Econometric Theory, **23**, 2007, 1233–1247. Printed in the United States of America.
DOI: 10.1017/S0266466607070491

NOTES AND PROBLEMS

LONG-RUN COVARIANCE MATRICES FOR FRACTIONALLY INTEGRATED PROCESSES

PETER C.B. PHILLIPS
Cowles Foundation for Research in Economics, Yale University
University of York
and
University of Auckland

CHANG SIK KIM
Sungkyunkwan University

An asymptotic expansion is given for the autocovariance matrix of a vector of stationary long-memory processes with memory parameters $d \in [0, \frac{1}{2})$. The theory is then applied to deliver formulas for the long-run covariance matrices of multivariate time series with long memory.

1. MOTIVATION

Stationary long-memory processes have extensive applications in economics and finance, particularly with regard to modeling financial variables such as volatility and trading volume. The autocovariances of such processes decay according to a power law, and the spectra are undefined at the origin. Correspondingly, conventional formulas and estimation procedures for long-run variance matrices that apply under weak dependence are no longer relevant under long-range dependence. Nonetheless, some modified versions of these (typically infinite-dimensional) quantities do exist and are useful in the development of asymptotics involving long-memory time series, for instance, in the estimation of fractionally cointegrated systems (Kim and Phillips, 1999; Robinson and Hualde, 2003; Velasco, 2003; Davidson, 2004; Davidson and Hashimzade, 2008). Henry and Zaffaroni (2003) provide a recent survey of the many applications of fractional integration and long-range dependence in macroeconomics and finance.

Phillips acknowledges partial support from a Kelly Fellowship and from the NSF under grant SES 04-142254.
Address correspondence to Peter C.B. Phillips, Department of Economics, Yale University, P.O. Box 208268, New Haven, CT 06520-8268, USA; e-mail: peter.phillips@yale.edu.

© 2007 Cambridge University Press 0266-4666/07 $15.00

https://doi.org/10.1017/S0266466607070491 Published online by Cambridge University Press

This note shows how to define long-run covariance matrices for general multivariate fractionally integrated processes, and we focus here on cases of long-range dependence, some results on overdifferenced processes being given in earlier work (Phillips, 1995, Lem. 8.1). We first develop a general form of asymptotic expansion for the autocovariance matrix of such a multivariate process. The approximation induced by this expansion is of independent interest. It has a simple form that gives the power law decay structure of the elements of the autocovariance matrix and reveals an interesting asymmetric structure for the cross autocovariances. In the scalar case, the result reduces to a formula obtained recently in Lieberman and Phillips (2006). The expansion is particularly useful in developing a limiting form of a standardized sum of the autocovariance matrices, allowing us to define the long-run variance matrix of a multivariate fractional process. In contrast to the autocovariance function, the long-run variance matrix has a simple symmetric form that depends on the long-memory parameters of the constituent processes and the long-run variance matrix of the short-memory components.

2. RESULTS

Let X_t be a real-valued covariance stationary m-vector time series generated by the system

$$\begin{pmatrix} (1-L)^{d_1} & & 0 \\ & \ddots & \\ 0 & & (1-L)^{d_m} \end{pmatrix} \begin{pmatrix} X_{1t} - EX_{1t} \\ \vdots \\ X_{mt} - EX_{mt} \end{pmatrix} = \begin{pmatrix} u_{1t} \\ \vdots \\ u_{mt} \end{pmatrix},$$

$$0 \leq d_1, \ldots, d_m < \frac{1}{2}, \tag{1}$$

where $u_t = (u_{1t}, \ldots, u_{mt})'$ is a covariance stationary process whose spectral density matrix $f_{uu}(\lambda)$ is assumed to be continuously differentiable on $[-\pi, \pi]$ and bounded away from zero (in the sense of positive definite matrices) at the zero frequency $\lambda = 0$. The smoothness condition on $f_{uu}(\lambda)$ is needed to develop an asymptotic expansion of the autocovariance function defined by a Fourier integral inversion of $f_{uu}(\lambda)$.

Here X_t is a multivariate extension of a scalar fractionally integrated process (the so-called $I(d)$ process), and each component X_{at} exhibits long-range dependence whenever $d_a > 0$. The time series X_t reduces to a multivariate autoregressive fractionally integrated moving average (ARFIMA) process when u_t is a vector autoregressive moving average (ARMA) process, but the specification (1) does not require u_t to be of this or any other parametric form.

https://doi.org/10.1017/S0266466607070491 Published online by Cambridge University Press

Let $f_{xx}(\lambda)$ denote the spectral density of X_t, so that the autocovariance matrix is given by

$$\Gamma_{xx}(k) = E(X_t - EX_t)(X_{t+k} - EX_t)' = \int_{-\pi}^{\pi} e^{ik\lambda} f_{xx}(\lambda)\, d\lambda.$$

Define

$$\Phi(\lambda) = \operatorname{diag}((1 - e^{i\lambda})^{-d_1}, \ldots, (1 - e^{i\lambda})^{-d_m}) = \operatorname{diag}((1 - e^{i\lambda})^{-d_a}),$$

and then the spectral density of X_t satisfies (e.g., Hannan, 1970, p. 61)

$$f_{xx}(\lambda) = \Phi(\lambda) f_{uu}(\lambda) \Phi^*(\lambda), \tag{2}$$

where the affix $*$ signifies complex conjugate transpose. As is well known, the memory parameters, d_a, govern the long-run dynamics of X_t and the behavior of its spectrum $f_{xx}(\lambda)$ around the origin. Often, when attention is focused on long-run dynamics, it is useful to specify the spectral density only locally in the vicinity of the origin and to avoid short-run dynamic specifications concerning u_t altogether. In the multivariate case, we also are interested in the behavior of cross spectra at the origin and the corresponding cross autocovariances at long lags, in addition to the individual spectra and autocovariance functions.

