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COPRIME COMMUTATORS IN THE SUZUKI GROUPS ${}^2B_2(q)$

GIOVANNI ZINI

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Abstract

In this note we show that every element of a simple Suzuki group ${}^2B_2(q)$ is a commutator of elements of coprime orders.

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1. Introduction

An element g of a group G is called a commutator if g equals $[a, b] = a^{-1}b^{-1}ab$ for some a and b in G. Whenever G is a nonabelian finite simple group, every element of G is a commutator. This was conjectured by Ore [4] and finally proved by Liebeck *et al.* [3].

A stronger property was conjectured by Shumyatsky. In [6], he conjectured that every element of a nonabelian finite simple group G is a coprime commutator, that is, a commutator of elements of coprime orders. Shumyatsky's conjecture was proved for the alternating groups [6, Theorem 3.1] and for the special linear groups $L_2(q)$, for any prime power q [5, Theorem 1.1]; it was also verified with MAGMA [1] for all nonabelian simple groups of order less than 10^7 (see [5]). Here we prove the conjecture for the simple Suzuki groups ${}^2B_2(q)$.

THEOREM 1.1. *Let q be an odd power of 2. Every element of ${}^2B_2(q)$ is a coprime commutator.*

More precisely, for every $g \in {}^2B_2(q)$, we prove that if the order of g is different from 4, then g is the commutator of an involution and an element of odd order; if the order of g is 4, then g is the commutator of an element of order 4 and an element of order dividing $q - 1$.

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Note that, according to the classification of Zassenhaus groups concluded by Suzuki [7], the simple groups $L_2(q)$ and ${}^2B_2(q)$ are the only perfect Zassenhaus groups. Therefore, Theorem 1.1 together with [5] shows that Shumyatsky's conjecture holds for the Zassenhaus groups in which it can be posed.

In order to prove Theorem 1.1, we first show that each nontrivial coset of the centraliser of an involution contains an element of odd order. In [5], this property was shown for the simple groups $L_2(q)$ and conjectured for all finite simple groups other than $L_n(4)$ with $n \geq 4$. Arguing as in [5], this allows us to write every strongly real element of ${}^2B_2(q)$ as a coprime commutator. The only elements of ${}^2B_2(q)$ that are not strongly real are those of order 4, which are considered separately.

2. Preliminaries

Let s be a positive integer, $q_0 = 2^s$, $q = 2q_0^2 = 2^{2s+1}$, $\mathbb{F}_q$ be the finite field with q elements and ${}^2B_2(q)$ be the simple Suzuki group associated with $\mathbb{F}_q$. We summarise here some notation and results on ${}^2B_2(q)$; for the proofs and further information, we refer the reader to [2, Ch. XI.3] and [7–9].

Let φ be the automorphism $x \mapsto x^{2q_0}$ of $\mathbb{F}_q$; note that φ^2 is the automorphism $x \mapsto x^2$ of $\mathbb{F}_q$. For any $a, c, d \in \mathbb{F}_q$ with $d \neq 0$, let $\sigma_{a,c}, \tau_d \in \mathrm{GL}(4, q)$ be defined by

$$\sigma_{a,c} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ c & a^\varphi & 1 & 0 \\ a^{\varphi+2} + ac + c^\varphi & a^{\varphi+1} + c & a & 1 \end{pmatrix}, \quad \tau_d = \begin{pmatrix} d^{-q_0-1} & 0 & 0 & 0 \\ 0 & d^{-q_0} & 0 & 0 \\ 0 & 0 & d^{q_0} & 0 \\ 0 & 0 & 0 & d^{q_0+1} \end{pmatrix}.$$

The matrix $\sigma_{a,c}$ has order 2 or 4 according as $a = 0$ or $a \neq 0$, respectively. We have

$$\sigma_{a,c}\,\sigma_{\breve{a},\breve{c}} = \sigma_{a+\breve{a},c+\breve{c}+a^\varphi\breve{a}}, \quad \sigma_{a,c}^{-1} = \sigma_{a,c+a^{\varphi+1}}.$$

The group $S = \{\sigma_{a,c} \colon a, c \in \mathbb{F}_q\}$ has order q^2. Its centre is generated by the involutions of S and is elementary abelian, that is, $Z(S) = \{\sigma_{0,c} \colon c \in \mathbb{F}_q\} \cong E_q$. Also, $S/Z(S)$ is elementary abelian of order q and hence $S \cong E_q^{1+1}$, that is, S has a normal subgroup isomorphic to E_q such that the quotient group is isomorphic to E_q.

