω^j with the inner product $\langle f, g \rangle = \sum_{\sigma \in X^j} f(\sigma) g(\sigma)$, which gives rise to a dual *coboundary operator* $\delta_j = \partial_j^* : \Omega^{j-1} \to \Omega^j$. The real *cohomology* of X is $H^j = Z^j / B^j$, where

$$B^j \stackrel{\text{def}}{=} \operatorname{im} \delta_j \subseteq \ker \delta_{j+1} \stackrel{\text{def}}{=} Z^j,$$

and by the fundamental theorem of linear algebra one has $B_j^\perp = Z^j$ and $Z_j^\perp = B^j$.

The *upper, lower* and *full Laplacians* in dimension j are $\Delta_j^+ = \partial_{j+1}\delta_{j+1}$, $\Delta_j^- = \delta_j \partial_j$ and $\Delta_j = \Delta_j^+ + \Delta_j^-$, respectively. All of the Laplacians are self-adjoint and decompose with respect to the orthogonal decompositions $\Omega^j = B^j \oplus Z_j = B_j \oplus Z^j$, and the following properties are simple exercises (unlike their Riemannian counterparts):

$$Z^j = \ker \Delta_j^+, \quad B_j = \operatorname{im} \Delta_j^+, \quad Z_j = \ker \Delta_j^-, \quad B^j = \operatorname{im} \Delta_j^-,$$
$$Z^j \cap Z_j = (B_j \oplus B^j)^\perp = \ker \Delta_j \cong H_j \cong H^j \quad \text{(discrete Hodge theorem)}.$$

The dimension of $\ker \Delta_j$ is the jth (reduced) *Betti number* of X, denoted by β_j.

The combinatorial meaning of the Laplacians is better understood via the following adjacency relations on oriented cells.

(1) For two oriented j-cells σ, σ', we denote $\sigma \Cap \sigma'$ if σ and σ' intersect in a common $(j-1)$-cell and induce the same orientation on it; for edges this means that they have a common origin or a common endpoint, and for vertices $v \Cap v'$ holds whenever $v \neq v'$.

(2) We denote $\sigma \sim \sigma'$ if $\sigma \Cap \sigma'$, and in addition the $(j+1)$-cell $\sigma \cup \sigma'$ is in X. For vertices this is the standard relation of neighbours in a graph.

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Using these relations, the Laplacians can be expressed as follows (here the degree of a j-cell is the number of $(j+1)$-cells which contain it):

$$(\Delta_j^+\varphi)(\sigma) = \deg(\sigma)\varphi(\sigma) - \sum_{\sigma'\sim\sigma}\varphi(\sigma'),$$
$$(\Delta_j^-\varphi)(\sigma) = (j+1)\varphi(\sigma) + \sum_{\sigma'\pitchfork\sigma}\varphi(\sigma'),$$
$$(\Delta_j\varphi)(\sigma) = (\deg\sigma + j + 1)\varphi(\sigma) + \sum_{\substack{\sigma'\pitchfork\sigma\\ \sigma'\nsim\sigma}}\varphi(\sigma').$$

We also define adjacency operators on Ω^j which correspond to the $\sim$ and $\pitchfork$ relations:

$$(\mathcal{A}_j^{\sim}\varphi)(\sigma) = \sum_{\sigma'\sim\sigma}\varphi(\sigma'), \quad (\mathcal{A}_j^{\pitchfork}\varphi)(\sigma) = \sum_{\sigma'\pitchfork\sigma}\varphi(\sigma'),$$

so that $\Delta_j^- = (j+1)\cdot I + \mathcal{A}_j^{\pitchfork}$ and $\Delta_j^+ = D_j - \mathcal{A}_j^{\sim}$, where D_j is the degree operator $(D_jf)(\sigma) = \deg(\sigma)f(\sigma)$. Let us remark that these operators can be used to define stochastic processes on j-cells whose properties relate to the homology of the complex: see [34, 36].

2.1. Spectrum

The spectra we are primarily interested in are those of Δ_j^+, for $0 \leqslant j \leqslant d-1$. Since (Ω^j, δ_j) is a co-chain complex, $B^j = \operatorname{im}\delta_j$ must be contained in the kernel of $\Delta_j^+ = \partial_{j+1}\delta_{j+1}$, and the zero eigenvalues which correspond to forms in B^j are considered to be the *trivial spectrum* of Δ_j^+. As $(B^j)^\perp = Z_j$, we call $\operatorname{Spec}\Delta_j^+|_{Z_j}$ the *non-trivial spectrum* of Δ_j^+. Note that zero is a non-trivial eigenvalue of Δ_j^+ precisely when $Z_j \cap Z^j \neq 0$, that is, $\beta_j \neq 0$. For example, the constant functions on V form the trivial eigenfunctions of Δ_0^+. The non-trivial spectrum of Δ_j^+ corresponds to Z_0, which are the functions whose sum on all vertices vanish, and zero is a non-trivial eigenvalue of Δ_0^+ if and only if the complex is disconnected.