A first-order approximation to the behavior of $f_{xx}(\lambda)$ at the origin is easily seen to be given by

$$f_{xx}(\lambda) \sim \operatorname{diag}(\lambda^{-d_a} e^{i\pi d_a/2}) f_{uu}(0) \operatorname{diag}(\lambda^{-d_a} e^{-i\pi d_a/2}), \qquad \lambda \to 0+, \tag{3}$$

and higher order approximations may be similarly obtained (e.g., Phillips and Shimotsu, 2004; Shimotsu, 2006). The factors involving the complex exponentials $e^{i\pi d_a/2}$ turn out to be important in the off diagonal elements of $f_{xx}(\lambda)$, and these figure in the analysis that follows. When $0 < d_a < \frac{1}{2}$, the individual time series X_{at} have long memory, and the j-lag autocovariances decrease slowly, according to the power law $j^{2d_a - 1}$ as $j \to \infty$. In this case, the autocovariances are not summable, and the usual formula for the long-run variance of X_{at} is undefined. However, as shown subsequently, upon suitable standardization, we may define the long-run variances and covariances of the elements of X_t.

We start with the following result, which gives an asymptotic approximation to the autocovariance matrix function for long lags. Lieberman and Phillips (2006) gave a complete asymptotic series expansion of the autocovariance function of a scalar long-memory time series. Under stronger smoothness conditions on the spectrum $f_{uu}(\lambda)$, similar asymptotic series may be developed here. The theorem that follows gives the leading term in the corresponding expansion for the multivariate case, which is sufficient for the present purpose of developing a formula for the long-run variance matrix. The proof is given in the Appendix.

https://doi.org/10.1017/S0266466607070491 Published online by Cambridge University Press

THEOREM 1. *If $d_a \in [0, \frac{1}{2})$ for all $a = 1, \ldots, m$, and the spectral density matrix $f_{uu}(\lambda)$ of u_t is continuously differentiable, then*

$$[\Gamma_{xx}(k)]_{ab} = \frac{2f_{u_a u_b}(0)\Gamma(1 - d_a - d_b)\sin(\pi d_b)}{k^{1-d_a-d_b}} + O\left(\frac{1}{k^{2-d_a-d_b}}\right). \tag{4}$$

Remark 1. Note that the asymptotic approximation (4) is asymmetric. Suppose for example that $d_a < d_b$ and $f_{u_a u_b}(0) > 0$. Then, for large k

$$\begin{aligned}\gamma_{ab}(k) &= \frac{2f_{u_a u_b}(0)\Gamma(1 - d_a - d_b)}{k^{1-d_a-d_b}} \sin\{\pi d_b\} + O\left(\frac{1}{k^{2-d_a-d_b}}\right) \\ &> \frac{2f_{u_b u_a}(0)\Gamma(1 - d_a - d_b)}{k^{1-d_a-d_b}} \sin\{\pi d_a\} + O\left(\frac{1}{k^{2-d_a-d_b}}\right) \\ &= \gamma_{ba}(k),\end{aligned}$$

because $\sin\{\pi d_a\} < \sin\{\pi d_b\}$. In particular, when $d_a = 0 < d_b < \frac{1}{2}$, $\sin\{\pi d_a\} = 0$, and we have the interesting phenomenon that $\gamma_{ba}(k)$ decays faster than the power law k^{d_b-1}, corresponding to the short-memory property of X_{at}, whereas $\gamma_{ab}(k)$ decays according to the power law k^{d_b-1} as $k \to \infty$, corresponding to the long-memory property of X_{bt}. This asymmetry is explained by the fact that $\gamma_{ab}(k)$ is dominated by the slow decay in the impulse responses affecting X_{bt+k}, whereas the impulse responses and autocovariances of X_{at+k} decay faster than any power rate,[1] thereby determining the different behavior of $\gamma_{ba}(k)$ when $0 = d_a < d_b < \frac{1}{2}$.

Remark 2. When $d_a = d_b$, we have

$$\gamma_{aa}(k) = \frac{2f_{u_a u_a}(0)\Gamma(1 - 2d_a)}{k^{1-2d_a}} \sin\{\pi d_a\} + O\left(\frac{1}{k^{2-2d_a}}\right), \tag{5}$$

corresponding to the leading term given in the asymptotic expansion of the autocovariance function for the scalar case in Lieberman and Phillips (2006).

We now define the standardization matrix $D_n = \mathrm{diag}(n^{d_1}, \ldots, n^{d_m})$ and the partial sum $S_t = \sum_{s=1}^{t} X_s$ and let $d = \min_{a \le m} d_a$ for the following theorem, whose proof is in the Appendix.

