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Stochastic stability of invariant subspaces

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Abstract. We prove that a relevant part of the Lyapunov spectrum and the corresponding Oseledets spaces of a quasi-compact linear cocycle are stable under a certain type of random perturbation. The basic approach is a graph-transform argument. The result applies to the spectrum of (not necessarily i.i.d.) randomly perturbed expanding maps and yields generalizations of results recently obtained by Baladi, Kondah and Schmitt [**5**] with different methods.

0. *Introduction*

Consider an expanding $\mathcal{C}^k$ map $\varphi : M \to M$ ($k > 1$) on a compact connected smooth Riemannian manifold M and a $\mathcal{C}^k$ function $g : M \to \mathbb{R}$. The thermodynamic formalism of Ruelle [**15**] yields, amongst others, the existence of a unique equilibrium state μ_g, i.e. μ_g is the unique φ-invariant probability measure for which the supremum in the variational principle

$$\pi_\varphi(g) = \sup\left\{h_\mu(\varphi) + \int g\, d\mu : \mu\ \varphi\text{-invariant}\right\}$$

is attained. Here $\pi_\varphi(g)$ denotes the pressure of φ at g and $h_\mu(\varphi)$ denotes the K-S entropy. Moreover, μ_g is exponentially mixing: for $\mathcal{C}^k$ functions f_1 and f_2 the rate of decay of correlation

$$\tau(f_1, f_2) := \limsup_{n\to\infty} \frac{1}{n} \log \left| \int (f_1 \circ \varphi^n) \cdot f_2\, d\mu_g - \left(\int f_1\, d\mu_g\right)\left(\int f_2\, d\mu_g\right)\right|$$

can be bounded from above by a negative number which is independent of f_1 and f_2. This, by the way, holds for Hölder observables, too.

The fundamental tool to prove all this is Ruelle's transfer operator $\mathcal{L}_g : \mathcal{C}^k(M, \mathbb{C}) \to \mathcal{C}^k(M, \mathbb{C})$, which is defined by

$$(\mathcal{L}_g f)(x) = \sum_{y\in\varphi^{-1}\{x\}} e^{g(y)} f(y).$$

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Corresponding notions and results for the generalized situation where φ^n is replaced by the successive application of randomly chosen expanding maps can be drawn from the thermodynamic formalism for random dynamical systems (cf. Khanin and Kifer [**10**], who more generally deal with expanding in average random maps, and the references therein). One can use this setting to model small random perturbations of an expanding map (cf. Bogenschütz [**8**]).

The present paper is motivated by the question of whether the rate of decay of correlation is stable under such perturbations. The door to an answer of this question has been opened by the ground breaking work of Baladi and Young [**6**]. They proved stability of the rate of decay of correlation of the SRB measure under small i.i.d. random perturbations by considering the spectra of 'averaged' random (i.e. deterministic) transfer operators. In the present paper, we suggest a different approach using some terminology of multiplicative ergodic theory: we consider the linear cocycle induced by random transfer operators and study its Lyapunov exponents and Oseledets spaces.

Our main result is formulated in an abstract setting (§1). Roughly speaking, it says that the 'good' part of the Lyapunov spectrum and the corresponding invariant subspaces of a quasi-compact linear cocycle are stable under a specific type of perturbation (taylor-made to be satisfied by transfer operators of randomly perturbed expanding maps). There are few stability results in this framework, the most important being due to Ledrappier and Young [**12**] (see also Ochs [**13**] for a recent attack), and none in infinite dimensions. The present work might be viewed as the first step in this direction.

The proof of the main result (§2) basically uses a 'graph-transform' argument. That something like this should work came into the author's mind when seeing an alternative proof of [**6**, Lemma 3] due to Young [**20**]. Further sources of inspiration have been Schaumlöffel [**16**] and Thieullen [**18**].

At the end of the paper we apply our result to randomly perturbed expanding maps (§3). The type of stochastic stability we obtain generalizes recent results of Baladi *et al*. [**5**] (where the SRB measure is studied). Baladi [**4**] has also related results for general weights, but using a different (integrated) notion of perturbed correlation.

1. *An abstract perturbation result*

1.1. *Set-up.* Let $\mathbb{B}(X, X)$ denote the space of bounded linear operators on a separable (real or complex) Banach space $(X, \|\cdot\|)$. Further, let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be a dynamical system in the sense of ergodic theory, i.e. $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and θ is a measure-preserving transformation of Ω. We assume θ to be ergodic and invertible.

A map $T : \Omega \to \mathbb{B}(X, X)$ is said to be *strongly measurable* if $\omega \mapsto T(\omega)x$ is $(\mathcal{F}, \mathcal{B}(X))$-measurable for each $x \in X$, where $\mathcal{B}(X)$ denotes the Borel-σ-algebra on X. This is equivalent to being measurable with respect to the strong operator topology on $\mathbb{B}(X, X)$. Any such map induces a linear cocycle on X over θ via

$$T(n, \omega) := T(\vartheta^{n-1}\omega) \circ \cdots \circ T(\omega).$$

In the application we have in mind T will be a suitably weighted random transfer operator associated to an expanding random dynamical system.

