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A NOTE ON FRAGMENTS OF UNIFORM REFLECTION IN SECOND ORDER ARITHMETIC

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Abstract. We consider fragments of uniform reflection for formulas in the analytic hierarchy over theories of second order arithmetic. The main result is that for any second order arithmetic theory T_0 extending RCA_0 and axiomatizable by a Π^1_{k+2} sentence, and for any $n \geq k+1$,

$$T_0 + \mathrm{RFN}_{\Pi^1_{n+2}}(T) \;=\; T_0 + \mathrm{TI}_{\Pi^1_n}(\varepsilon_0),$$

$$T_0 + \mathrm{RFN}_{\Sigma^1_{n+1}}(T) \;=\; T_0 + \mathrm{TI}_{\Pi^1_n}(\varepsilon_0)^-,$$

where T is T_0 augmented with full induction, and $\mathrm{TI}_{\Pi^1_n}(\varepsilon_0)^-$ denotes the schema of transfinite induction up to ε_0 for Π^1_n formulas without set parameters.

§1. Introduction. The following benchmark results showcase the relationship between uniform reflection and induction in the context of first order arithmetic:

$$\mathsf{EA} + \mathrm{RFN}(\mathsf{EA}) \;=\; \mathsf{PA}, \tag{1}$$

where EA is Kalmár elementary arithmetic, and

$$\mathsf{PA} + \mathrm{RFN}(\mathsf{PA}) \;=\; \mathsf{PA} + \mathrm{TI}(\varepsilon_0). \tag{2}$$

Equation (1) is usually attributed to [12]. Equation (2) is a special case of [12, Theorem 12].

This note is concerned with uniform reflection in the context of second order arithmetic.[1] It can be gleaned from Kreisel and Lévy [12, Theorem 14] that

$$\mathsf{RCA} + \mathrm{RFN}(\mathsf{RCA}) \;=\; \mathsf{RCA} + \mathrm{TI}(\varepsilon_0),$$

where RCA is RCA_0 together with full induction.[2]

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[1] For an extensive study of iterated uniform reflection in second order arithmetic, see [16, 17].

[2] Kreisel and Lévy prove the result for a slightly different system, denoted Z_1, which appears in [10]. The theory Z_1 consists of number and function variables, axioms for the constants zero, successor, pairing, projections, axioms expressing the closure of all functions under

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The same proof-theoretic methods of [12], namely, formalized cut elimination, show that Equations (1) and (2) lift up to any theory T of second order arithmetic of the form T_0 together with full induction, where T_0 is any finitely axiomatizable extension of RCA_0. We thus have (see Theorem 3.1)

$$T_0 + \mathrm{RFN}(T_0) \;=\; T, \tag{3}$$

$$T_0 + \mathrm{RFN}(T) \;=\; T_0 + \mathrm{TI}(\varepsilon_0). \tag{4}$$

More in general (cf. [12, Theorem 12]),

$$T_0 + \mathrm{RFN}(T + \mathrm{TI}(\alpha)) \;=\; T_0 + \mathrm{TI}(\varepsilon(\alpha)),$$

where $\varepsilon(\alpha)$ is the least ε-number beyond α. The schema $\mathrm{TI}(\alpha)$ of transfinite induction along α consists of formulas

$$\forall x\,(\forall y \prec x\, \varphi(y) \to \varphi(x)) \to \forall x\, \varphi(x),$$

where $\varphi(x)$ ranges over all formulas and $\prec$ is a primitive recursive ordinal notation system for α.

Anyone familiar with (a bit of) ordinal analysis will readily realize that (3) and (4) hold because of the unlimited amount of induction, respectively transfinite induction up to ε_0, available. The point we are trying to make is that the so-called Π^1_1 proof-theoretic ordinals of T_0 and T may be way bigger than ε_0. Recall that the Π^1_1 proof-theoretic ordinal $|S|$ of a second order arithmetic theory S can be defined as the supremum of the provably well-founded ordinals of S. These are the ordinals α such that $S \vdash \mathrm{WF}(\alpha)$, where $\mathrm{WF}(\alpha)$ expresses the well-foundedness of α and corresponds to the following single instance of transfinite induction:

$$\forall X\,(\forall \beta\,(\forall \gamma < \beta\,(\gamma \in X) \to \beta \in X) \to \forall \beta < \alpha\,(\beta \in X)).$$

Actually, one always works with a given primitive recursive ordinal notation system, so that the above definition is relative to the choice of such notation system (see [18] for an overview). Now, under certain conditions (cf. [18, Proposition 2.13]), $|S|$ coincides with the least ordinal α such that

$$S + \mathrm{WF}(\alpha) \vdash \mathrm{Con}(S).$$

Again, the above definition refers to a given ordinal notation system. In particular, for theories of great proof-theoretic strength, it is certainly not the case that $T_0 + \mathrm{WF}(\varepsilon_0) \vdash \mathrm{Con}(T_0)$, let alone $T_0 + \mathrm{WF}(\varepsilon_0) \vdash \mathrm{RFN}(T)$. So much for that.

A more subtle point is that the restriction to finitely axiomatizable theories is a necessary one, as the result does not apply to recursively enumerable theories T_0 of bounded complexity. The assumption plays a role in the direction from (transfinite) induction to reflection. The reason, roughly speaking, is that an infinite theory may not prove that all its axioms are true. In fact, for sufficiently strong theories, such condition entails finite

composition and primitive recursion, and full induction (induction for all formulas in the language).

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axiomatizability. Note that main subsystems of reverse mathematics, the so-called *Big Five*, are finitely axiomatizable [21]. We refer to Section 4 for a discussion of this issue.