THEOREM 2. *If $d_a \in (0, \frac{1}{2})$ for all $a = 1, \ldots, m$, and the spectral density matrix $f_{uu}(\lambda)$ of u_t is continuously differentiable, then as $n \to \infty$*

$$\left[\frac{1}{n} D_n^{-1} E\{S_n S_n'\} D_n^{-1}\right]_{ab} \to \frac{2\pi f_{u_a u_b}(0)\Gamma(1 - d_a - d_b)\{\sin(\pi d_b) + \sin(\pi d_a)\}}{\pi(d_a + d_b)(1 + d_a + d_b)}. \tag{6}$$

https://doi.org/10.1017/S0266466607070491 Published online by Cambridge University Press

Remark 3. The diagonal elements of (6) are

$$\left[\frac{1}{n} D_n^{-1} E\{S_n S_n'\} D_n^{-1}\right]_{aa} \to \frac{2\pi f_{u_a u_a}(0)\Gamma(1-2d_a)\sin(\pi d_a)}{\pi d_a(1+2d_a)}$$

$$= \mathrm{lrvar}(u_{at}) \frac{\Gamma(1-2d_a)\sin(\pi d_a)}{\pi d_a(1+2d_a)}, \tag{7}$$

and, as $d_a \to 0$, this formula tends to $2\pi f_{uu}(0) = \mathrm{lrvar}(u_t)$, the limiting variance of the standardized partial sum $n^{-1/2}S_n = n^{-1/2}\sum_{t=1}^{n} u_t$.

Remark 4. In the scalar case with $u_t \sim iid(0, \sigma_u^2)$, Sowell (1990) showed that

$$\lim_{n\to\infty} \frac{\mathrm{var}(S_n)}{n^{1+2d}} = \sigma_u^2 \frac{\Gamma(1-2d)}{(1+2d)\Gamma(1+d)\Gamma(1-d)}. \tag{8}$$

We may compare this formula with (7). By the reflection formula for the gamma function we have $\Gamma(d)\Gamma(1-d) = \pi/\sin(\pi d)$, so that (7) may be written in the alternate form

$$\frac{2\pi f_{uu}(0)\sin\{\pi d\}\Gamma(1-2d)}{\pi d(1+2d)} = \frac{2\pi f_{uu}(0)\Gamma(1-2d)}{d(1+2d)\Gamma(d)\Gamma(1-d)}$$

$$= \frac{2\pi f_{uu}(0)\Gamma(1-2d)}{(1+2d)\Gamma(d+1)\Gamma(1-d)}, \tag{9}$$

which clearly reduces to (8) in the case of *iid* u_t.

Remark 5. Formula (7) for the asymptotic variance corresponds to that delivered by the covariance kernel of the limiting fractional Brownian motion. In particular, it is well known (e.g., Chan and Terrin, 1995; Marinucci and Robinson, 2000) that under certain regularity conditions we have the weak convergence

$$\frac{1}{n^{1/2+d_a}} \sum_{t=1}^{[nr]} X_{at} \xrightarrow{d} B_{d_a}(r).$$

It is often convenient to define the limiting fractional Brownian motions $B_{d_a}(r)$ in terms of their harmonizable representations (see Samorodnitsky and Taqqu, 1994; Chan and Terrin, 1995; Davidson and Hashimzade, 2008) as follows:

$$B_{d_a}(r) = \frac{1}{\sqrt{2\pi}} \int_R \frac{e^{i\lambda r} - 1}{i\lambda} (i\lambda)^{-d_a} \, dW_a(\lambda), \tag{10}$$

https://doi.org/10.1017/S0266466607070491 Published online by Cambridge University Press

where $\{W_a(\lambda): a = 1, \ldots, m\}$ are complex-valued Gaussian random measures satisfying

$$dW_a(\lambda) = d\overline{W_a(-\lambda)},$$

$$\mathbf{E}[dW_a(\lambda)] = 0,$$

$$\mathbf{E}[dW_a(\lambda)d\overline{W_b(\mu)}] = \begin{cases} \omega_{ab}\,d\lambda, & \lambda = \mu \\ 0, & \lambda \neq \mu \end{cases}, \quad a, b = 1, \ldots, m,$$

for $\lambda \in [-\pi, \pi]$, where $\omega_{ab} = 2\pi f_{u_a u_a}(0)$ and where the bar denotes complex conjugation. Observe that

$$E\{B_{d_a}(1)^2\} = \frac{\omega_{aa}}{2\pi}\int_R \left|\frac{e^{i\lambda}-1}{i\lambda}\right|^2 |\lambda|^{-2d_a}\,d\lambda$$

$$= \frac{\omega_{aa}}{2\pi}\int_R \frac{2-2\cos\lambda}{\lambda^{2+2d_a}}\,d\lambda = \frac{2\omega_{aa}}{\pi}\int_0^\infty \frac{1-\cos\lambda}{\lambda^{2+2d_a}}\,d\lambda$$

$$= \frac{4\omega_{aa}}{\pi}\int_0^\infty \frac{\sin^2\dfrac{\lambda}{2}}{\lambda^{2+2d_a}}\,d\lambda$$

$$= -\frac{2\omega_{aa}}{\pi}\Gamma(-1-2d_a)\cos\left(\frac{(1+2d_a)\pi}{2}\right) \tag{11}$$

$$= \frac{2\omega_{aa}}{\pi}\Gamma(-1-2d_a)\sin(d_a\pi)$$

$$= \frac{2\omega_{aa}}{\pi}\frac{\pi\sin(d_a\pi)}{\sin(\pi(2+2d_a))\Gamma(2+2d_a)}$$

$$= \frac{2\omega_{aa}\sin(d_a\pi)}{\sin(2\pi d_a)\Gamma(2+2d_a)} = \frac{\omega_{aa}}{\cos(\pi d_a)\Gamma(2+2d_a)} \tag{12}$$