Definition. The complex X is a (j,k,ε)-*expander* if $\varepsilon < 1$ and $\operatorname{Spec}\Delta_j^+|_{Z_j} \subseteq [k(1-\varepsilon), k(1+\varepsilon)]$. Given $\overline{k} = (k_0,\ldots,k_{d-1})$ and $\overline{\varepsilon} = (\varepsilon_0,\ldots,\varepsilon_{d-1})$, we say that X is a $(\overline{k},\overline{\varepsilon})$-*expander* if it is a (j,k_j,ε_j)-expander for all j.

The restriction $\varepsilon_j < 1$ in the definition ensures that X has trivial jth homology, that is, $\beta_j = 0$. While some of our results hold for general ε (*e.g.* Lemma 1.4), or for any global bound on it (*e.g.* Theorem 1.3), we shall need the stronger assumption for later applications.

We remark that it is sometimes convenient to consider the Laplacian Δ_{-1}^+ as well. This operator acts on $\Omega^{-1} \cong \mathbb{R}$ as multiplication by $n = |V|$, so that every complex is automatically a $(-1,n,0)$-expander.

3. The main theorems

In this section we assume that X is a d-complex on n vertices, and prove the Descent Lemma (Lemma 1.4) and the mixing lemmas it implies.

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Proof of the Descent Lemma. To any sets of vertices $A_0,\dots,A_j$, we associate the characteristic j-form $\mathbb{1}_{A_0\cdots A_j}\in\Omega^j$, which takes ±1 on j-cells in $F(A_0,\dots,A_j)$ (according to their orientation), and vanishes elsewhere:

$$\mathbb{1}_{A_0\cdots A_j}(\sigma)=\begin{cases}\operatorname{sgn}(\pi) & \exists\pi\in\operatorname{Sym}_{\{0\cdots j\}}\text{ with }\sigma_i\in A_{\pi(i)}\text{ for }0\leqslant i\leqslant j,\\ 0 & \text{otherwise.}\end{cases}$$

Restriction of j-forms to $F(A_0,\dots,A_j)$ gives an orthogonal projection operator in Ω^j, which we denote by $\mathbb{P}_{A_0\cdots A_j}$:

$$(\mathbb{P}_{A_0\cdots A_j}\varphi)(\sigma)=\varphi(\sigma)\cdot|(\mathbb{1}_{A_0\cdots A_j})(\sigma)|=\begin{cases}\varphi(\sigma) & \sigma\in F(A_0,\dots,A_j),\\ 0 & \text{otherwise.}\end{cases}\tag{3.1}$$

We start our analysis by observing that for pairwise disjoint sets $A_0,\dots,A_{j+1}$ the form

$$(-1)^j\mathbb{P}_{A_0\cdots A_j}\mathcal{A}_j^{\sim}\mathbb{1}_{A_1\cdots A_{j+1}}$$

vanishes outside $F(A_0,\dots,A_j)$, and assigns to each j-cell therein its number of $\sim$-neighbours in $F(A_1,\dots,A_{j+1})$. As these neighbours are in correspondence with $(j+1)$-cells in $F(A_0,\dots,A_{j+1})$, we obtain

$$|F(A_0,\dots,A_{j+1})|=|\langle\mathbb{1}_{A_0\cdots A_j},\mathbb{P}_{A_0\cdots A_j}\mathcal{A}_j^{\sim}\mathbb{1}_{A_1\cdots A_{j+1}}\rangle|.\tag{3.2}$$

Next, let φ be a j-form which is supported on $F(A_1,\dots,A_{j+1})$, and which assigns to each j-cell σ the number of $(j+1)$-galleries in $A_1,\dots,A_\ell$ whose first cell contains σ. By the same considerations as above, $(-1)^j\mathbb{P}_{A_0\cdots A_j}\mathcal{A}_j^{\sim}\varphi$ assigns to every j-cell τ in $F(A_0,\dots,A_j)$ the number of $(j+1)$-galleries in $A_0,\dots,A_\ell$ whose first $(j+1)$ cell contains τ. Therefore,

$$|\langle\mathbb{1}_{A_0\cdots A_j},\mathbb{P}_{A_0\cdots A_j}\mathcal{A}_j^{\sim}\varphi\rangle|=|F^{j+1}(A_0,\dots,A_\ell)|,$$

and we conclude by induction that

$$|F^{j+1}(A_0,\dots,A_\ell)|=\left|\left\langle\mathbb{1}_{A_0\cdots A_j},\left(\prod_{i=0}^{\ell-j-1}\mathbb{P}_{A_i\cdots A_{i+j}}\mathcal{A}_j^{\sim}\right)\mathbb{1}_{A_{\ell-j}\cdots A_\ell}\right\rangle\right|.\tag{3.3}$$