In this note we obtain the following refinement of (3) and (4). We show in Theorem 5.1 that if T_0 is any theory of second order arithmetic extending RCA_0 and axiomatized by a Π^1_2 (in general, Π^1_{k+2}) sentence, T is T_0 plus full induction, and $n \geq 1$ (in general, $n \geq k+1$), then over T_0

$$\mathrm{RFN}_{\Pi^1_{n+2}}(T_0) \;=\; \mathrm{I}\Pi^1_n \;\supseteq\; (\mathrm{I}\Pi^1_n)^- \;=\; \mathrm{RFN}_{\Sigma^1_{n+1}}(T_0), \tag{5}$$

$$\mathrm{RFN}_{\Pi^1_{n+2}}(T) \;=\; \mathrm{TI}_{\Pi^1_n}(\varepsilon_0) \;\supseteq\; \mathrm{TI}_{\Pi^1_n}(\varepsilon_0)^- \;=\; \mathrm{RFN}_{\Sigma^1_{n+1}}(T). \tag{6}$$

The superscript $^-$ refers to the restriction of the pertaining schema to formulas without set parameters. We prove in Theorem 5.3 that, under certain conditions, the inclusions in (5) and (6) are strict. This is obtained by a standard application of Gödel's second incompleteness theorem.

In first order arithmetic we have a neat correspondence between uniform reflection and induction (see [13, 15]): for all $n \geq 1$,

$$\mathsf{EA} + \mathrm{RFN}_{\Pi_{n+2}}(\mathsf{EA}) \;=\; \mathsf{EA} + \mathrm{RFN}_{\Sigma_{n+1}}(\mathsf{EA}) \;=\; \mathsf{EA} + \mathrm{I}\Pi_n.$$

If anything, we can say that a less clear-cut picture emerges when we move to second order arithmetic.[3] On the other hand (cf. [17, Section 3.1]), one could recover the equivalence between Π^1_{n+2} and Σ^1_{n+1} uniform reflection by considering a relativized version of uniform reflection with set parameters. More precisely, given a set X of natural numbers and a second order arithmetic theory T, let $T(X)$ be the extension of T with *oracle axioms* $\bar{n} \in \bar{X}$ for $n \in X$ and $\bar{n} \notin \bar{X}$ for $n \notin X$, where $\bar{X}$ is a new set constant. Note that this construction makes sense in RCA_0 since the theory $T(X)$ is recursive in T and X. One can then introduce the schema $\mathrm{RFN}^*(T)$ given by formulas

$$\forall x\, \forall X\, (\mathrm{Pr}_{T(X)}(\ulcorner \varphi(\dot{x}, \dot{X}) \urcorner) \to \varphi(x, X)).$$

Now, the proofs of Theorems 3.1 and 5.1 go through *mutatis mutandis*. Note that

$$\mathrm{Pr}_T(\ulcorner \forall X\, \varphi(X) \urcorner) \to \forall X\, \mathrm{Pr}_{T(X)}(\ulcorner \varphi(\dot{X}) \urcorner)$$

is provable in RCA_0. In particular, one obtains

$$T_0 + \mathrm{RFN}^*_{\Pi^1_{n+2}}(T_0) = \; T_0 + \mathrm{I}\Pi^1_n \;=\; T_0 + \mathrm{RFN}^*_{\Sigma^1_{n+1}}(T_0),$$
$$T_0 + \mathrm{RFN}^*_{\Pi^1_{n+2}}(T) = \; T_0 + \mathrm{TI}_{\Pi^1_n}(\varepsilon_0) \;=\; T_0 + \mathrm{RFN}^*_{\Sigma^1_{n+1}}(T),$$

where T_0, T, and n are as above.

[3] It should be noted that parameter free versions of induction in first order arithmetic, considered, e.g., in [11] from a model-theoretic point of view, can be characterized by using *relativized* forms of local reflection (see [3, Section 3]).

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§2. Basics.

2.1. First order arithmetic. For the purposes of this paper, let the language of first order arithmetic consist of finitely many symbols including $0, 1, +, \times, =, <$. As usual, $\Delta_0 = \Pi_0 = \Sigma_0$ denotes the class of formulas built up from atomic formulas by means of Boolean connectives and bounded quantifiers of the form $\exists x < t$ and $\forall x < t$, where t is a term. The arithmetic hierarchy for $n > 0$ is thus defined: $\Pi_{n+1} = \{\forall x\, \varphi(x) \mid \varphi(x) \in \Sigma_n\}$ and dually $\Sigma_{n+1} = \{\exists x\, \varphi(x) \colon \varphi(x) \in \Pi_n\}$.

2.2. Reflection. We assume a primitive recursive Gödel numbering $\ulcorner \varphi \urcorner$ of formulas. The uniform reflection principle $\mathrm{RFN}(T)$ for a theory T is a schema consisting of formulas

$$\forall x\, (\mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \varphi(x)),$$

where $\varphi(x)$ is a formula with at most the displayed free variable and $\mathrm{Pr}_T(y)$ is a canonical provability predicate for T (cf. [5, 6]). The formula $\mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner)$ can be seen as an abbreviation of

$$\exists y\, (\mathrm{Pr}_T(y) \wedge \sigma(\ulcorner \varphi \urcorner, x, y)).$$

Here, $\sigma(u, x, y)$ is a Σ_1 formula defining the primitive recursive operation $(\ulcorner \varphi \urcorner, n) \mapsto \ulcorner \varphi(\bar{n}/v) \urcorner$, where v is the only free variable of φ and $\varphi(\bar{n}/v)$ denotes the substitution of the numeral $\bar{n}$ for v in φ. For ease of notation, we omit the overline notation for numerals when writing $\ulcorner \varphi \urcorner$ inside a formula.

2.3. Second order arithmetic. The language of second order arithmetic is two-sorted, with first order variables $x, y, z, \ldots$ intended to range over natural numbers, and second order variables $X, Y, Z, \ldots$ intended to range over sets of natural numbers. It is obtained from the language of first order arithmetic by adding a membership relation symbol $\in$ connecting the two sorts. The equality relation symbol $=$ only applies to first order terms.