using Gradshteyn and Ryzhik (2000, formula 3.823) in (11); cf. Davidson and Hashimzade (2008). Applying the reflection formula $\Gamma(1-z)\Gamma(z) = \pi/\sin(\pi z)$ with $z = 2 + 2d_a$ so that $1 - z = -1 - 2d_a$, we have

https://doi.org/10.1017/S0266466607070491 Published online by Cambridge University Press

$$\Gamma(2+2d_a)\cos\pi d_a$$

$$= \frac{\pi}{\sin(2\pi d_a + 2\pi)\Gamma(-1-2d_a)}\cos\pi d_a$$

$$= \frac{\pi}{\sin(2\pi d_a)\Gamma(-1-2d_a)}\cos\pi d_a$$

$$= \frac{\pi(-2d_a)(-1-2d_a)}{2\sin(\pi d_a)\Gamma(1-2d_a)} = \frac{\pi d_a(1+2d_a)}{\sin(\pi d_a)\Gamma(1-2d_a)}, \tag{13}$$

using $\sin(2\pi d_a) = 2\sin\pi d_a\cos\pi d_a$. It now follows from (12) and (13) that

$$E\{B_{d_a}(1)^2\} = \frac{\omega_{aa}}{\Gamma(2+2d_a)\cos(\pi d_a)} = \frac{\omega_{aa}\Gamma(1-2d_a)\sin(\pi d_a)}{\pi d_a(1+2d_a)}$$

$$= \frac{\omega_{aa}\Gamma(1-2d_a)}{d_a(1+2d_a)\Gamma(1-d_a)\Gamma(d_a)} = \frac{\omega_{aa}\Gamma(1-2d_a)}{(1+2d_a)\Gamma(1-d_a)\Gamma(1+d_a)},$$

where we use the reflection formula again in the form $\Gamma(1-d_a)\Gamma(d_a) = \pi/\sin(\pi d_a)$, leading to the stated correspondence with (7) and (9). Expression (12) was also obtained in Davidson and Hashimzade (2008, formula 2.6 with $\kappa = 1$). Similar arguments show that

$$E\{B_{d_a}(1)B_{d_b}(1)\} = \frac{\omega_{ab}\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}}{\pi(d_a+d_b)(1+d_a+d_b)},$$

corresponding to (6).

3. DISCUSSION AND APPLICATION

Some estimation procedures such as fully modified estimation in a fractional cointegration model (Kim and Phillips, 1999; Davidson, 2004) involve unknown long-run variance and covariance matrices for fractional processes such as those given in Theorem 2, which need to be estimated consistently for these procedures to be implemented. Consistent estimation of these long-run covariances can be accomplished by a stepwise procedure that involves separate consistent estimation of the memory parameters and the long-run variance matrix, $\Omega_{uu} = 2\pi f_{uu}(0)$ of u_t. The memory parameters $\{d_a : a = 1,\dots,m\}$ can be estimated by any consistent semiparametric method, such as the local Whittle (Robinson, 1995) or exact local Whittle (Shimotsu and Phillips, 2005) procedures. Using estimates $\hat{d}_a$ obtained in this way, estimates of the residuals u_{at} can be constructed by the truncated filtering operation

https://doi.org/10.1017/S0266466607070491 Published online by Cambridge University Press

$$\sum_{k=0}^{t} \hat{\pi}_k (X_{at-k} - \bar{X}_a) = \hat{u}_{at}, \tag{14}$$

where $\hat{\pi}_k = \Gamma(k - \hat{d}_a)/\{\Gamma(k+1)\Gamma(-\hat{d}_a)\}$. Using $\hat{u}_{at}$, the long-run variance matrix Ω_{uu} of u_t may then be consistently estimated by any conventional heteroskedasticity and autocorrelation consistent (HAC) procedure.[2] These estimates of Ω_{uu} and d_a may then be plugged into formulas such as those given in Theorem 2 to produce consistent estimates of the required long-run covariances of the fractional processes.

NOTES

1. This may be proved directly using a Fourier integral asymptotic expansion when the spectrum of the short-memory component is analytic.

2. Demonstration of consistency requires attention to the effect of the finite length filtering operation (14), as in Velasco (2003) and Robinson and Hualde (2003), and the use of residuals $\hat{u}_{at}$ in the estimation of Ω_{uu}, as in conventional HAC estimation.

REFERENCES

Bleistein, N. & R.A. Handelsman (1986) *Asymptotic Expansions of Integrals*. Dover.

Chan, N.H. & N. Terrin (1995) Inference for long-memory processes with application to fractional unit root autoregression. *Annals of Statistics* 23, 1662–1683.

Davidson, J. (2004) Convergence to Stochastic Integrals with Fractionally Integrated Integrator Processes: Theory and Applications to Cointegrating Regression. Mimeo, Cardiff University.

Davidson, J. & N. Hashimzade (2008) Alternative frequency and time domain versions of fractional Brownian motion. *Econometric Theory* (forthcoming).

Erdélyi, A. (1956) *Asymptotic Expansions*. Dover.

Gradshteyn, I.S. & I.M. Ryzhik (2000) *Tables of Integrals, Series and Products*, 6th ed., A. Jeffrey & D. Zwillinger (eds.). Academic Press.

Hannan, E.J. (1970) *Multiple Time Series.* Wiley.

Henry, M. & P. Zaffaroni (2003) The Long Range Dependence Paradigm for Macroeconomics and Finance. Mimeo, Columbia University.

Kim, C. & P.C.B. Phillips (1999) Fully-Modified Estimation of Fractional Cointegration Models. Mimeo, Yale University.

