Note on Continuous Functionals. By S. VERBLUNSKY, Ph.D., Magdalene College.

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1. Let $\mathfrak{F}$ be a family of functions $X(t)$ defined in $(0, 2\pi)$. The functionals to be considered are of a particular class, namely those which define a correspondence between each function X of $\mathfrak{F}$ and a number which we denote by $F(X)$. It is convenient to refer to $F(X)$ as the functional.

Some interesting results in this connection were obtained by Steinhaus*, whose treatment is devoted to the proof of three main theorems, their converses and of three parallel theorems and their converses. Steinhaus thus proved three groups of four theorems. The proofs of the first two groups leave little to be desired. The third proof, however, is much too long. In this note we give a short proof of this group of theorems (§§ 4, 5) and also a generalisation of a theorem due to Fréchet (§ 6).

2. We first explain the notation and definitions. The functional $F(X)$ is said to be additive in $\mathfrak{F}$ if, given $X(t)$, $Y(t)$, $X(t)+Y(t)$ all belonging to $\mathfrak{F}$, then

$$F(X)+F(Y)=F(X+Y).$$

We define a functional of two variable functions in $\mathfrak{F}$, the value of which, $D(X, Y)$, we call the distance function. The family $\mathfrak{F}$ in which the function $D(X, Y)$ has been defined is called the field $\mathfrak{F}D$.

We denote by B the family of bounded functions. Let M be a positive number. Let $f(t)$ be a function which is equivalent to a function $\phi(t)$ satisfying $|\phi(t)| \leqslant M$. We denote by B_M the aggregate of functions $\phi(t)$. By $(B_M, 1)$ we mean the aggregate B_M where

$$D(X, Y)=\int_0^{2\pi} |X(t)-Y(t)|\,dt.$$

We denote by $L_p\,(p>1)$ the aggregate of functions of integrable pth power, and by (L_p, p) this aggregate with the distance function

$$D(X, Y)=\int_0^{2\pi} |X(t)-Y(t)|^p\,dt.$$

We write L for L_1. We denote by q the number which satisfies

$$\frac{1}{p}+\frac{1}{q}=1.$$

* *Math. Zeit.* 5, pp. 186–221 (1919).

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A functional is said to be continuous in a given field if, $X(t)$ being a function of the field and $X_n(t)$ a sequence of functions of the field which satisfy

$$\lim_{n\to\infty} D(X, X_n) = 0,$$

then

$$\lim_{n\to\infty} F(X_n) = F(X).$$

We denote by $\phi_k(t)$ $(k = 1, 2, 3, \ldots)$ the normal orthogonal system

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{\cos x}{\sqrt{\pi}}, \quad \frac{\sin x}{\sqrt{\pi}}, \ldots.$$

We shall assume the following result*:

If $\{G_n(t)\}$ be a sequence of bounded functions, and if, for every bounded function $X(t)$,

$$\lim_{n\to\infty} \int_0^{2\pi} G_n(t)\, X(t)\, dt$$

exists, then the sequence

$$\int_0^{2\pi} |G_n(t)|\, dt$$

is bounded.

3. We first prove a generalisation of the result mentioned at the end of the preceding section.

THEOREM I. *If $\{G_n(t)\}$ be a sequence of bounded functions with the property that*

$$\lim_{n\to\infty} \int_0^{2\pi} G_n(t)\, X(t)\, dt$$

exists for every bounded function $X(t)$, then the functions

$$H_n(x) = \int_0^x G_n(t)\, dt$$

converge to an absolutely continuous function.

By the result mentioned at the end of § 2, the functions $H_n(x)$ are uniformly of bounded variation. Further, if we take $X(t) = 1$ in $(0 \leqslant t \leqslant x)$, $X(t) = 0$ in $(x < t \leqslant 2\pi)$, it follows that the sequence $\{H_n(x)\}$ converges everywhere, and therefore converges to a function of bounded variation $H(x)$.

If $H(x)$ is not absolutely continuous, we may without loss of generality suppose that there is an $\eta > 0$ and a sequence $\{e_n\}$ of sets such that $e_n \supset e_{n+1}$, $\lim me_n = 0$ and

$$\int_{e_n} G_{n_r}(t)\, dt > \eta$$

* Steinhaus, *loc. cit.*, *Hilfsatz* 3.

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for some increasing sequence $\{n_r\}$. By altering the notation, we may suppose that

$$\int_{e_n} G_n(t)\,dt > \eta. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

We next observe that as a consequence of the hypothesis of the theorem

$$\lim_{n\to\infty} \int_e G_n(t)\,X(t)\,dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

exists for any assigned set e in $(0, 2\pi)$ and any assigned bounded $X(t)$.

On the hypothesis (1), we shall construct a function $X(t)$ in $(0, 2\pi)$ which satisfies

$$|X(t)| \leqslant 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

and for which the limit (2) does not exist. This will prove the theorem.