The classes of formulas Δ^0_0, Π^0_n, and Σ^0_n are defined just as in the first order case, but now set parameters are allowed. In this context, a formula is arithmetic if it does not contain set quantifiers, that is, quantifications of the form $\forall X$ and $\exists X$. Note that an arithmetic formula may contain free set variables. A formula is Π^1_n if it is of the form $\forall X_1\, \exists X_2 \cdots Q X_n\, \varphi$, where φ is arithmetic. The definition of Σ^1_n formulas is dual.

The theory known as (full) second order arithmetic, often denoted Z_2, is given by the usual first order axioms for zero, successor, addition, multiplication, and less than relation, together with comprehension and induction schemata

$$\exists X\, \forall x\, (x \in X \leftrightarrow \varphi(x)), \qquad \text{(Comprehension)}$$

$$\varphi(0) \wedge \forall x\, (\varphi(x) \to \varphi(x+1)) \to \forall x\, \varphi(x), \qquad \text{(Induction)}$$

where φ ranges over all formulas. Note that every instance of induction can be obtained from comprehension and the induction axiom

$$0 \in X \wedge \forall x\, (x \in X \to x + 1 \in X) \to \forall x\, (x \in X).$$

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The main subsystems of reverse mathematics [21], also known as the Big Five, are obtained by prescribing how much induction one can use and what sets one can form (in terms of comprehension or set existence axioms). These are RCA_0 (Recursive Comprehension Axiom), WKL_0 (Weak König's Lemma), ACA_0 (Arithmetical Comprehension Axiom), ATR_0 (Arithmetical Transfinite Recursion), and Π^1_1-CA_0 (Comprehension for Π^1_1 formulas). RCA_0, the standard base theory of reverse mathematics, consists of Δ^0_1 comprehension and Σ^0_1 induction. It is well known that RCA_0 is a conservative extension of $\mathrm{I}\Sigma_1$, that is, Peano arithmetic PA with induction restricted to Σ_1 formulas. The theory ACA_0 consists of comprehension and induction for all arithmetic formulas.

2.4. Partial truth definitions. In first order arithmetic one has access for any $n \geq 1$ to a Σ_n (resp. Π_n) partial truth definition for the class Σ_n (resp. Π_n) over $\mathrm{I}\Sigma_1$. The construction involves a partial truth definition for Δ_0 formulas, of complexity Δ_1 over $\mathrm{I}\Sigma_1$ (cf. [9, Chapter 1, Section 1(d)]). By the same token, one can construct partial truth definitions for formulas of bounded complexity, say of bounded rank, over $\mathrm{I}\Sigma_1$ (cf. [22, Chapter 1, Section 5.4]).

In the context of second order arithmetic, one can define truth predicates for arithmetic formulas and Π^1_n formulas but they fail to be provably so over weak fragments such as RCA_0. However, for the purposes of this paper, it will be sufficient to work with partial truth definitions for formulas of bounded complexity. Let R_m denote the class of formulas of rank at most m and let $P_{n,m}$ denote the (closure under subformulas of the) class of Π^1_n formulas with arithmetic matrix of rank at most m. We define the rank as usual: the rank is a natural number and the rank of a compound formula is bigger than the rank of its components. Notice that we make no distinction between number and set quantifiers. The construction of partial truth definitions in first order arithmetic easily extends to arithmetic formulas with set parameters. In particular, for any choice of n and m, one obtains a partial truth predicate for R_m and a Π^1_n truth predicate for $P_{n,m}$ formulas over RCA_0.

We will use partial truth definitions of the first kind (for R_m) in the proof of Theorem 3.1. In Theorem 5.1 we will choose a Π^1_n partial truth definition for sufficiently many Π^1_n formulas, namely, for a sufficiently large class of the form $P_{n,m}$.

At one point (see Lemma 5.2) we will use the fact that Π^1_n (resp. Σ^1_n) uniform reflection can be finitely axiomatized over ACA_0 for each $n \geq 1$. This can be attained by means of universal formulas. We say that a Π^1_n formula $\varphi_U(e, x)$ with displayed free variables is universal Π^1_n in a theory T if for all Π^1_n formulas $\varphi(x)$, T proves $\forall x\,(\varphi(x) \leftrightarrow \varphi_U(\ulcorner\varphi\urcorner, x))$. Universal Σ^1_n formulas are defined in the exact same way. The existence in ACA_0 of universal Π^1_n and Σ^1_n formulas for every $n \geq 1$ is basically folklore. There is some leeway in designing universal formulas. For example, one possible route involves the following two steps in the construction of a universal Π^1_1 formula with set parameters; it is then routine to build up universal Π^1_n and Σ^1_n formulas for all $n \geq 1$. First, given a Σ^0_1 formula $\varphi(x, f, X_1, \ldots, X_k)$, where f denotes a function from ω to ω, one can primitive recursively find

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an index e such that

$$\varphi(x, f, X_1, \dots, X_k) \leftrightarrow \{e\}^{f \oplus X_1 \oplus \cdots \oplus X_k}(x) \downarrow$$
$$\leftrightarrow \exists s\, \vartheta(e, x, s, \bar{f}(s), \bar{X}_1(s), \dots, \bar{X}_k(s)),$$

where $\vartheta(e, x, s, z)$ is a fixed Δ_0^0 formula, $\bar{f}(s)$ encodes the sequence $\langle f(0), \dots, f(s-1)\rangle$, and $\bar{X}_i(s)$ encodes the first s bits of the characteristic function of X_i. This version of the enumeration theorem can be formalized in RCA_0 (cf. [21, Theorem II.2.7, p. 68]). Second, given a Π_1^1 formula $\varphi(x, X_1, \dots, X_k)$, one can primitive recursively construct a Σ_1^0 formula $\varphi_0(x, f, X_1, \dots, X_k)$ such that $\varphi(x, X_1, \dots, X_k) \leftrightarrow \forall f\, \varphi_0(x, f, X_1, \dots, X_k)$. This is Kleene normal form theorem for Π_1^1 formulas and can be formalized in ACA_0 (cf. [21, Lemma V.1.4, p. 169]). Another approach (cf. [16, Section 2.1]) is to further formalize the second step with the help of a partial truth definition for Σ_1^0 formulas.