Lieberman, O. & P.C.B. Phillips (2006) A Complete Asymptotic Series for the Autocovariance Function of a Long Memory Process. Mimeo, Yale University.

Marinucci, D. & P.M. Robinson (2000) Weak convergence of multivariate fractional processes. *Stochastic Processes and Their Applications* 86, 103–120.

Phillips, P.C.B. (1995) Fully modified least squares and vector autoregression. *Econometrica* 63, 1023–1078.

Phillips, P.C.B. & K. Shimotsu (2004) Local Whittle estimation in nonstationary and unit root cases. *Annals of Statistics* 32, 656–692.

Robinson, P.M. (1995) Gaussian semiparametric estimation of long range dependence. *Annals of Statistics* 23, 1630–1661.

Robinson, P.M. & J. Hualde (2003) Cointegration in fractional systems with unknown integration orders. *Econometrica* 71, 1727–1766.

Samorodnitsky, G. & M.S. Taqqu (1994) *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. Chapman and Hall.

https://doi.org/10.1017/S0266466607070491 Published online by Cambridge University Press

Shimotsu, K. (2006) Gaussian Semiparametric Estimation of Multivariate Fractionally Integrated Processes. Mimeo, Queens University.
Shimotsu, K. & P.C.B. Phillips (2005) Exact local Whittle estimation of fractional integration. *Annals of Statistics* 33, 1890–1933.
Sowell, F.B. (1990) The fractional unit root distribution. *Econometrica* 58, 495–505.
Velasco, C. (2003) Gaussian semiparametric estimation of fractional cointegration. *Journal of Time Series Analysis* 24, 345–378.

APPENDIX: Proofs

Proof of Theorem 1. The derivation of (4) uses an asymptotic expansion of the Fourier inversion formula for $\Gamma_{xx}(k)$, which can be written as

$$\Gamma_{xx}(k) = \int_{-\pi}^{\pi} e^{iwk} L(e^{iw}) f_{uu}(w) L(e^{-iw})\, dw, \tag{A.1}$$

where

$$L(e^{iw}) = \operatorname{diag}\{(1 - e^{iw})^{-d_1}, \ldots, (1 - e^{iw})^{-d_m}\}.$$

In what follows, we will work with the abth element of $\Gamma_{xx}(k)$ denoted by

$$\gamma_{ab}(k) = E\{(X_{at} - EX_{at})(X_{bt+k} - EX_{bt+k})'\}.$$

Observe that $|1 - e^{iw}| = |2\sin(w/2)|$ and

$$\arg(1 - e^{iw}) = \begin{cases} (w - \pi)/2 & \text{for } 0 \le w < \pi \\ (\pi - |w|)/2 & \text{for } -\pi \le w < 0, \end{cases}$$

so that

$$1 - e^{iw} = \begin{cases} |2\sin(w/2)| e^{i(w-\pi)/2} & 0 \le w < \pi \\ |2\sin(w/2)| e^{i(\pi-|w|)/2} & -\pi \le w < 0, \end{cases}$$

and then

$$(1 - e^{iw})^{\theta} = \begin{cases} |2\sin(w/2)|^{\theta} e^{i(w-\pi)\theta/2} & 0 \le w < \pi \\ |2\sin(w/2)|^{\theta} e^{i(\pi-|w|)\theta/2} & -\pi \le w < 0 \end{cases}$$

$$= \begin{cases} |w|^{\theta} \left| \dfrac{2\sin(w/2)}{w} \right|^{\theta} e^{i(w-\pi)\theta/2} & 0 \le w < \pi \\ |w|^{\theta} \left| \dfrac{2\sin(w/2)}{w} \right|^{\theta} e^{i(\pi-|w|)\theta/2} & -\pi \le w < 0. \end{cases}$$

https://doi.org/10.1017/S0266466607070491 Published online by Cambridge University Press

It follows that for $0 \leq w < \pi$

$$
\begin{aligned}
&(1 - e^{iw})^{-d_a}(1 - e^{-iw})^{-d_b} f_{u_a u_b}(w) \\
&\quad = w^{-d_a}\left(\frac{2\sin(w/2)}{w}\right)^{-d_a} e^{-(i(w-\pi)d_a)/2} \\
&\qquad \times w^{-d_b}\left(\frac{2\sin(w/2)}{w}\right)^{-d_b} e^{-(i(\pi-w)d_b)/2} f_{u_a u_b}(w) \\
&\quad = e^{(i\pi(d_a-d_b))/2} w^{-d_a-d_b}\left(\frac{2\sin(w/2)}{w}\right)^{-d_a-d_b} e^{-(iw(d_a-d_b))/2} f_{u_a u_b}(w),
\end{aligned}
$$