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Let $\mathcal{V}(X)$ denote the set of closed subspaces of X. A map $E : \Omega \to \mathcal{V}(X)$ is called a *measurable subspace* of X, if $\omega \mapsto \inf\{\|x - y\| : y \in E(\omega), \|y\| = 1\}$ is measurable for each $x \in X$.

Definition 1.2. Measurable subspaces $E_1, \ldots, E_p, F$ of X ($p \geq 1$) are said to form a *semi-strictly invariant splitting* for T if $\mathbb{P}$-a.s. we have:
(1) $\dim E_i(\omega) \equiv d_i < \infty$ for $i = 1, \ldots, p$;
(2) $X = E_1(\omega) \oplus \cdots \oplus E_p(\omega) \oplus F(\omega)$;
(3) $T(\omega)F(\omega) \subset F(\theta\omega)$ and $T(\omega)E_i(\omega) = E_i(\theta\omega)$ for $i = 1, \ldots, p$.

Given a semi-strictly invariant splitting for T define its strictly invariant component E by $E(\omega) := E_1(\omega) \oplus \cdots \oplus E_p(\omega)$. Then $T_E(\omega) := T(\omega)|E(\omega)$ is a linear isomorphism between $E(\omega)$ and $E(\theta\omega)$ (since $\dim E(\omega) = d_1 + \cdots + d_p < \infty$). It thus makes sense to define $T_E(-n, \omega) : E(\omega) \to E(\theta^{-n}\omega)$ as the inverse of $T_E(n, \theta^{-n}\omega)$. (In other words, we are considering T_E as an invertible linear bundle random dynamical system with fiber mappings $T_E(\omega) : E(\omega) \to E(\theta\omega)$.)

Definition 1.3. We say that T *admits a spectral theory* if there exist a θ-invariant set $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$, real numbers $\lambda_1 > \cdots > \lambda_p > \kappa \geq -\infty$, and a semi-strictly invariant splitting $E_1, \ldots, E_p, F$ of X such that for all $\omega \in \Omega_0$ we have:
(1) $\lim_{n\to\pm\infty} n^{-1} \log \|T_E(n, \omega)x\| = \lambda_i \Leftrightarrow x \in E_i(\omega) \setminus \{0\}$;
(2) $\limsup_{n\to\infty} n^{-1} \log \|T(n, \omega)x\| \leq \kappa$ for $x \in F(\omega)$.
In this case we say that $E_1, \ldots, E_p, F$ is a *quasi-Oseledets splitting*, the numbers $\lambda_1 > \cdots > \lambda_p$ are *Lyapunov exponents*, and κ is a *split index*.

Note that the limit for $n \to -\infty$ in statement (1) of this definition is well-defined because $\dim E(\omega) < \infty$.

Note also that if T admits a spectral theory with Lyapunov exponents $\lambda_1 > \cdots > \lambda_p$ and split index κ then it also admits a spectral theory with Lyapunov exponents $\lambda_1 > \cdots > \lambda_i$ and split index $(\lambda_i + \lambda_{i+1})/2$ for $1 \leq i \leq p$. It will be useful in the following to not require that the split index is optimal.

Example 1.4. If we take $(\Omega, \mathcal{F}, \mathbb{P})$ to be the trivial probability space consisting of one point and θ is the identity transformation, then our set-up reduces to looking at the iterates of a single bounded linear operator T. The essential spectral radius $\rho_{\text{ess}}(T)$ of T is the smallest number $\rho > 0$ such that the spectrum $\Sigma(T)$ of T outside the disc of radius ρ consists of a finite or countable set of isolated eigenvalues of finite multiplicity. Assume T is quasi-compact, i.e. $\rho_{\text{ess}}(T)$ is strictly less than the spectral radius $\rho_\Sigma(T)$ of T. Fix $\rho > 0$ with $\rho_\Sigma(T) > \rho > \rho_{\text{ess}}(T)$ and let $\gamma_1 > \cdots > \gamma_p$ denote the absolute values of the eigenvalues outside the disk of radius ρ. (These are only finitely many because any limit point of $\Sigma(T)$ has absolute value $\leq \rho_{\text{ess}}(T)$.) Put $E_i = \bigoplus_{|\gamma|=\gamma_i} E(\gamma)$, where $E(\gamma)$ denotes the generalized eigenspace for the eigenvalue γ. Finally, let F be the invariant subspace belonging to the rest of the spectrum. With $\lambda_i = \log|\gamma_i|$ and $\kappa = \sup\{\log|z| : z \in \Sigma(T), |z| \leq \rho\}$, then all objects of Definition 1.3 are identified.

Example 1.5. If X is finite dimensional we are looking at random matrices. Put $F(\omega) = \{x \in X : \limsup_{n\to\infty} n^{-1} \log \|T(n, \omega)x\| = -\infty\}$ and $E(\omega) = F(\omega)^\perp$. The classical

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multiplicative ergodic theorem of Oseledets [**14**] says that if $\omega \mapsto \log^+ \|T(\omega)\|$ and $\omega \mapsto \log^+ \|T_E(\omega)^{-1}\|$ are integrable then T always admits a spectral theory (cf. Walters [**19**]).