Since $A_i,\dots,A_{i+j+1}$ are pairwise disjoint, $\mathbb{1}_{A_i\cdots A_{i+j}}$ and $\mathbb{1}_{A_{i+1}\cdots A_{i+j+1}}$ are supported on different cells, hence by (3.1) we have $\mathbb{P}_{A_i\cdots A_{i+j}}T\mathbb{P}_{A_{i+1}\cdots A_{i+j+1}}=0$ for any diagonal operator T. Thus, all the $\mathcal{A}_j^{\sim}$ in (3.3) can be replaced by $\mathcal{A}_j^{\sim}+T$, and taking $T=k_jI-D_j$ we obtain

$$|F^{j+1}(A_0,\dots,A_\ell)|=\left|\left\langle\mathbb{1}_{A_0\cdots A_j},\left(\prod_{i=0}^{\ell-j-1}\mathbb{P}_{A_i\cdots A_{i+j}}(k_jI-\Delta_j^+)\right)\mathbb{1}_{A_{\ell-j}\cdots A_\ell}\right\rangle\right|.\tag{3.4}$$

Our next step is to approximate this quantity using the lower jth Laplacian. Denoting

$$E=k_jI-\Delta_j^+-\frac{k_j}{k_{j-1}}\Delta_j^-,$$

the orthogonal decomposition $\Omega^j=Z_j\oplus B^j$ gives

$$E=k_j(\mathbb{P}_{Z_j}+\mathbb{P}_{B^j})-\Delta_j^+-\frac{k_j}{k_{j-1}}\Delta_j^-=k_j\mathbb{P}_{Z_j}-\Delta_j^++\frac{k_j}{k_{j-1}}(k_{j-1}\mathbb{P}_{B^j}-\Delta_j^-).$$

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We first observe that $\|k_j\mathbb{P}_{Z_j} - \Delta_j^+\| \leqslant k_j\varepsilon_j$ follows from $\operatorname{Spec}\Delta_j^+|_{Z_j} \subseteq [k_j(1-\varepsilon_j), k_j(1+\varepsilon_j)]$ and $\Delta_j^+|_{B^j} \equiv 0$. For the lower Laplacian, we have

$$\begin{aligned}\operatorname{Spec}\Delta_j^-|_{B^j} &= \operatorname{Spec}\Delta_j^-|_{Z_j^\perp} = \operatorname{Spec}\Delta_j^-\backslash\{0\} \overset{(*)}{=} \operatorname{Spec}\Delta_{j-1}^+\backslash\{0\} = \operatorname{Spec}\Delta_{j-1}^+|_{(Z^{j-1})^\perp} \\ &= \operatorname{Spec}\Delta_{j-1}^+|_{B_{j-1}} \subseteq \operatorname{Spec}\Delta_{j-1}^+|_{Z_{j-1}} \subseteq [k_{j-1}(1-\varepsilon_{j-1}), k_{j-1}(1+\varepsilon_{j-1})], \end{aligned} \tag{3.5}$$

where $(*)$ follows from the fact that $\Delta_j^- = \partial_j^*\partial_j$ and $\Delta_{j-1}^+ = \partial_j\partial_j^*$. As Δ_j^- vanishes on Z_j, we have in total $\|k_{j-1}\mathbb{P}_{B^j} - \Delta_j^-\| \leqslant k_{j-1}\varepsilon_{j-1}$, so that

$$\|E\| \leqslant \|k_j\mathbb{P}_{Z_j} - \Delta_j^+\| + \frac{k_j}{k_{j-1}}\|k_{j-1}\mathbb{P}_{B^j} - \Delta_j^-\| \leqslant k_j(\varepsilon_{j-1} + \varepsilon_j). \tag{3.6}$$

For brevity let us denote $\mathbb{P}_i = \mathbb{P}_{A_i,\dots,A_{i+j}}$. Using

$$k_jI - \Delta_j^+ = \frac{k_j}{k_{j-1}}\Delta_j^- + E,$$

and allowing ourselves to translate Δ_j^- by diagonal (in fact, scalar) operators, we find that

$$\begin{aligned}\prod_{i=0}^{\ell-j-1}\mathbb{P}_i(k_jI - \Delta_j^+) &= \prod_{i=0}^{\ell-j-1}\mathbb{P}_i\left(\frac{k_j}{k_{j-1}}\Delta_j^- + E\right) = \left(\frac{k_j}{k_{j-1}}\right)^{\ell-j}\cdot\prod_{i=0}^{\ell-j-1}\mathbb{P}_i\Delta_j^- \\ &\quad + \sum_{m=1}^{\ell-j}\left(\frac{k_j}{k_{j-1}}\right)^{\ell-j-m}\left(\prod_{i=0}^{\ell-j-m-1}\mathbb{P}_i\Delta_j^-\right)\mathbb{P}_{\ell-j-m}E\left(\prod_{i=\ell-j-m+1}^{\ell-j-1}\mathbb{P}_i\left(\frac{k_j}{k_{j-1}}\Delta_j^- + E\right)\right) \\ &= \left(\frac{k_j}{k_{j-1}}\right)^{\ell-j}\cdot\prod_{i=0}^{\ell-j-1}\mathbb{P}_i\mathcal{A}_j^{\pitchfork} \\ &\quad + \sum_{m=1}^{\ell-j}\left(\frac{k_j}{k_{j-1}}\right)^{\ell-j-m}\left(\prod_{i=0}^{\ell-j-m-1}\mathbb{P}_i(\Delta_j^- - k_{j-1}I)\right)\mathbb{P}_{\ell-j-m}E\left(\prod_{i=\ell-j-m+1}^{\ell-j-1}\mathbb{P}_i(k_jI - \Delta_j^+)\right),\end{aligned}$$