and for $-\pi < w \leq 0$

$$
\begin{aligned}
&(1 - e^{iw})^{-d_a}(1 - e^{-iw})^{-d_b} f_{u_a u_b}(w) \\
&\quad = |w|^{-d_a}\left|\frac{2\sin(w/2)}{w}\right|^{-d_a} e^{-(i(w+\pi)d_a)/2} \\
&\qquad \times |w|^{-d_b}\left|\frac{2\sin(w/2)}{w}\right|^{-d_b} e^{-(i(-w-\pi)d_b)/2} \\
&\quad = e^{-(i\pi(d_a-d_b))/2}|w|^{-d_a-d_b}\left|\frac{2\sin(w/2)}{w}\right|^{-d_a-d_b} e^{-(iw(d_a-d_b))/2} f_{u_a u_b}(w).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\gamma_{ab}(k) &= \int_{-\pi}^{\pi} e^{iwk}(1 - e^{iw})^{-d_a}(1 - e^{-iw})^{-d_b} f_{u_a u_b}(w)\, dw \\
&= \left\{\int_0^{\pi} + \int_{-\pi}^{0}\right\} e^{iwk}(1 - e^{iw})^{-d_a}(1 - e^{-iw})^{-d_b} f_{u_a u_b}(w)\, dw \\
&= e^{(i\pi(d_a-d_b))/2}\int_0^{\pi} e^{iwk}\left[\left(\frac{2\sin(w/2)}{w}\right)^{-d_a-d_b} f_{u_a u_b}(w) e^{-(iw(d_a-d_b))/2}\right] w^{-d_a-d_b}\, dw \\
&\quad + e^{-(i\pi(d_a-d_b))/2}\int_{-\pi}^{0} e^{iwk}\left[\left|\frac{2\sin(w/2)}{w}\right|^{-d_a-d_b}\right. \\
&\qquad\qquad \left. \times f_{u_a u_b}(w) e^{-(iw(d_a-d_b))/2}\right] |w|^{-d_a-d_b}\, dw \\
&= e^{(i\pi(d_a-d_b))/2}\int_0^{\pi} e^{iwk}\left[\left(\frac{2\sin(w/2)}{w}\right)^{-d_a-d_b} f_{u_a u_b}(w) e^{-(iw(d_a-d_b))/2}\right] w^{-d_a-d_b}\, dw \\
&\quad + e^{-(i\pi(d_a-d_b))/2}\int_0^{\pi} e^{-iwk}\left[\left(\frac{2\sin(w/2)}{w}\right)^{-d_a-d_b}\right. \\
&\qquad\qquad \left. \times f_{u_a u_b}(-w) e^{(iw(d_a-d_b))/2}\right] w^{-d_a-d_b}\, dw \\
&= e^{(i\pi(d_a-d_b))/2}\int_0^{\pi} e^{iwk} F_{u_a u_b}(w) w^{-d_a-d_b}\, dw \\
&\quad + e^{-(i\pi(d_a-d_b))/2}\int_0^{\pi} e^{-iwk} F_{u_a u_b}(-w) w^{-d_a-d_b}\, dw, \qquad \textbf{(A.2)}
\end{aligned}
$$

where $F_{u_a u_b}(w) = ((2\sin(w/2))/w)^{-d_a-d_b} f_{u_a u_b}(w) e^{-(iw(d_a-d_b))/2} \in C^1[-\pi, \pi]$.

https://doi.org/10.1017/S0266466607070491 Published online by Cambridge University Press

When at least one of d_a or $d_b > 0$, the two integrals in (A.2) have critical points (singularities in the integrand) at the origin $w = 0$. For Fourier integrals of this type asymptotic expansions for large k were originally developed by Erdélyi (1956), a convenient reference being Bleistein and Handelsman (1986). In particular, if $F(w) \in C^{\infty}[a,b]$, and α and β are not integers, then Erdélyi's result implies that the integral

$$I(k) = \int_a^b e^{ikw}(w-a)^{\alpha-1}(b-w)^{\beta-1}F(w)\,dw \tag{A.3}$$

has the following complete asymptotic series representation as $k \to \infty$:

$$I(k) = I_a(k) + I_b(k),$$

where

$$I_a(k) \sim \sum_{n=0}^{\infty} \frac{d^n}{da^n}\{(b-a)^{\beta-1}F(a)\}\frac{\Gamma(n+\alpha)}{n!k^{n+\alpha}}e^{(\pi i/2)(n+\alpha)+ika} \tag{A.4}$$

and

$$I_b(k) \sim \sum_{n=0}^{\infty} \frac{d^n}{db^n}\{(b-a)^{\alpha-1}F(b)\}\frac{\Gamma(n+\beta)}{n!k^{n+\beta}}e^{(\pi i/2)(n-\beta)+ikb}. \tag{A.5}$$

These expansions hold to the first term $n = 0$ provided $F(w) \in C^1[a,b]$, and this degree of smoothness is all that is required for the present application.

Specializing the expansion of (A.3) to the present case, we first consider the integral

$$\int_0^{\pi} e^{iwk}F_{u_a u_b}(w)w^{-d_a-d_b}\,dw$$

and set $a = 0$, $b = \pi$, $\beta = 1$, and $\alpha = 1 - d_a - d_b$ in formula (A.4). We deduce that

$$\begin{aligned}\int_0^{\pi} e^{ikw}w^{\alpha-1}F_{u_a u_b}(w)\,dw &= \sum_{n=0}^{\infty}\frac{d^n}{da^n}\{F_{u_a u_b}(a)\}_{a=0}\frac{\Gamma(n+\alpha)}{n!k^{n+\alpha}}e^{(\pi i/2)(n+\alpha)+ika} \\ &= \frac{\Gamma(1-d_a-d_b)e^{(\pi i/2)(1-d_a-d_b)}}{k^{1-d_a-d_b}}F_{u_a u_b}(0) + O\left(\frac{1}{k^{2-d_a-d_b}}\right).\end{aligned} \tag{A.6}$$

https://doi.org/10.1017/S0266466607070491 Published online by Cambridge University Press