We refer the reader to [16, Section 2.1] for more background on partial truth predicates in second order arithmetic.

§3. The uniform reflection principle in analysis. In this section we outline a proof of the following theorem.

THEOREM 3.1 [12]. *Let $T_0 \supseteq \mathsf{RCA}_0$ be a finitely axiomatizable second order arithmetic theory. Let T be T_0 plus the schema of full induction. Then*

$$T_0 + \mathrm{RFN}(T_0) = T, \tag{3}$$

$$T_0 + \mathrm{RFN}(T) = T_0 + \mathrm{TI}(\varepsilon_0). \tag{4}$$

PROOF. (3) For the forward direction note that for every standard n the formula

$$\varphi(0) \wedge \forall x\, (\varphi(x) \to \varphi(x+1)) \to \varphi(\bar{n})$$

is provable in classical logic, and hence in any theory whatsoever. The construction of such a proof is primitive recursive (indeed elementary recursive) in n. This can be formalized in RCA_0. Therefore

$$T_0 \vdash \forall x\, \mathrm{Pr}_{T_0}(\ulcorner \varphi(0) \wedge \forall x\, (\varphi(x) \to \varphi(x+1)) \to \varphi(\dot{x}) \urcorner).$$

It follows that

$$T_0 + \mathrm{RFN}(T_0) \vdash \forall x\, (\varphi(0) \wedge \forall x\, (\varphi(x) \to \varphi(x+1)) \to \varphi(x)),$$

which is classically equivalent to induction for φ.

For the other direction, we use cut elimination for classical logic, which is available in RCA_0. More in detail, fix a sentence ψ axiomatizing T_0 and let $\varphi(x)$ be given. We aim to show

$$T \vdash \mathrm{Pr}_{T_0}(\ulcorner \varphi(\dot{x}) \urcorner) \to \varphi(x).$$

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For convenience, let us consider a Tait style sequent calculus for classical logic.[4] We reason informally in T. If $\varphi(\bar{n})$ is provable in T_0, then there is a finite cut-free proof of the sequent $\neg\psi, \varphi(\bar{n})$. By induction on the height of the proof, one shows that every sequent in the proof is true. Now, ψ is true, and so $\varphi(\bar{n})$ must be true. The induction argument can be formalized in RCA with the aid of a partial truth predicate for R_m such that both $\neg\psi$ and $\varphi(x)$ are in R_m. Note in fact that by the subformula property a cut-free proof consists only of subformulas of the end sequent.

(4) The proof that $T_0 + \mathrm{RFN}(T)$ entails $\mathrm{TI}(\varepsilon_0)$ is based on a straightforward generalization of Gentzen's lifting result [8, p. 293] (i.e., the closure of provable transfinite induction under $\alpha \mapsto \omega^\alpha$ over PA). Let $\mathrm{Prog}(\varphi)$ be a shorthand for $\forall x\,(\forall y \prec x\, \varphi(y) \to \varphi(x))$. Then for every formula $\varphi(x)$ and for every standard number k, T proves

$$\mathrm{Prog}(\varphi) \to \forall x \prec \omega_k\, \varphi(x),$$

where $\omega_0 = 1$ and $\omega_{k+1} = \omega^{\omega_k}$. The construction of such a proof is uniform and gives a primitive recursive function that accepts a Π^1_n formula $\varphi(x)$ and a standard number $k > 0$ as inputs and outputs a proof in RCA_0 plus induction for Π^1_{n+k} formulas of $\mathrm{Prog}(\varphi) \to \forall x \prec \omega_{k+1}\, \varphi(x)$. This formalizes in RCA_0. The upshot is that

$$T_0 \vdash \forall z\, \mathrm{Pr}_T(\ulcorner \mathrm{Prog}(\varphi) \to \forall x \prec \omega_{\dot{z}}\, \varphi(x) \urcorner).$$

Therefore

$$T_0 + \mathrm{RFN}(T) \vdash \forall z\, (\mathrm{Prog}(\varphi) \to \forall x \prec \omega_z\, \varphi(x)).$$

The conclusion follows.

The converse direction (cf. the proof of [12, Theorem 14]) is based on the fact that in a suitable infinitary system for ω-arithmetic one can eliminate cuts and then prove by transfinite induction on ε_0 that every sequent in a cut-free ω-proof is true.

More in detail, let ψ be a sentence axiomatizing T_0. Given a formula φ, we aim to show

$$T_0 + \mathrm{TI}(\varepsilon_0) \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \varphi(x).$$

Again, let us consider a two-sorted extension of a finitary Tait sequent calculus for PA, call it PA^2, and its infinitary counterpart with the ω-rule, call it PA^2_∞.[5] We reason informally in $T_0 + \mathrm{TI}(\varepsilon_0)$. If T proves $\varphi(\bar{n})$, then there is a proof of the sequent $\neg\psi, \varphi(\bar{n})$ in PA^2. We first convert such a finite proof into an ω-proof of $\neg\psi, \varphi(\bar{n})$ in PA^2_∞ of height $< \omega \cdot 2$ and cut formulas

[4] Tait one-sided calculi for first order classical logic (with equality) and cut elimination thereof are presented in, e.g., [23, Section 3.6] and [1, Sections 1.2, 2.1.2, and 3.1.1]. Here, of course, we are dealing with a two-sorted variant of any such calculus.

[5] Tait one-sided calculi for PA and its infinitary counterpart, call it PA_∞, are described in [2, 4]. One obtains a two-sorted extension by simply adding the appropriate axioms and set quantifier rules to such calculi (cf. [17, Section 3.1]).

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of rank $<r$, for some natural number r.[6] By applying full cut elimination, we then obtain a cut-free ω-proof of height $<\varepsilon_0$. We can now show that every sequent in the proof is true by transfinite induction on ε_0. As before, since ψ is true, $\varphi(\bar{n})$ must be true. This concludes the proof.