and plugging this into (3.4) gives

$$\begin{aligned}|F^{j+1}(A_0,\dots,A_\ell)| &= \left|\left(\frac{k_j}{k_{j-1}}\right)^{\ell-j}\left\langle\mathbb{1}_{A_0\cdots A_j}, \left(\prod_{i=0}^{\ell-j-1}\mathbb{P}_{A_i\cdots A_{i+j}}\mathcal{A}_j^{\pitchfork}\right)\mathbb{1}_{A_{\ell-j}\cdots A_\ell}\right\rangle\right. \\ &\quad \left. + \sum_{m=1}^{\ell-j}\left(\frac{k_j}{k_{j-1}}\right)^{\ell-j-m}\left\langle\mathbb{1}_{A_0\cdots A_j}, \begin{array}{l}\left(\displaystyle\prod_{i=0}^{\ell-j-m-1}\mathbb{P}_{A_i\cdots A_{i+j}}(\Delta_j^- - k_{j-1}I)\right)\mathbb{P}_{A_{\ell-j-m}\cdots A_{\ell-m}}E\cdot \\ \cdot\left(\displaystyle\prod_{i=\ell-j-m+1}^{\ell-j-1}\mathbb{P}_{A_i\cdots A_{i+j}}(k_jI - \Delta_j^+)\right)\mathbb{1}_{A_{\ell-j}\cdots A_\ell}\end{array}\right\rangle\right|.\end{aligned}$$

Let us call the term on the first line the main term, and the one on the second line the error term. Note that the form $(-1)^j\mathbb{P}_{A_0\cdots A_j}\mathcal{A}_j^{\pitchfork}\mathbb{1}_{A_1\cdots A_{j+1}}$ assigns to every j-cell in $F(A_0,\dots,A_j)$ the number of j-cells in $F(A_1,\dots,A_{j+1})$ with which it intersects, so that by definition

$$|\langle\mathbb{1}_{A_0\cdots A_j}, \mathbb{P}_{A_0\cdots A_j}\mathcal{A}_j^{\pitchfork}\mathbb{1}_{A_1\cdots A_{j+1}}\rangle| = |F^j(A_0,\dots,A_{j+1})|$$

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(compare this with (3.2)). By the same arguments as for $\mathcal{A}^{\sim}$, one sees that

$$|F^j(A_0,\dots,A_\ell)| = \left|\left\langle \mathbb{1}_{A_0\cdots A_j}, \left(\prod_{i=0}^{\ell-j-1} \mathbb{P}_{A_i\cdots A_{i+j}}\mathcal{A}_j^{\pitchfork}\right)\mathbb{1}_{A_{\ell-j}\cdots A_\ell}\right\rangle\right|,$$

so that the main term is precisely

$$\left(\frac{k_j}{k_{j-1}}\right)^{\ell-j}|F^j(A_0,\dots,A_\ell)|,$$

our estimate for $|F^{j+1}(A_0,\dots,A_\ell)|$. Turning to the error term, since $\operatorname{Spec}\Delta_j^+ \subseteq [0,k_j(1+\varepsilon_j)]$ we have $\|k_jI-\Delta_j^+\|\leqslant k_j$, and similarly from (3.5) we obtain $\|\Delta_j^- - k_{j-1}I\|\leqslant k_{j-1}$. Together with (3.6) this implies that the error term is bounded by

$$\sum_{m=1}^{\ell-j}\left(\frac{k_j}{k_{j-1}}\right)^{\ell-j-m}\|\mathbb{1}_{A_0\cdots A_j}\|k_{j-1}^{\ell-j-m}k_j(\varepsilon_{j-1}+\varepsilon_j)k_j^{m-1}\|\mathbb{1}_{A_{\ell-j}\cdots A_\ell}\|$$
$$=(\ell-j)k_j^{\ell-j}(\varepsilon_{j-1}+\varepsilon_j)\sqrt{|F(A_0,\dots,A_j)||F(A_{\ell-j},\dots,A_\ell)|},$$

which concludes the proof.

We remark that a slightly better bound is possible here: one can replace $k_jI-\Delta_j^+$ in the error term by $(k_j(1+\varepsilon_j)/2)I-\Delta_j^+$, which is bounded by $k_j(1+\varepsilon_j)/2$, and likewise for Δ_j^- and k_{j-1}. For example, putting $\varepsilon=\max(\varepsilon_{j-1},\varepsilon_j)$ this gives the bound

$$\frac{(\ell-j)}{2^{\ell-j-2}}k_j^{\ell-j}\varepsilon(1+\varepsilon)^{\ell-j-1}\sqrt{|F(A_0,\dots,A_j)||F(A_{\ell-j},\dots,A_\ell)|}$$

which might be useful when ε is small and $\ell\gg j$. □

Using the Descent Lemma repeatedly gives Proposition 1.5, which for $j=d-1$, $\ell=d$ implies Theorem 1.3, as follows.