The matrices τ_d with $d \in \mathbb{F}_q^*$ form a cyclic group T of order $q - 1$. Also,

$$\tau_d^{-1}\,\sigma_{a,c}\,\tau_d = \sigma_{a/d,c/d^{\varphi+1}},$$

and hence the subgroup F of $\mathrm{GL}(4, q)$ generated by S and T is a semidirect product $F = S \rtimes T \cong E_q^{1+1} : (q-1)$ of order $q^2(q-1)$. Let $\omega \in \mathrm{GL}(4, q)$ be the involution defined by the anti-diagonal matrix with all elements 1 on the anti-diagonal. The Suzuki group ${}^2B_2(q)$ associated with $\mathbb{F}_q$ is the subgroup of $\mathrm{GL}(4, q)$ generated by F and ω.

The group ${}^2B_2(q)$ has a natural (left) action on $\mathrm{PG}(3, q) = \{P = \langle \underline{v} \rangle \colon \underline{v} \in \mathbb{F}_q^4 \setminus \{\underline{0}\}\}$ defined by $\alpha(P) = \langle \alpha \underline{v} \rangle$. In this action, G acts as a 2-transitive permutation group of degree $q^2 + 1$ on the set

$$\Omega = \{P_\infty\} \cup \{\langle (1, u, v, u^{\varphi+2} + uv + v^\varphi) \rangle \colon u, v \in \mathbb{F}_q\}.$$

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The set Ω is an ovoid of $\mathrm{PG}(3,q)$, known as the Suzuki–Tits ovoid. In its 2-transitive action on Ω, G is a Zassenhaus group. The stabiliser ${}^2B_2(q)_{P_\infty}$ of P_∞ in ${}^2B_2(q)$ is F, and acts as a Frobenius group on $\Omega \setminus \{P_\infty\}$, with Frobenius kernel S and Frobenius complement T. The kernel S acts regularly on $\Omega \setminus \{P_\infty\}$, while the complement T is the two-point stabiliser ${}^2B_2(q)_{P_\infty,O}$ of P_∞ and $0 = \langle (1,0,0,0) \rangle$ in ${}^2B_2(q)$. Recall that every element of $F \setminus S$ is conjugate to an element of T; the two-point stabilisers $Sz(q)_{P_\infty,Q}$ in $Sz(q)$, where Q runs in $\Omega \setminus \{P_\infty\}$, are exactly the conjugates of T in F. Recall also that the centraliser in F of an element of S is contained in S; hence, the centraliser in $Sz(q)$ of an involution of S coincides with $S \cong E_q^{1+1}$.

An element g of a group G is called real if it is conjugate to its inverse, and strongly real if it is conjugate to its inverse by an involution of G. Note that an element of G is strongly real if and only if it is the product of two involutions of G. The group ${}^2B_2(q)$ satisfies the following further properties.

- All involutions of ${}^2B_2(q)$ are conjugate (see [7, Proposition 7]).
- An element of ${}^2B_2(q)$ is real if and only if it is strongly real (see the discussion after [7, Proposition 15]).
- In ${}^2B_2(q)$ there are exactly two conjugacy classes of elements of order 4, namely the class of γ and the class of γ^{-1} for any $\gamma \in {}^2B_2(q)$ of order 4 (see [7, Proposition 18] and its proof).
- Every element of ${}^2B_2(q)$ whose order is different from 4 is strongly real (see [7, proof of Theorem 4]).

3. Coprime commutators

We start by proving the existence of elements of odd order in any nontrivial coset of a centraliser of an involution.

LEMMA 3.1. *Each nontrivial coset of the centraliser of an involution in* ${}^2B_2(q)$ *contains an element of odd order.*

PROOF. Let α be an involution in $G = {}^2B_2(q)$, $C \cong E_q^{1+1}$ be the centraliser of α in G, and $\beta \in G \setminus C$. Let $P \in \Omega$ be the fixed point of α, so that C is the Sylow 2-subgroup S of G_P. Let $Q \in \Omega$ be such that $\beta(Q) = P$, so that all elements of $C\beta$ map Q to P. If $Q = P$, then $C\beta$ is contained in $G_P \setminus S$, and hence every element of $C\beta$ has odd order. Therefore, we can assume that $Q \neq P$.

Suppose by contradiction that all elements of $C\beta$ have order 2 or 4. Then $C\beta$ contains q^2 elements mapping Q to P and fixing exactly one point in the set $\Omega \setminus \{P, Q\}$ of size $q^2 - 1$. Thus $C\beta$ contains two distinct elements $\alpha_1 = \gamma_1\beta$ and $\alpha_2 = \gamma_2\beta$ fixing the same point $R \in \Omega \setminus \{P, Q\}$. Then $\alpha_1\alpha_2^{-1}$ fixes both P and R. In addition, we have $\alpha_1\alpha_2^{-1} = \gamma_1\gamma_2^{-1} \in S \setminus \{1\}$. This is a contradiction, because S has trivial intersection with the two-point stabilisers. The claim is proved. □

Lemma 3.1 allows us to use an element of odd order to conjugate any pair of involutions.