Let us notice that the embedding of PA into an infinitary system with the ω-rule, call it PA_∞, and cut elimination thereof (see, e.g., [19, Theorems 22.8 and 28.5]) extends to PA^2 and PA^2_∞ in a straightforward manner. On the other hand, it is by now clear how to formalize all this in, e.g., RCA_0 by considering ω-proofs with ordinal tags below ε_0.[7] In particular, ω-proofs can be directly represented as infinite trees (see [7, 14]) or suitably coded by numbers (see [4, 20]). What matters is that being (a code of) an ω-proof of height less than ε_0 can be expressed in an arithmetic way, whether this is a property about sets or about numbers (it can be Π^0_1 [20] or even primitive recursive [4]). We point out that for the purposes of cut elimination a key element of any such coding appears to be Mint's repetition rule [14]. Finally, the argument by transfinite induction can be formalized by using a partial truth predicate as before, thanks to the fact that cut-free proofs (provably) enjoy the subformula property. ⊣

§4. Some remarks. The proof of uniform reflection from (transfinite) induction relies on partial truth definitions and the assumption that T_0 is finitely axiomatizable. We have already addressed in Section 2 how to deal with partial truth definitions over RCA_0. With regard to finite axiomatizability, it is clear that any sufficiently strong finite theory can prove its own truth. In fact, if $T_0 \supseteq \mathsf{RCA}_0$ is axiomatized by a sentence ψ, then

$$T_0 \vdash \mathrm{Ax}_{T_0}(x) \to \mathrm{Tr}(x),$$

by letting $\mathrm{Ax}_{T_0}(x)$ be $x = \ulcorner\psi\urcorner$, where $\mathrm{Tr}(x)$ is a partial truth definition for a sufficiently large class of sentences including ψ. Conversely, under mild conditions, every theory capable of proving its own truth must be finitely axiomatizable.

PROPOSITION 4.1. *Let $T \supseteq S$ be a recursively enumerable theory and $S \supseteq \mathsf{RCA}_0$ be finitely axiomatizable. Suppose there is predicate $\mathrm{T}(x)$ such that*

$$T \vdash \mathrm{Ax}_T(x) \to \mathrm{T}(x),$$

[6] The bound $\omega \cdot 2$ arises from a sequent style formalization of Peano arithmetic with induction axioms instead of rules for induction. Induction rules would yield the bound ω^2. The rank is defined as usual. Literals (atomic and negated atomic formulas) receive rank 0, $\mathrm{rk}(\varphi \circ \psi) = \max(\mathrm{rk}(\varphi), \mathrm{rk}(\psi)) + 1$ for $\circ \in \{\vee, \wedge\}$, $\mathrm{rk}(\circ\, x\, \varphi(x)) = \mathrm{rk}(\varphi(x)) + 1$, and $\mathrm{rk}(\circ\, X\, \varphi(X)) = \mathrm{rk}(\varphi(X)) + 1$ for $\circ \in \{\exists, \forall\}$.

[7] The reader should inspect Kreisel and Lévy's *hints and tips* explanation on how to deal with infinite proof trees and cut elimination (see [12, pp. 127–128]).

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where Ax_T *is a* Σ_1 *definition of* T*, and suppose that*

$$S \vdash \mathrm{T}(\ulcorner\varphi\urcorner) \to \varphi,$$

for every axiom $\varphi \in T$*. Then* T *is finitely axiomatizable.*

PROOF. The theory $S + \{\forall x\,(\mathrm{Ax}_T(x) \to \mathrm{T}(x))\}$ is finitely axiomatizable and equivalent to T. Note that, by Σ_1 completeness, $\mathsf{RCA}_0 \vdash \mathrm{Ax}_T(\ulcorner\varphi\urcorner)$ for every $\varphi \in T$. ⊣

In general, Theorem 3.1 fails for infinite theories of bounded complexity (e.g., axioms with a bound on the number of set quantifiers or even axioms with bounded rank). The following simple example is a case in point.

EXAMPLE 4.2. Let T_0 be $\mathsf{RCA}_0 + \{\emptyset^{(\bar{n})}\ \text{exists} : n \in \omega\}$. Note that RCA_0 is finitely axiomatizable and hence T_0 is equivalent to a theory of bounded rank.[8] We claim that $T_0 + \mathrm{TI}(\varepsilon_0)$ does not prove uniform reflection over T_0. In fact, $T_0 \vdash \forall x\,\mathrm{Pr}_{T_0}(\ulcorner\emptyset^{(\dot{x})}\ \text{exists}\urcorner)$. By reflection one would obtain

$$T_0 + \mathrm{TI}(\varepsilon_0) \vdash \forall x\,(\emptyset^{(x)}\ \text{exists}).$$

A standard compactness argument shows that there is a model of $T_0 + \mathrm{TI}(\varepsilon_0)$ where $\forall x\,(\emptyset^{(x)}\ \text{exists})$ fails. In the compactness argument use the fact that for every n the ω-model $\{X \subseteq \omega\colon X \leq_T \emptyset^{(n)}\}$ is a model of $\mathsf{RCA}_0 + \mathrm{TI}(\varepsilon_0)$ that satisfies

$$\emptyset^{(\bar{n})}\ \text{exists and}\ \exists x\,(\emptyset^{(x)}\ \text{does not exist}).$$