Proof of Proposition 1.5. We denote $m=\max|A_i|$ and assume by induction that the proposition holds for $j-1$ (and any $j\leqslant\ell$), that is,

$$\left||F^j(A_0,\dots,A_\ell)|-\frac{k_0\cdots k_{j-2}k_{j-1}^{\ell-j+1}}{n^\ell}\prod_{i=0}^{\ell}|A_i|\right|\leqslant c_{j-1,\ell}k_0\cdots k_{j-2}k_{j-1}^{\ell-j+1}(\varepsilon_0+\cdots+\varepsilon_{j-1})m. \tag{3.7}$$

For $j=0$ this indeed holds, in the sense that

$$\left||F^0(A_0,\dots,A_\ell)|-\frac{k_{-1}^\ell}{n^\ell}\prod_{i=0}^{\ell}|A_i|\right|=0 \tag{3.8}$$

(recall that every complex is a $(-1,n,0)$-expander). Let $\mathcal{E}$ denote the discrepancy

$$\left||F^{j+1}(A_0,\dots,A_\ell)|-\frac{k_0k_1\cdots k_{j-1}k_j^{\ell-j}}{n^\ell}\prod_{i=0}^{\ell}|A_i|\right|.$$

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Combining the Descent Lemma with (3.7) (or (3.8), for $j=0$) multiplied by $(k_j/k_{j-1})^{\ell-j}$ gives

$$\begin{aligned}\mathcal{E} &\leqslant (\ell-j)k_j^{\ell-j}(\varepsilon_j+\varepsilon_{j-1})\sqrt{|F(A_0,\dots,A_j)||F(A_{\ell-j},\dots,A_\ell)|}\\ &\quad + c_{j-1,\ell}k_0k_1\cdots k_{j-1}k_j^{\ell-j}(\varepsilon_0+\cdots+\varepsilon_{j-1})m.\end{aligned}$$

To bound $|F(A_0,\dots,A_j)|$ we use (3.7) with $\ell=j$, which gives

$$\begin{aligned}|F^j(A_0,\dots,A_j)| &\leqslant \frac{k_0\cdots k_{j-1}}{n^j}\prod_{i=0}^{j}|A_i| + c_{j-1,j}k_0\cdots k_{j-1}(\varepsilon_0+\cdots+\varepsilon_{j-1})m\\ &\leqslant [1+c_{j-1,j}(\varepsilon_0+\cdots+\varepsilon_{j-1})]k_0\cdots k_{j-1}m\\ &\leqslant (1+jc_{j-1,j})k_0\cdots k_{j-1}m\end{aligned}$$

(we have used $\varepsilon_i<1$, but any global bound for ε_i would do). The same holds for $|F(A_{\ell-j},\dots,A_\ell)|$, hence

$$\begin{aligned}\mathcal{E} &\leqslant (\ell-j)k_j^{\ell-j}(\varepsilon_j+\varepsilon_{j-1})(1+jc_{j-1,j})k_0\cdots k_{j-1}m\\ &\quad + c_{j-1,\ell}k_0k_1\cdots k_{j-1}k_j^{\ell-j}(\varepsilon_0+\cdots+\varepsilon_{j-1})m\\ &= k_0k_1\cdots k_{j-1}k_j^{\ell-j}[c_{j-1,\ell}(\varepsilon_0+\cdots+\varepsilon_{j-1})+(\ell-j)(1+jc_{j-1,j})(\varepsilon_j+\varepsilon_{j-1})]m\\ &\leqslant \underbrace{[c_{j-1,\ell}+(\ell-j)(1+jc_{j-1,j})]}_{c_{j,\ell}}k_0k_1\cdots k_{j-1}k_j^{\ell-j}(\varepsilon_0+\cdots+\varepsilon_j)m.\end{aligned}$$

as desired. □

4. Applications

4.1. Geometric overlap

The following notion of geometric expansion for graphs and complexes originates in Gromov's work [20] (see also [13, 33]).

Definition. Let X be a d-dimensional simplicial complex. The *geometric overlap* of X is

$$\operatorname{overlap} X = \min_{\varphi:V\to\mathbb{R}^d}\max_{x\in\mathbb{R}^d}\frac{\#\{\sigma\in X^d \mid x\in\operatorname{conv}\{\varphi(v)\mid v\in\sigma\}\}}{|X^d|}.$$

In words, X has overlap $\geqslant\varepsilon$ if, for every affine map of X into $\mathbb{R}^d$ (a mapping induced linearly by the images of the vertices), some point in $\mathbb{R}^d$ is covered by at least an ε-fraction of the d-cells of X.

A theorem of Pach [35] relates pseudo-randomness and geometric overlap, and allows us to show the following proposition.