Using the same settings $a = 0$, $b = \pi$, $\beta = 1$, and $\alpha = 1 - d_a - d_b$ again, we next find that

$$\begin{aligned}
&\int_0^\pi e^{-iwk} F_{u_a u_b}(-w) w^{-d_a-d_b}\, dw \\
&\quad = \sum_{n=0}^{\infty} \frac{d^n}{da^n} \{F_{u_a u_b}(a)\}_{a=0} \frac{\Gamma(n+\alpha)}{n!(-k)^{n+\alpha}} e^{(\pi i/2)(n+\alpha)+ika} \\
&\quad = \frac{\Gamma(1-d_a-d_b) e^{(\pi i/2)(1-d_a-d_b)}}{k^{1-d_a-d_b} e^{\pi i(1-d_a-d_b)}} F_{u_a u_b}(0) + O\left(\frac{1}{k^{2-d_a-d_b}}\right) \\
&\quad = \frac{\Gamma(1-d_a-d_b) e^{-(\pi i/2)(1-d_a-d_b)}}{k^{1-d_a-d_b}} F_{u_a u_b}(0) + O\left(\frac{1}{k^{2-d_a-d_b}}\right). \qquad \textbf{(A.7)}
\end{aligned}$$

Combining results (A.6) and (A.7) in (A.2) we obtain

$$\begin{aligned}
\gamma_{ab}(k) &= e^{(i\pi(d_a-d_b))/2} \int_0^\pi e^{iwk} F_{u_a u_b}(w) w^{-d_a-d_b}\, dw \\
&\quad + e^{-(i\pi(d_a-d_b))/2} \int_0^\pi e^{-iwk} F_{u_a u_b}(-w) w^{-d_a-d_b}\, dw \\
&= e^{(i\pi(d_a-d_b))/2} \frac{\Gamma(1-d_a-d_b) e^{(\pi i/2)(1-d_a-d_b)}}{k^{1-d_a-d_b}} F_{u_a u_b}(0) \\
&\quad + e^{-(i\pi(d_a-d_b))/2} \frac{\Gamma(1-d_a-d_b) e^{-(\pi i/2)(1-d_a-d_b)}}{k^{1-d_a-d_b}} F_{u_a u_b}(0) + O\left(\frac{1}{k^{2-d_a-d_b}}\right) \\
&= \frac{\Gamma(1-d_a-d_b)}{k^{1-d_a-d_b}} F_{u_a u_b}(0) \{e^{(\pi i/2)(1-2d_b)} + e^{-(\pi i/2)(1-2d_b)}\} + O\left(\frac{1}{k^{2-d_a-d_b}}\right) \\
&= \frac{2\Gamma(1-d_a-d_b)}{k^{1-d_a-d_b}} F_{u_a u_b}(0) \cos\left\{\frac{\pi}{2}(1-2d_b)\right\} + O\left(\frac{1}{k^{2-d_a-d_b}}\right) \\
&= \frac{2\Gamma(1-d_a-d_b)}{k^{1-d_a-d_b}} F_{u_a u_b}(0) \sin\{\pi d_b\} + O\left(\frac{1}{k^{2-d_a-d_b}}\right),
\end{aligned}$$

which gives the stated result because $F_{u_a u_b}(0) = f_{u_a u_b}(0)$. ■

Proof of Theorem 2. From Theorem 1, as $k \to \infty$

$$\gamma_{ab}(k) = \frac{2 f_{u_a u_b}(0) \Gamma(1-d_a-d_b) \sin(\pi d_b)}{k^{1-d_a-d_b}} + O\left(\frac{1}{k^{2-d_a-d_b}}\right). \qquad \textbf{(A.8)}$$

https://doi.org/10.1017/S0266466607070491 Published online by Cambridge University Press

Correspondingly, as $k \to -\infty$, we have, because $\gamma_{ab}(k) = \gamma_{ba}(-k)$,

$$\gamma_{ab}(k) = \frac{2f_{u_b u_a}(0)\Gamma(1-d_a-d_b)\sin(\pi d_a)}{|k|^{1-d_a-d_b}} + O\left(\frac{1}{k^{2-d_a-d_b}}\right). \tag{A.9}$$

Then,

$$\begin{aligned}
&\frac{1}{n} D_n^{-1} E\{S_n S_n'\} D_n^{-1} \\
&\quad = \frac{1}{n} D_n^{-1} E\left(\sum_{t=1}^{n} X_t\right)\left(\sum_{t=1}^{n} X_t\right)' D_n^{-1} = \frac{1}{n} D_n^{-1} \sum_{t,s=1}^{n} (EX_t X_s') D_n^{-1} \\
&\quad = \frac{1}{n} D_n^{-1} \sum_{t,s=1}^{n} \Gamma_{xx}(s-t) D_n^{-1} = D_n^{-1} \sum_{h=-n+1}^{n-1} \left(1 - \frac{|h|}{n}\right)\Gamma_{xx}(h) D_n^{-1} \\
&\quad = D_n^{-1} \sum_{h=-n+1}^{n-1} \Gamma_{xx}(h) D_n^{-1} - n^{-1} D_n^{-1} \sum_{h=-n+1}^{n-1} |h| \Gamma_{xx}(h) D_n^{-1}.
\end{aligned} \tag{A.10}$$