§5. Fragments. Let $\mathrm{RFN}_{\Pi^1_n}(T)$ be the restriction of $\mathrm{RFN}(T)$ to Π^1_n formulas. The schema $\mathrm{RFN}_{\Sigma^1_n}(T)$ is defined similarly. Observe that the uniform reflection schema applies to formulas $\varphi(x)$ with no free variables other than x. Therefore, the schemata $\mathrm{RFN}_{\Pi^1_n}(T)$ and $\mathrm{RFN}_{\Sigma^1_n}(T)$ refer to Π^1_n and Σ^1_n formulas $\varphi(x)$ with no free variables, in particular no free set variables, other than x. However, we can apply uniform reflection to formulas with finitely many number variables. Note that in first order arithmetic the schema

$$\mathrm{Pr}_T(\ulcorner\varphi(\dot{x}_1, \dots, \dot{x}_k)\urcorner) \to \varphi(x_1, \dots, x_k)$$

is equivalent (over EA) to the schema with only one variable (see, e.g., [6]). The same applies to restrictions of uniform reflection in the Π_n and Σ_n hierarchy. Similarly, Π^1_n uniform reflection is equivalent (over RCA_0) to its multivariate version. The same holds for Σ^1_n uniform reflection. For example, given a Π^1_n formula $\varphi(x_1, \dots, x_k)$ with the free variables shown, consider the formula

$$\psi(x) \ =_{\mathrm{def}}\ \forall x_1 \cdots \forall x_k\,((x)_1 = x_1 \wedge \cdots \wedge (x)_k = x_k \to \varphi(x_1, \dots, x_k)).$$

[8] One can express the existence of the n-th jump by a single formula $\exists Z\,\exists X\,((X)_0 = \emptyset \wedge (\forall m < n)\,(X)_{m+1} = (X)'_m \wedge Z = (X)_n)$, where $(X)_m = \{n \mid (m, n) \in X\}$ is the m-th column of X.

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Then $\psi(x)$ is Π^1_n in RCA_0,

$$\mathsf{RCA}_0 \vdash \forall x_1 \cdots \forall x_k \, \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}_1, \dots, \dot{x}_k) \urcorner) \leftrightarrow \forall x \, \mathrm{Pr}_T(\ulcorner \psi(\dot{x}) \urcorner)$$

and

$$\mathsf{RCA}_0 \vdash \forall x_1 \cdots \forall x_k \, (\mathrm{Pr}_T(\ulcorner \varphi(\dot{x}_1, \dots, \dot{x}_k) \urcorner) \to \varphi(x_1, \dots, x_k)) \leftrightarrow \\ \leftrightarrow \forall x \, (\mathrm{Pr}_T(\ulcorner \psi(\dot{x}) \urcorner) \to \psi(x)).$$

For a fine characterization of uniform reflection, we need to consider lightface versions of induction and transfinite induction up to ε_0. Let $(\mathrm{I}\Pi^1_n)^-$ be the restriction of induction to Π^1_n formulas with no set parameters. The same definition applies to $\mathrm{TI}_{\Pi^1_n}(\varepsilon_0)$.

THEOREM 5.1. *Let $T_0 \supseteq \mathsf{RCA}_0$ be a second order arithmetic theory axiomatized by a Π^1_2 sentence, and let $n \geq 1$. Let T denote T_0 plus the schema of full induction. Over T_0,*

$$\mathrm{RFN}_{\Pi^1_{n+2}}(T_0) = \mathrm{I}\Pi^1_n \supseteq (\mathrm{I}\Pi^1_n)^- = \mathrm{RFN}_{\Sigma^1_{n+1}}(T_0), \tag{5}$$

$$\mathrm{RFN}_{\Pi^1_{n+2}}(T) = \mathrm{TI}_{\Pi^1_n}(\varepsilon_0) \supseteq \mathrm{TI}_{\Pi^1_n}(\varepsilon_0)^- = \mathrm{RFN}_{\Sigma^1_{n+1}}(T). \tag{6}$$

More in general, if T_0 is axiomatized by a Π^1_{k+2} sentence, then the above relations hold for all $n \geq k+1$.

PROOF. We sketch a proof of (6).

1. From (Σ^1_{n+1}) Π^1_{n+2} uniform reflection to (lightface) Π^1_n transfinite induction. Note that for a Π^1_n formula $\varphi(x)$, the formula

$$\psi(z) =_{\mathrm{def}} \forall x \, (\forall y \prec x \, \varphi(y) \to \varphi(x)) \to \forall x \prec \omega_z \, \varphi(x)$$

is Σ^1_{n+1} within RCA_0 (by simple quantifier manipulations).

1.1. Let us consider the lightface case. Suppose that $\varphi(x)$ has no set parameters. Note that $\varphi(x)$ may contain free number variables other than x, say $x_1, \dots, x_i$. We have

$$T_0 \vdash \forall z \, \forall x_1 \cdots \forall x_i \, \mathrm{Pr}_T(\ulcorner \psi(\dot{z}, \dot{x}_1, \dots, \dot{x}_i) \urcorner).$$

Then one can apply (the multivariate version of) Σ^1_{n+1} uniform reflection, and obtain

$$T_0 + \mathrm{RFN}_{\Sigma^1_{n+1}}(T) \vdash \forall z \, \forall x_1 \cdots \forall x_i \, \psi(z, x_1, \dots, x_i).$$

1.2. For the boldface version, suppose that $\varphi(x)$ has parameters $x_1, \dots, x_i$ and $X_1, \dots, X_j$. Then one must apply (the multivariate version of) uniform reflection to

$$\forall X_1 \cdots \forall X_j \, \psi(z, x_1, \dots, x_i, X_1, \dots, X_j),$$

which is Π^1_{n+2}. Therefore,

$$T_0 \vdash \forall z \, \forall x_1 \cdots \forall x_i \, \mathrm{Pr}_T(\ulcorner \forall X_1 \cdots \forall X_j \, \psi(\dot{z}, \dot{x}_1, \dots, \dot{x}_i, X_1, \dots, X_j) \urcorner)$$

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and hence

$$T_0 + \mathrm{RFN}_{\Pi^1_{n+2}}(T) \vdash \forall z\, \forall x_1 \cdots \forall x_i\, \forall X_1 \cdots \forall X_j\, \psi(z, x_1, \dots, x_i, X_1, \dots, X_j).$$