Proposition 4.1. *If X is a d-dimensional $(\bar{k},\bar{\varepsilon})$-expander with $\mathcal{E}=\varepsilon_0+\cdots+\varepsilon_{d-1}<1$, then*

$$\operatorname{overlap} X > \mathcal{P}_d d!(1-\mathcal{E})\left(\left(\frac{\mathcal{P}_d}{d+1}\right)^d - c_d\mathcal{E}\right),$$

where $\mathcal{P}_d$ is Pach's constant [35]*, and c_d is the constant in Theorem 1.3.*

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Thus, a family of d-complexes with $\varepsilon_0 + \cdots + \varepsilon_{d-1}$ small enough is a family of geometric expanders. For the proof we shall need the following lemma, which relates the Laplace spectrum to cell density.

Lemma 4.2. *Let X be a d-complex with $\beta_j = 0$ for $0 \leqslant j \leqslant d-1$, and let λ_j be the average non-trivial eigenvalue of Δ_j^+, for $-1 \leqslant j \leqslant d-1$ (in particular $\lambda_{-1} = n$). Then, for $0 \leqslant m \leqslant d$ the number of m-cells is*

$$|X^m| = \frac{\lambda_{m-1}}{m+1} \cdot \prod_{j=-1}^{m-2} \left(\frac{\lambda_j}{j+2} - 1 \right) = \frac{\lambda_{m-1}(n-1)}{m+1} \cdot \prod_{j=0}^{m-2} \left(\frac{\lambda_j}{j+2} - 1 \right), \tag{4.1}$$

and the average degree of an m-cell is

$$\operatorname{avg}\{\deg \sigma \mid \sigma \in X^m\} = \lambda_m \left(1 - \frac{m+1}{\lambda_{m-1}} \right). \tag{4.2}$$

Proof. Since the trivial spectrum of Δ_j^+ consists of zeros,

$$|X^m| = \frac{1}{m+1} \sum_{\sigma \in X^{m-1}} \deg \sigma = \frac{1}{m+1} \operatorname{tr} D_{m-1} = \frac{1}{m+1} \operatorname{tr} \Delta_{m-1}^+ = \frac{\lambda_{m-1}}{m+1} \dim Z_{m-1}.$$

Thus, (4.1) is equivalent to the assertion that

$$\dim Z_{m-1} = \prod_{j=-1}^{m-2} \left(\frac{\lambda_j}{j+2} - 1 \right).$$

This is true for $m = 0$, and by induction, together with the triviality of the $(m-2)$th homology we find that

$$\begin{aligned} \dim Z_{m-1} &= \dim \Omega^{m-1} - \dim B_{m-2} = |X^{m-1}| - \dim Z_{m-2} \\ &= \frac{\lambda_{m-2}}{m} \prod_{j=-1}^{m-3} \left(\frac{\lambda_j}{j+2} - 1 \right) - \prod_{j=-1}^{m-3} \left(\frac{\lambda_j}{j+2} - 1 \right) = \prod_{j=-1}^{m-2} \left(\frac{\lambda_j}{j+2} - 1 \right) \end{aligned}$$

as desired. Formula (4.2) follows from (4.1), as

$$\operatorname{avg}\{\deg \sigma \mid \sigma \in X^m\} = \frac{(m+2)|X^{m+1}|}{|X^m|}. \qquad \square$$

We return to the proof of the proposition.

Proof of Proposition 4.1. Let φ be a simplicial map $X \to \mathbb{R}^d$, and divide $V = X^0$ arbitrarily into parts $P_0, \ldots, P_{d+1}$ of size $n/(d+1)$. Pach's theorem [35] then states that there exist $Q_i \subseteq P_i$ of size $|Q_i| = \mathcal{P}_d |P_i|$ and a point $x \in \mathbb{R}^{d+1}$, such that $x \in \operatorname{conv}\{\varphi(v) \mid v \in \sigma\}$ for all $\sigma \in F(Q_0, \ldots, Q_d)$. Denoting $\mathcal{K} = k_0 \cdot \ldots \cdot k_{d-1}$ and $\mathcal{E} = \varepsilon_0 + \cdots + \varepsilon_{d-1}$, we

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have by Theorem 1.3

$$|F(Q_0,\dots,Q_d)| \geqslant \frac{\mathcal{K}}{n^d}\left(\frac{\mathcal{P}_d n}{d+1}\right)^{d+1} - \frac{c_d \mathcal{K}\mathcal{E}\mathcal{P}_d n}{d+1} = \frac{\mathcal{K}\mathcal{P}_d n}{d+1}\left[\left(\frac{\mathcal{P}_d}{d+1}\right)^d - c_d\mathcal{E}\right],$$

and by the lemma above

$$|X^d| = \frac{\lambda_{d-1}}{d+1}\cdot\prod_{j=-1}^{d-2}\left(\frac{\lambda_j}{j+2}-1\right) \leqslant \prod_{j=-1}^{d-1}\frac{\lambda_j}{j+2} \leqslant n\prod_{j=0}^{d-1}\frac{k_j(1+\varepsilon_j)}{j+2} \leqslant \frac{n\mathcal{K}}{(d+1)!(1-\mathcal{E})}.$$

This means x is covered by the φ-images of at least a $\mathcal{P}_d d!(1-\mathcal{E})((\mathcal{P}_d/(d+1))^d - c_d\mathcal{E})$-fraction of the d-cells, and as this is true for all φ the proposition follows. □

4.2. Crossing numbers

This is another invariant, of somewhat similar flavour to the overlap constant. The *m-dimensional crossing number* of a d-complex X is the minimal c such that for any affine map $X \to \mathbb{R}^m$ there are at least c pairs of disjoint d-cells in X whose images in $\mathbb{R}^m$ intersect. In [23, § 8.1] Gundert and Wagner use the mixing lemma from [37] to give a lower bound for the $2d$-dimensional crossing number of a d-complex with a complete skeleton, and their arguments hold for the general case using Theorem 1.3 and Lemma 4.2.