In Ce_1, the complement of e_1 with respect to $(0, 2\pi)$, put $X(t) = 0$. Let

$$\int_{e_1} G_1(t)\,dt = l_1,$$

where $l_1 > \eta$. We can find an n_1 such that

$$\int_{e_{n_1}} |G_1(t)|\,dt < \epsilon,$$

where $\epsilon < \frac{1}{2}\eta$ is arbitrarily assigned. In $e_1 - e_{n_1}$ put $X(t) = 1$. Then

$$\left| \int_0^{2\pi} G_1(t)\,X(t)\,dt - l_1 \right| < \epsilon, \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

whatever be the definition of $X(t)$ in e_{n_1} subject to (3). Now

$$\lim_{n\to\infty} \int_{e_1 - e_{n_1}} G_n(t)\,X(t)\,dt = l_2,$$

say. We can then find an $N_1 > n_1$ such that

$$\left| \int_{e_1 - e_{n_1}} G_{N_1}(t)\,X(t)\,dt - l_2 \right| < \frac{\epsilon}{2}.$$

Further,

$$\int_{e_{N_1}} G_{N_1}(t)\,dt > \eta.$$

We can find an $n_2 > N_1$ such that

$$\int_{e_{n_2}} |G_{N_1}(t)|\,dt < \frac{\epsilon}{2}.$$

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Put $X(t)=0$ in $e_{n_1}-e_{N_1}$. In $e_{N_1}-e_{n_2}$ we can choose $X(t)$ equal to either $+1$ or to -1 so that, whatever be the definition in e_{n_2} subject to (3),

$$\left|\int_{e_1} G_1(t)X(t)\,dt-\int_{e_1} G_{N_1}(t)\,X(t)\,dt\right|>\eta-2\epsilon.$$

It is now clear that the process can be continued so as to give a sequence $\{N_r\}$ with the property

$$\left|\int_{e_1} G_{N_r}(t)X(t)\,dt-\int_{e_1} G_{N_{r+1}}(t)\,X(t)\,dt\right|>\eta-2\epsilon.$$

4. THEOREM II*. *If the sequence $\{h_k\}$ has the property that*

$$\sum_{k=1}^{\infty} h_k x_k \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

is summable $(C, 1)$, where $\{x_k\}$ are the normalised Fourier constants of any bounded function $X(t)$, then $\{h_k\}$ are the normalised Fourier constants of a function $H(t)\subset L$, and the sum $(C, 1)$ of (5) is the value of a functional $F(X)$ which is additive in B and continuous in $(B_M, 1)$ for every M.

Write $$H_n^{(1)}(t)=\frac{1}{n}\sum_{k=1}^{n}(n-k+1)\,h_k\phi_k(t).$$

Then, by hypothesis,

$$\lim_{n\to\infty}\int_0^{2\pi} H_n^{(1)}(t)\,X(t)\,dt$$

exists for every bounded X. If we write

$$F(X)=\lim_{n\to\infty}\int_0^{2\pi} H_n^{(1)}(t)\,X(t)\,dt, \qquad \ldots\ldots\ldots\ldots(6)$$

then $F(X)$ is additive in B and

$$F(X)=\sum_{k=1}^{\infty} h_k x_k \qquad (C, 1).$$

By Theorem I, the function

$$A(t)=\lim_{n\to\infty}\int_0^t H_n^{(1)}(u)\,du$$

is absolutely continuous. But by definition

$$A(t)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(n-k+1)\,h_k\int_0^t\phi_k(u)\,du$$

$$=\sum_{k=1}^{\infty} h_k\int_0^t\phi_k(u)\,du.$$

* *Loc. cit., Umkehrung des Parallelsatzes* 3.

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It follows that $\sum_{k=1}^{\infty} h_k \phi_k(t)$

is the normalised Fourier series of the absolutely integrable function $A'(t) = H(t)$. Hence also

$$F(X) = \int_0^{2\pi} H(t)\, X(t)\, dt.$$

It remains to prove that $F(X)$ is additive in $(B_M, 1)$. Let $X_n(t)$, $X(t)$ satisfy

$$|X_n(t)| \leqslant M, \quad |X(t)| \leqslant M,$$

$$\lim_{n\to\infty} \int_0^{2\pi} |X(t) - X_n(t)|\, dt = 0.$$

From and after a certain n,

$$\int_0^{2\pi} |X(t) - X_n(t)|\, dt < 2\pi\epsilon,$$

and the measure of the set $E(\epsilon)$ in which

$$|X(t) - X_n(t)| > \epsilon^{\frac{1}{2}}$$

is less than $2\pi\epsilon^{\frac{1}{2}}$. Hence

$$|F(X_n) - F(X)| < 2M \int_{E(\epsilon)} |H(t)|\, dt + \epsilon \int_0^{2\pi} |H(t)|\, dt \to 0.$$

This proves the theorem.

THEOREM III*. *If the value of a functional is given by*

$$F(X) = \int_0^{2\pi} H(t)\, X(t)\, dt,$$

where $H(t) \subset L$ is independent of $X(t)$, then the functional is additive in B and continuous in $(B_M, 1)$.

The first part is obvious, and the second is contained in the proof of Theorem II.