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COROLLARY 3.2. *Any two involutions in ${}^2B_2(q)$ are conjugate by an element of odd order.*

PROOF. Any two distinct involutions $\alpha_1, \alpha_2 \in {}^2B_2(q)$ are conjugate in ${}^2B_2(q)$, say $\alpha_2 = \beta^{-1}\alpha_1\beta$ with $\beta \in Sz(q)$ not in the centraliser C of α_1. By Lemma 3.1, $C\beta$ contains an element γ of odd order. Then $\alpha_2 = \gamma^{-1}\alpha_1\gamma$. □

From Corollary 3.2 the claim follows for any strongly real element.

PROPOSITION 3.3. *Every element of ${}^2B_2(q)$ of order different from 4 is a coprime commutator. In particular, it is the commutator of an involution and an element of odd order.*

PROOF. Let $\sigma \in {}^2B_2(q)$ have order different from 4. As recalled in Section 2, σ is a strongly real element, and hence $\sigma = \alpha_1\alpha_2$ for some involutions $\alpha_1, \alpha_2 \in {}^2B_2(q)$. By Corollary 3.2, $\alpha_2 = \gamma^{-1}\alpha_1\gamma$ for some $\gamma \in {}^2B_2(q)$ of odd order, so that $\sigma = [\alpha_1, \gamma]$ and the claim is proved. □

Finally, we prove the claim explicitly for the nonstrongly real elements, which have order 4.

PROPOSITION 3.4. *Every element of ${}^2B_2(q)$ of order 4 is a coprime commutator. In particular, it is the commutator of an element of order 4 and an element of order dividing $q-1$.*

PROOF. Let $\sigma \in {}^2B_2(q)$ have order 4 and $P \in \Omega$ be the fixed point of α. Since ${}^2B_2(q)$ is transitive on Ω, we can assume up to conjugation that $P = P_\infty$. As recalled in Section 2, ${}^2B_2(q)$ has exactly two conjugacy classes of elements of order 4, namely the class of σ and the class of σ^{-1}. Therefore it is enough to prove the claim for the element σ.

Let $a, c \in \mathbb{F}_q$, with $a \neq 0$, be such that $\sigma = \sigma_{a,c}$. Choose $\bar{a} \in \mathbb{F}_q \setminus \{0, a\}$; such an element $\bar{a}$ exists, as $q > 2$. Define

$$d = \bar{a}/(a + \bar{a}) \in \mathbb{F}_q \setminus \{0, 1\}. \tag{3.1}$$

Since $2q_0 + 1$ is coprime with $q-1$, the map $x \mapsto x^{\varphi+1}$ is one-to-one on $\mathbb{F}_q$. Therefore, $d \neq 1$ implies $d^{\varphi+1} \neq 1$. Now define

$$\bar{c} = \frac{d^{\varphi+1}c + \bar{a}^{\varphi+1}(1 + d^{\varphi})}{1 + d^{\varphi+1}} \in \mathbb{F}_q. \tag{3.2}$$

By direct checking,

$$\begin{aligned}[\sigma_{\bar{a},\bar{c}}^{-1}, \tau_d] &= \sigma_{\bar{a},\bar{c}}\,\tau_d^{-1}\,\sigma_{\bar{a},\bar{c}}^{-1}\,\tau_d = \sigma_{\bar{a},\bar{c}}\,(\tau_d^{-1}\,\sigma_{\bar{a},\bar{c}+\bar{a}^{\varphi+1}}\,\tau_d) \\ &= \sigma_{\bar{a},\bar{c}}\,\sigma_{\bar{a}/d,(\bar{c}+\bar{a}^{\varphi+1})/d^{\varphi+1}} = \sigma_{\bar{a}+\bar{a}/d,\bar{c}+(\bar{c}+\bar{a}^{\varphi+1})/d^{\varphi+1}+\bar{a}^{\varphi}\bar{a}/d}\,.\end{aligned}$$

Also, by direct computation, (3.1) and (3.2) imply that

$$\bar{a} + \frac{\bar{a}}{d} = a$$

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and

$$\bar{c} + \frac{\bar{c} + \bar{a}^{\varphi+1}}{d^{\varphi+1}} + \bar{a}^{\varphi}\frac{\bar{a}}{d} = c.$$

Therefore $\sigma = [\sigma_{\bar{a},\bar{c}}^{-1}, \tau_d]$. The order of $\sigma_{\bar{a},\bar{c}}^{-1}$ is 4 as $\bar{a} \neq 0$, and the order of τ_d is a nontrivial divisor of $q-1$ as $d \neq 1$. The claim follows. □

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GIOVANNI ZINI, Dipartimento di Matematica e Fisica,
Università degli Studi della Campania 'Luigi Vanvitelli',
viale Lincoln 5, 81100 Caserta, Italy
e-mail: giovanni.zini@unicampania.it

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