4.3. Chromatic numbers

A d-complex X is *weakly c-colourable* if there is a colouring of its vertices by c colours so that no d-cell is monochromatic. The *weak chromatic number* of X, denoted $\chi(X)$, is the smallest c for which X is c-colourable. We will use the mixing property to show that spectral expansion implies a chromatic bound, as is done for graphs in [29]. These results are weaker than Hoffman's chromatic bound for graphs [24], as they require a two-sided spectral bound, and the chromatic bound obtained is not optimal. A chromatic bound for complexes which generalizes Hoffman's result was recently obtained in [18].

Proposition 4.3. *If X is a d-dimensional $(\overline{k},\overline{\varepsilon})$-expander, then*

$$\chi(X) \geqslant \frac{1}{(d+1)\sqrt[d]{c_d(\varepsilon_0+\cdots+\varepsilon_{d-1})}},$$

where c_d is the constant from Theorem 1.3.

Proof. Colouring X by $\chi = \chi(X)$ colours, there is necessarily a monochromatic set of vertices of size at least n/χ. Take n/χ of these vertices and partition them arbitrarily into $d+1$ sets $A_0,\dots,A_d$ of equal size. By assumption we have $F(A_0,\dots,A_d) = \varnothing$, so that Theorem 1.3 reads

$$\frac{k_0\cdots k_{d-1}}{n^d}\prod_{i=0}^{d}|A_i| \leqslant c_d k_0\cdots k_{d-1}(\varepsilon_0+\cdots+\varepsilon_{d-1})\max|A_i|,$$

and since $|A_i| = n/(\chi\cdot(d+1))$, the conclusion follows. □

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4.4. Isoperimetric bounds

The Cheeger inequalities for graphs relate $\min \operatorname{Spec} \Delta_0^+|_{Z_0}$ to the Cheeger constant, which is the minimum of $|E(A,B)|/(|A||B|)$ over all partitions of the vertices into (non-empty) sets A and B. In [37] one side of these inequalities is generalized to complexes with a complete skeleton. Namely, for any partition $A_0,\dots,A_d$ of the vertices in a d-complex, the quantity

$$\frac{|F(A_0,\dots,A_d)|}{|A_0|\cdot\ldots\cdot|A_d|}$$

is bounded from below in terms of $\min \operatorname{Spec} \Delta_{d-1}^+|_{Z_{d-1}}$. In [19] we generalize this result to complexes with non-complete skeleton, obtaining the following result.

Theorem 4.4 ([19]). *Let X be a d-complex on n vertices, which is a (j,k_j,ε_j)-expander for $0 \leqslant j \leqslant d-2$, and let*

$$\lambda_{d-1} = \min \operatorname{Spec} \Delta_{d-1}^+|_{Z_{d-1}}.$$

Then, for any partition $A_0,\dots,A_d$ of the vertices of X,

$$\frac{|F(A_0,\dots,A_d)|n^d}{|A_0|\cdot\ldots\cdot|A_d|} \geqslant k_0\cdot\ldots\cdot k_{d-2}\cdot\lambda_{d-1}\left(1-\varepsilon_{d-2}-C_d(\varepsilon_0+\cdots+\varepsilon_{d-2})\frac{n^{d+1}}{|A_0|\cdot\ldots\cdot|A_d|}\right),$$

where C_d depends only on d.

The proof relies on both Theorem 1.3 and Proposition 1.5 of the present paper, which serve to replace the assumption of a complete $(d-1)$-skeleton with a pseudo-random one.

4.5. Ramanujan complexes

Ramanujan graphs, constructed in [29, 31, 32], are spectrally optimal expanders. Ramanujan complexes, a natural high-dimensional counterpart, were defined and studied in [7, 27, 30, 39], but not as yet from the point of view of the Hodge Laplacian. It is natural to suspect that they are high-dimensional expanders in the sense of the current paper, but it turns out that the picture is more complicated. For the two-dimensional case, the non-trivial one-dimensional spectrum is concentrated in two narrow strips.

Theorem 4.5 ([19]). *Let X be a two-dimensional Ramanujan complex with n vertices, constant vertex degree $k_0 = 2(q^2+q+1)$ and constant edge degree $k_1 = q+1$ (q is any prime power). If X is not three-colourable, then*

$$\operatorname{Spec} \Delta_0^+|_{Z_0} \subseteq \left[k_0\left(1-\frac{3}{q}\right), k_0\left(1+\frac{3}{q}\right)\right],$$

$$\operatorname{Spec} \Delta_1^+|_{Z_1} \subseteq \left[k_1\left(1-\frac{2}{\sqrt{q}}\right), k_1\left(1+\frac{2}{\sqrt{q}}\right)\right] \cup \left[2k_1\left(1-\frac{4}{q}\right), 2k_1\left(1+\frac{4}{q}\right)\right] \cup \{3k_1\},$$

and if X is three-colourable then Δ_0^+ has in addition the eigenvalue $3k_0/2$.