5. THEOREM IV†. *A functional which is additive in B and continuous in $(B_M, 1)$ is of the form*

$$F(X) = \int_0^{2\pi} H(t)\, X(t)\, dt,$$

where $H(t) \subset L$ is a function independent of $X(t)$.

* *Loc. cit., Umkehrung des Satzes* 3. † *Loc. cit., Satz* 3.

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If $S_n(t)$ denote the nth Cesàro mean of order 1 of the normalised Fourier series of $X(t)$, then

$$\int_0^{2\pi} |X(t) - S_n(t)|\, dt \to 0,$$

and
$$|S_n(t)| \leqslant M.$$

Hence
$$F(S_n) \to F(X).$$

By the additivity and continuity of the functional

$$F(S_n) = \frac{1}{n} \sum_{k=1}^{n} (n-k+1)\, x_k\, F(\phi_k).$$

Writing $F(\phi_k) = h_k$, we have

$$F(X) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (n-k+1)\, h_k x_k.$$

Hence, by Theorem II, $\{h_k\}$ are the normalised Fourier constants of a function $H(t) \subset L$, and therefore

$$F(X) = \int_0^{2\pi} H(t)\, X(t)\, dt.$$

THEOREM V*. *The value of a functional which is additive in B and continuous in $(B_M, 1)$ is given by*

$$F(X) = \sum_{k=1}^{\infty} h_k x_k \qquad (C, 1),$$

where $\{x_k\}$ are the normalised Fourier constants of $X(t)$, and $\{h_k\}$ are the normalised Fourier constants of a function $H(t) \subset L$, which is independent of $X(t)$.

This follows from the preceding proof.

6. THEOREM VI†. *A functional which is additive and continuous in (L_p, p) is of the form*

$$F(X) = \int_0^{2\pi} H(t)\, X(t)\, dt,$$

where $H(t) \subset L_q$ is independent of $X(t)$.

If
$$s_n(t) = \sum_{k=1}^{n} x_k \phi_k(t),$$

then, by a theorem of M. Riesz,

$$\lim_{n\to\infty} \int_0^{2\pi} |X(t) - s_n(t)|^p\, dt = 0.$$

* *Loc. cit., Parallelsatz* 3.

† The case $p=2$ is due to Fréchet; *loc. cit., Satz* 2.

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Hence $$F(s_n) \to F(X).$$

But $$F(s_n) = \sum_{k=1}^{n} x_k F(\phi_k).$$

Writing $F(\phi_k) = h_k$, we infer that

$$\sum_{k=1}^{\infty} h_k x_k$$

is convergent and that

$$F(X) = \sum_{k=1}^{\infty} h_k x_k. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

The functional $F(X)$ is additive in B and continuous in $(B_M, 1)$. For if $X_n \to X$ in $(B_M, 1)$, i.e.

$$\int_0^{2\pi} |X_n(t) - X(t)|\, dt \to 0,$$

then

$$\int_0^{2\pi} |X_n(t) - X(t)|^p\, dt < (2M)^{p-1} \int_0^{2\pi} |X_n(t) - X(t)|\, dt \to 0.$$

Hence if $\xi(t)$ is a bounded function, we have, by Theorem IV,

$$F(\xi) = \int_0^{2\pi} H(t)\, \xi(t)\, dt,$$

where $H(t) \subset L$ is independent of $\xi(t)$. Putting $\xi(t) = \phi_k(t)$, we infer that $\{h_k\}$ are the normalised Fourier constants of $H(t)$. If now $X(t) \subset L_p$, we can find a sequence $\{X_n(t)\}$ of bounded functions such that

$$\int_0^{2\pi} |X(t) - X_n(t)|^p\, dt \to 0.$$

Since $$F(X_n) = \int_0^{2\pi} H(t)\, X_n(t)\, dt,$$

we infer that $$F(X) = \lim_{n\to\infty} \int_0^{2\pi} H(t)\, X_n(t)\, dt.$$

Now this implies $H(t) \subset L_q$. For choose an $X(t) \subset L_p$ such that

$$\operatorname{sign} X(t) = \operatorname{sign} H(t), \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

and put $$X_n(t) = \Big[X(t)\Big]_{-n}^{n}.$$

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Then the existence of

$$\lim_{n\to\infty} \int_0^{2\pi} H(t)\, X_n(t)\, dt$$

implies the existence of

$$\int_0^{2\pi} H(t)\, X(t)\, dt. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Thus (9) exists for all $X(t) \subset L_p$ which satisfy (8). It therefore exists for all $X(t) \subset L_p$, and this implies $H(t) \subset L_q$. Hence

$$F(X) = \int_0^{2\pi} H(t)\, X(t)\, dt.$$

We have proved incidentally:

THEOREM VII. *The value of a functional which is continuous and additive in* (L_p, p) *is given by*

$$F(X) = \sum_{k=1}^{\infty} h_k x_k,$$

where $\{h_k\}$ *are the normalised Fourier constants of a function* $H(t) \subset L_q$ *independent of* $X(t)$, *and* $\{x_k\}$ *are the normalised Fourier constants of* $X(t)$.

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