2. From Π^1_n transfinite induction to Π^1_{n+2} uniform reflection.

Fix a Π^1_2 axiomatization $\forall X\, \psi(X)$ of T_0. Let $\forall X\, \exists Y\, \varphi(x, X, Y)$ be a Π^1_{n+2} formula with no free variables other than x. By definition, $\varphi(x, X, Y)$ is Π^1_n. We aim to show

$$T_0 + \mathrm{TI}_{\Pi^1_n}(\varepsilon_0) \vdash \mathrm{Pr}_T(\ulcorner \forall X\, \exists Y\, \varphi(\dot{x}, X, Y) \urcorner) \to \forall X\, \exists Y\, \varphi(x, X, Y).$$

Work in $T_0 + \mathrm{TI}_{\Pi^1_n}(\varepsilon_0)$. Suppose that $\forall X\, \exists Y\, \varphi(\bar{x}, X, Y)$ is provable in T. By embedding and cut elimination, we obtain a cut-free ω-proof of height $< \varepsilon_0$ of the sequent

$$\Xi \;=\; \exists X\, \neg\psi(X), \forall X\, \exists Y\, \varphi(\bar{x}, X, Y).$$

We aim to prove that $\forall X\, \exists Y\, \varphi(\bar{x}, X, Y)$ is true. Suppose, towards a contradiction, that there is a set X_0 such that $\neg\exists Y\, \varphi(x, X_0, Y)$ is true. We are going to argue as in [13] (cf. also [15, Lemma 4.3]) and note that in order to analyze a cut-free proof of a false Π^1_{n+2} formula we only need a partial truth predicate for (enough) Π^1_n formulas.

By the subformula property, every sequent Γ appearing in a cut-free ω-proof of Ξ is of the form Π, Λ, where Π consists of Π^1_n formulas and $\Lambda \subseteq \Xi \cup \{\exists Y\, \varphi(\bar{x}, U, Y) \mid U \text{ free variable}\}$. Call a sequent of this form a Ξ-sequent. Now, given a cut-free ω-proof of height $< \varepsilon_0$ of a Ξ-sequent, we prove by Π^1_n transfinite induction on ε_0 that:

"For every Ξ-sequent Π, Λ appearing in such a proof, the formula $\bigvee \Pi$ is true for any evaluation of the free variables, with the proviso that the eigenvariables of a $\forall$-introduction of $\forall X\, \exists Y\, \varphi(\bar{x}, X, Y)$ are evaluated by the set X_0."

The property in quotation marks can be expressed by a Π^1_n formula. For this we use a Π^1_n truth predicate for a sufficiently large class of Π^1_n formulas, one of the form $P_{n,m}$ so that both $\neg\psi(X)$ and $\varphi(x, X, Y)$ are in $P_{n,m}$. Note also that Π^1_n induction suffices to prove that Π^1_n formulas are closed under bounded quantifiers, so that the segment "...the formula $\bigvee \Pi$ is true..." is indeed equivalent to a Π^1_n formula.

We consider two key cases. Suppose we have an inference of the form

$$\frac{\Pi, \neg\psi(W), \Lambda}{\Pi, \Lambda, \exists X\, \neg\psi(X)}.$$

Here, $\neg\psi(W)$ is Π^1_1 and hence Π^1_n. Given an evaluation of all free variables as above, we want to show that $\bigvee \Pi$ is true. By possibly extending the evaluation to W, say by the set X, we can assume by the induction hypothesis that $\bigvee \Pi \vee \neg\psi(X)$ is true. By the assumption $\forall X\, \psi(X)$ we can clearly rule out $\neg\psi(X)$ from being true. Hence we are done.

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The other interesting case is when we have an inference of the form

$$\frac{\Pi, \varphi(\bar{x}, U, V), \Lambda}{\Pi, \Lambda, \exists Y\, \varphi(\bar{x}, U, Y)}.$$

Fix an evaluation of all variables such that the set variable U is evaluated by the set X_0. If necessary, we can extend the evaluation to the set variable V, say by the set Y. By the induction hypothesis, $\bigvee \Pi \vee \varphi(\bar{x}, X_0, Y)$ is true. On the other hand, from the assumption, $\varphi(\bar{x}, X_0, Y)$ is not true. Thereby, $\bigvee \Pi$ is true.

We now have a contradiction since the Π part of the sequent Ξ is empty or possibly consisting of $\exists X\, \neg\psi(X)$. In each case, the sequent fails to satisfy the required property.

3. From lightface Π^1_n transfinite induction to Σ^1_{n+1} uniform reflection.

We work in $T_0 + \mathrm{TI}_{\Pi^1_n}(\varepsilon_0)^-$. Suppose that $\exists X\, \forall Y\, \varphi(\bar{x}, X, Y)$, where $\varphi(x, X, Y)$ is Σ^1_{n-1}, is provable in T. Then we have a cut-free ω-proof of height $< \varepsilon_0$ of the sequent

$$\Xi \;=\; \exists X\, \neg\psi(X), \exists X\, \forall Y\, \varphi(\bar{x}, X, Y).$$

We now say that a Ξ-sequent is one of the form Σ, Λ, where Σ consists of Σ^1_{n-1} formulas, and $\Lambda \subseteq \Xi \cup \{\forall Y\, \varphi(\bar{x}, U, Y) \mid U \text{ free variable}\}$. Suppose, towards a contradiction, that $\neg\exists X\, \forall Y\, \varphi(x, X, Y)$. Again, we follow Leivant's *strategy* by considering only true Σ^1_{n-1} formulas in analyzing a cut-free proof of a false Σ^1_{n+1} formula. As before, every sequent in a cut-free ω-proof of Ξ is a Ξ-sequent. We want to show that:

"For every Ξ-sequent Σ, Λ of a cut-free ω-proof, the formula $\bigvee \Sigma$ is true for every evaluation of the free variables."