The Δ_1^+-eigenform corresponding to the eigenvalue $3k_1$ is a *disorientation* (see [19, 25, 36]), a two-dimensional analogue of graph bipartition. It is well understood, and would

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not prevent us from conducting an analysis like the one carried out in this paper. What prevents such an analysis is the strip around $2k_1$, which requires different arguments. In [19] we circumvent this problem by observing a quadratic polynomial in Δ_1^+, and conclude pseudo-random behaviour for 2-galleries of length four, namely $F^2(A,B,C,D)$.

4.6. Ideal expanders

This is not quite an application, but rather a useful intuition. Let us say that X is an *ideal $\bar{k}$-expander* if it is a $(j,k_j,0)$-expander for $0 \leqslant j \leqslant d-1$. In this case, the Descent Lemma tells us that

$$F^{j+1}(A_0,\dots,A_\ell) = \left(\frac{k_j}{k_{j-1}}\right)^{\ell-j} |F^j(A_0,\dots,A_\ell)|,$$

and the number of j-galleries between pairwise disjoint sets of vertices is completely determined by their sizes:

$$|F^j(A_0,\dots,A_\ell)| = \frac{k_0 k_1 \cdots k_{j-2} k_{j-1}^{\ell-j+1}}{n^\ell} \prod_{i=0}^{\ell} |A_i|; \tag{4.3}$$

in particular,

$$|F(A_0,\dots,A_d)| = \frac{k_0 \cdots k_{d-1}}{n^d} |A_0| \cdots |A_d|.$$

For

$$k_j = \begin{cases} n & 0 \leqslant j < m, \\ 0 & m \leqslant j < d, \end{cases}$$

an example of an ideal $\bar{k}$-expander is given by $K_n^{(m)}$, the mth skeleton of the complete complex on n vertices. For this complex (4.3) holds trivially, and perhaps disappointingly, these are the only examples of ideal expanders: if X is an ideal $\bar{k}$-expander on n vertices, and $X^{(j)} = K_n^{(j)}$ (which holds for $j=0$), one has $k_0 = \cdots = k_{j-1} = n$, and also $k_j \leqslant n$ by [37, Proposition 3.2(2)]. For any vertices $v_0,\dots,v_{j+1}$,

$$\frac{k_0 \cdots k_j}{n^{j+1}} = |F(\{v_0\},\dots,\{v_{j+1}\})| \in \{0,1\}$$

then forces either $k_j = n$, which implies that $X^{(j+1)} = K_n^{(j+1)}$ as well, or $k_j = 0$, which means that X has no $(j+1)$-cells at all.

While ideal $\bar{k}$-expanders do not actually exist, save for the trivial examples $\bar{k} = (n,\dots,n,0,\dots)$, they provide a conceptual way to think of expanders in general: $(\bar{k},\bar{\varepsilon})$-expanders spectrally approximate the ideal $\bar{k}$-expander, and the mixing lemma asserts that they approximate it combinatorially as well. This point of view seems close in spirit to that of *spectral sparsification* [40], which has proved to be fruitful in both graphs and complexity theory.

5. Questions

Several natural questions arise from this study.

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- In [22] it is shown that random complexes in the Linial–Meshulam model [28] have spectral concentration for appropriate parameters (see also [37, §4.5]). These are complexes with a complete skeleton, which are high-dimensional analogues of Erdős–Rényi graphs. Is there a similar model for general complexes, for which the skeletons are not complete (preferably, where the expected degrees of cells are only logarithmic in the number of vertices), with concentrated spectrum?
- A well-known source of excellent expanders is random regular graphs (see *e.g.* [15, 38]). Can one construct a model for random regular complexes, and are these complexes high-dimensional expanders? This is interesting even for a weak notion of regularity, such as having a bounded fluctuation of degrees, or having all links of vertices isomorphic.
- Bilu and Linial [6] have established a converse to the Expander Mixing Lemma, which shows that pseudo-randomness and two-sided spectral concentration are (almost) equivalent. For complexes with a complete skeleton, a converse to the mixing lemma from [37] was recently established in [8]. Do these converse theorems admit a generalization to the general case?
- Another generalization of the high-dimensional Cheeger inequality for complexes without a complex skeleton appears in [21]. Rather than comparing $|F(A_0,\ldots,A_d)|$ with $|A_0|\cdot\ldots\cdot|A_d|$, it is compared with the number of $(d-1)$-spheres with one vertex at each A_i (which for a complete skeleton is $|A_0|\cdot\ldots\cdot|A_d|$), and the smallest non-trivial eigenvalue of Δ^+_{d-1} is used to relate them. Can one prove a mixing lemma along this line of thought, using $\operatorname{Spec}\Delta^+_{d-1}$ alone?

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