For some L such that $(1/L) + (L/n^{2d}) \to 0$ as $n \to \infty$, we decompose the first sum in (A.10) as follows:

$$\begin{aligned}
D_n^{-1} \sum_{h=-n+1}^{n-1} \Gamma_{xx}(h) D_n^{-1} &= D_n^{-1}\left[\sum_{-L+1}^{L-1} \Gamma_{xx}(h) + \sum_{L}^{n-1} \Gamma_{xx}(h) + \sum_{-n+1}^{-L} \Gamma_{xx}(h)\right] D_n^{-1} \\
&= D_n^{-1}\left[\sum_{L}^{n-1} \Gamma_{xx}(h) + \sum_{-n+1}^{-L} \Gamma_{xx}(h)\right] D_n^{-1} + o(1).
\end{aligned}$$

The abth element of this matrix is

$$\begin{aligned}
&\frac{1}{n^{d_a+d_b}}\left[\sum_{L}^{n-1} \gamma_{ab}(h) + \sum_{-n+1}^{-L} \gamma_{ab}(h)\right] + o(1) \\
&\quad = \frac{1}{n^{d_a+d_b}}\left[\sum_{L}^{n-1} \gamma_{ab}(h) + \sum_{-n+1}^{-L} \gamma_{ba}(-h)\right] + o(1) \\
&\quad = \frac{1}{n^{d_a+d_b}} \sum_{L}^{n-1} \{\gamma_{ab}(h) + \gamma_{ba}(h)\} + o(1) \\
&\quad = \frac{1}{n^{d_a+d_b}} \sum_{L}^{n-1}\left[\frac{2f_{u_a u_b}(0)\Gamma(1-d_a-d_b)\{\sin(\pi d_b) + \sin(\pi d_a)\}}{h^{1-d_a-d_b}}\right. \\
&\qquad \left. + O\left(\frac{1}{h^{2-d_a-d_b}}\right)\right] + o(1)
\end{aligned}$$

https://doi.org/10.1017/S0266466607070491 Published online by Cambridge University Press

$$= 2f_{u_a u_b}(0)\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}\frac{1}{n^{d_a+d_b}}\sum_1^n \frac{1}{h^{1-d_a-d_b}} + o(1)$$

$$= 2f_{u_a u_b}(0)\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}\frac{1}{n^{d_a+d_b}}\int_1^n \frac{dh}{h^{1-d_a-d_b}} + o(1)$$

$$= 2f_{u_a u_b}(0)\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}\frac{1}{n^{d_a+d_b}}\left[\frac{h^{d_a+d_b}}{d_a+d_b}\right]_1^n + o(1)$$

$$= \frac{2f_{u_a u_b}(0)\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}}{d_a+d_b} + o(1), \tag{A.11}$$

by Euler summation. It follows that

$$\left[D_n^{-1}\sum_{h=-n+1}^{n-1}\Gamma_{xx}(h)D_n^{-1}\right]_{ab} \to \frac{2f_{u_a u_b}(0)\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}}{d_a+d_b}.$$

Next consider the second sum in (A.10). The abth element is

$$\frac{1}{n^{1+d_a+d_b}}\sum_{h=-n+1}^{n-1}|h|\gamma_{ab}(h)$$

$$= \frac{1}{n^{1+d_a+d_b}}\sum_{h=-L+1}^{L-1}|h|\gamma_{ab}(h) + \frac{1}{n^{1+d_a+d_b}}\sum_{h=L}^{n-1}h\gamma_{ab}(h) + \frac{1}{n^{1+d_a+d_b}}\sum_{-n+1}^{-L}|h|\gamma_{ab}(h)$$

$$= \frac{1}{n^{1+d_a+d_b}}\sum_{h=-L+1}^{L-1}|h|\gamma_{ab}(h) + \frac{1}{n^{1+d_a+d_b}}\sum_{h=L}^{n-1}h\{\gamma_{ab}(h)+\gamma_{ba}(h)\}$$

$$= \frac{1}{n^{1+d_a+d_b}}\sum_{h=L}^{n-1}h\left\{\frac{2f_{u_a u_b}(0)\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}}{h^{1-d_a-d_b}} + O\left(\frac{1}{h^{2-d_a-d_b}}\right)\right\} + o(1)$$

$$= \frac{1}{n^{1+d_a+d_b}}\sum_{h=1}^{n-1}\left\{\frac{2f_{u_a u_b}(0)\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}}{h^{-d_a-d_b}}\right\} + o(1)$$

$$= 2f_{u_a u_b}(0)\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}\frac{1}{n^{1+d_a+d_b}}\int_1^n h^{d_a+d_b}\,dh + o(1)$$

$$= \frac{2f_{u_a u_b}(0)\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}}{1+d_a+d_b} + o(1), \tag{A.12}$$

by Euler summation again.

https://doi.org/10.1017/S0266466607070491 Published online by Cambridge University Press

We now combine (A.11) and (A.12) in (A.10) giving, as $n \to \infty$,

$$\left[\frac{1}{n} D_n^{-1} E\{S_n S_n'\} D_n^{-1}\right]_{ab}$$

$$= \frac{2f_{u_a u_b}(0)\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}}{d_a+d_b}$$

$$- \frac{2f_{u_a u_b}(0)\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}}{1+d_a+d_b} + o(1)$$

$$= \frac{2f_{u_a u_b}(0)\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}}{(d_a+d_b)(1+d_a+d_b)} + o(1)$$

$$\to \frac{2f_{u_a u_b}(0)\Gamma(1-d_a-d_b)\{\sin(\pi d_b)+\sin(\pi d_a)\}}{(d_a+d_b)(1+d_a+d_b)},$$

which corresponds to the stated result. ■

https://doi.org/10.1017/S0266466607070491 Published online by Cambridge University Press