A moment's reflection shows that this can be formalized by a Π^1_n formula with no set parameters. Note that if we use trees (second order objects) to represent ω-proofs, we must quantify over all possible cut-free ω-proofs in order to apply lightface Π^1_n transfinite induction. Let us consider the only key case. Suppose we have an inference of the form

$$\frac{\Sigma, \varphi(\bar{x}, U, V), \Lambda}{\Sigma, \Lambda, \forall Y\, \varphi(\bar{x}, U, Y)},$$

where V is the eigenvariable, and so does not appear in the lower sequent. Given an evaluation of the free variables by sets $X, \ldots$, where X is the evaluation of U, we want to show that $\bigvee \Sigma(X, \ldots)$ is true. By the induction hypothesis, for any evaluation of the extra variable V, say by the set Y, we have that $\bigvee \Sigma(X, \ldots) \vee \varphi(\bar{x}, X, Y)$ is true. By the eigenvariable condition, this implies that $\bigvee \Sigma(X, \ldots) \vee \forall Y\, \varphi(\bar{x}, X, Y)$ is true. By the assumption, we can rule out the case where $\forall Y\, \varphi(\bar{x}, X, Y)$ is true, as desired. As before, one obtains a contradiction since the end sequent fails to satisfy the required property. ⊣

5.1. Separation results. The local reflection principle $\mathrm{Rfn}(T)$ for a theory T is the schema consisting of sentences

$$\mathrm{Pr}_T(\ulcorner\varphi\urcorner) \to \varphi.$$

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We now show, under mild assumptions, that the inclusions in Theorem 5.1 are strict. This will be an immediate consequence of the following.

LEMMA 5.2. *Let T_0 be a Π^1_2 finitely axiomatizable theory extending* ACA_0 *and $n \geq 1$. Then*

$$T_0 + \mathrm{RFN}_{\Pi^1_{n+2}}(T_0) \nvdash \mathrm{Rfn}_{\Sigma^1_{n+2}}(T_0). \tag{7}$$

If, moreover, $T_0 + \mathrm{RFN}_{\Sigma^1_{n+1}}(T_0) + \Sigma^1_{n+1}$-$\mathsf{AC}$ *is consistent, then*

$$T_0 + \mathrm{RFN}_{\Sigma^1_{n+1}}(T_0) \nvdash \mathrm{Rfn}_{\Pi^1_{n+1}}(T_0). \tag{8}$$

The same holds with respect to reflection over T instead of T_0, where T is T_0 plus the schema of full induction.

PROOF. By using a universal Π^1_{n+2} formula one can see that, over ACA_0, the schema $\mathrm{RFN}_{\Pi^1_{n+2}}(T_0)$ can be axiomatized by a Π^1_{n+2} sentence, call it ψ. Now,

$$T_0 + \mathrm{Rfn}_{\Sigma^1_{n+2}}(T_0) \vdash \psi \to \neg \mathrm{Pr}_{T_0}(\ulcorner \neg \psi \urcorner).$$

By the second incompleteness theorem, $T_0 + \mathrm{RFN}_{\Pi^1_{n+2}}(T_0) \nvdash \mathrm{Rfn}_{\Sigma^1_{n+2}}(T_0)$. This proves (7).

Let us show (8). By using a universal Σ^1_{n+1} formula, the schema $\mathrm{RFN}_{\Sigma^1_{n+1}}(T_0)$ can be axiomatized in ACA_0 by a sentence of the form $\forall x\, \vartheta(x)$, where ϑ is Σ^1_{n+1}. This formula is not Σ^1_{n+1} yet. Now, let φ be the Σ^1_{n+1} sentence obtained by applying choice to $\forall x\, \vartheta(x)$. By the consistency assumption, $T_0 + \{\varphi\}$ is consistent. As before,

$$T_0 + \mathrm{Rfn}_{\Pi^1_{n+1}}(T_0) \vdash \varphi \to \neg \mathrm{Pr}_{T_0}(\ulcorner \neg \varphi \urcorner).$$

By the second incompleteness theorem, $T_0 + \{\varphi\} \nvdash \mathrm{Rfn}_{\Pi^1_{n+1}}(T_0)$. Note that φ implies $\forall x\, \vartheta(x)$, over, say, RCA_0. It follows that $T_0 + \mathrm{RFN}_{\Sigma^1_{n+1}}(T_0) \nvdash \mathrm{Rfn}_{\Pi^1_{n+1}}(T_0)$. ⊣

By combining Theorem 5.1 with Lemma 5.2 we then obtain the following.

THEOREM 5.3. *Under the hypotheses of Lemma 5.2,*

$$T_0 + \mathrm{I}\Pi^1_n \nvdash (\mathrm{I}\Pi^1_{n+1})^-, \tag{5$^\neg$}$$

$$T_0 + (\mathrm{I}\Pi^1_{n+1})^- \nvdash \mathrm{I}\Pi^1_n. \tag{6$^\neg$}$$

The same holds with respect to $\mathrm{TI}_{\Pi^1_n}(\varepsilon_0)$ *and* $\mathrm{TI}_{\Pi^1_{n+1}}(\varepsilon_0)^-$.

REMARK 5.4. The proof of Theorem 5.1 shows that

$$T_0 + (\mathrm{I}\Pi^1_n)^{--} \vdash \mathrm{Rfn}_{\Sigma^1_{n+1}}(T_0),$$

where $^{--}$ denotes the restriction of induction to Π^1_n formulas with no parameters at all. Hence, we can strengthen (5$^\neg$) by replacing $(\mathrm{I}\Pi^1_{n+1})^-$

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with its $^{\frown}$ sibling. Similarly,

$$T_0 + \mathrm{TI}_{\Pi^1_n}(\varepsilon_0)^{\frown} \vdash \mathrm{Rfn}_{\Sigma^1_{n+1}}(T)$$

and hence $T_0 + \mathrm{TI}_{\Pi^1_n}(\varepsilon_0) \nvdash \mathrm{TI}_{\Pi^1_{n+1}}(\varepsilon_0)^{\frown}$.

QUESTION. Can we drop the assumptions of Lemma 5.2? What is the relation between local reflection, induction, transfinite induction up to ε_0, and corresponding parameter free variants $^{-}$ and $^{\frown}$?

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