As pointed out to the author by H. Crauel, however, in infinite dimensions the existence of a measurable splitting $E(\omega) \oplus F(\omega) = X$ with $\dim E(\omega) \equiv d < \infty$, $T(\omega)E(\omega) = E(\theta\omega)$, and $T(\omega)F(\omega) \subset F(\theta\omega)$ does not automatically imply that T admits a spectral theory: further conditions have to be imposed, e.g. the existence of a Lyapunov cohomology between $E(\omega)$ and $\mathbb{R}^d$ (cf. Arnold [**2**, Ch. 3.7]), which is the case, in particular, if X is a Hilbert space. For multiplicative ergodic theorems in infinite dimensions see also Schaumlöffel [**17**] and the references therein. The index of compactness occurring there can be considered as the analogue of $\log \rho_{\text{ess}}(T)$ in Example 1.4 in the sense that it is an optimal split index.

1.6. *The perturbation model.* In the following, we assume that a given T admits a spectral theory and refer to it as the unperturbed cocycle. Put $\tilde{T}(\omega) := e^{-\lambda_1} T(\omega)$. Then $\tilde{T}$ also admits a spectral theory with the same quasi-Oseledets spaces and Lyapunov exponents $0 > \lambda_2 - \lambda_1 > \cdots > \lambda_p - \lambda_1$. We may thus assume $\lambda_1 = 0$ without loss of generality.

Definition 1.7. A family $\{T_\epsilon : \Omega \to \mathbb{B}(X, X) : \epsilon > 0\}$ of strongly measurable random linear operators on X is said to be an *asymptotically small random perturbation* of T for the split index κ if it satisfies the following.

(1) (Strong convergence) For each $x \in X$ and $n \in \mathbb{N}$ we have

$$\lim_{\epsilon \to 0} \|T_\epsilon(n, \omega)x - T(n, \omega)x\| = 0, \quad \text{uniformly in } \omega.$$

(2) (Asymptotic norm convergence) There exists a $\tilde{\lambda}$ with $\kappa < \tilde{\lambda} < \lambda_p$ and an $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$ there is $\epsilon(n)$ independent of ω with

$$\|T_\epsilon(n, \omega) - T(n, \omega)\| < e^{n\tilde{\lambda}}, \quad \text{for all } 0 < \epsilon < \epsilon(n) \text{ and all } \omega \in \Omega.$$

Note that if $\{T_\epsilon\}$ is an asymptotically small random perturbation of T for the split index κ then it is an asymptotically small random perturbation of T for any split index $\geq \kappa$.

Remark 1.8. Conditions of this type have been introduced by Baladi and Young [**6**], who discovered its use to obtain abstract perturbation results and its validity for transfer operators coming from small perturbations of expanding maps in a finitely differentiable setting.

1.9. *The main result.* For any measurable splitting $E(\omega) \oplus F(\omega) = X$ we let $\Pi_{E\|F}(\omega)$ denote the projection with kernel $F(\omega)$ and range $E(\omega)$. Note that $\omega \mapsto \Pi_{E\|F}(\omega)$ is strongly measurable.

THEOREM 1.10. *Assume T admits a spectral theory with quasi-Oseledets splitting $E_1, \ldots, E_p, F$, Lyapunov exponents $0 = \lambda_1 > \cdots > \lambda_p$, and split index κ. Further, assume:*

(A1) *the mappings $\omega \mapsto \|\Pi_{E_i\|W_i}(\omega)\|$ are uniformly bounded in ω for all $1 \leq i \leq p$, where $W_i(\omega) := \bigoplus_{i=1,\, k\neq i}^{p} E_k(\omega) \oplus F(\omega)$,*

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(A2) *the convergences in Definition 1.3 are uniform in* ω.

If $\{T_\epsilon\}$ *is an asymptotically small random perturbation of* T *for* κ *with* $\log^+ \|T_\epsilon(\cdot)\| \in \mathbb{L}^1(\mathbb{P})$, *then the following holds.*

(1) *There exist (for sufficiently small* ϵ*) quasi-Oseledets splittings* $E_1^\epsilon, \ldots, E_p^\epsilon, F^\epsilon$ *for* T_ϵ *with* $\dim E_i^\epsilon = \dim E_i$ *and such that*

$$\lim_{\epsilon\to 0} E_i^\epsilon(\omega) = E_i(\omega), \quad \lim_{\epsilon\to 0} F^\epsilon(\omega) = F(\omega),$$

where convergence takes place in the topology on the set $\mathcal{V}(X)$ *of closed subspaces of* X *induced by the Hausdorff distance and is uniform in* ω.

(2) *If, for each* ϵ, λ_i^ϵ *is one of the (at most* d_i *different possible) values of* $\lim_{n\to\pm\infty} n^{-1} \log \|T_\epsilon(n,\omega)x\|$ *for* $x \in E_i^\epsilon(\omega) \setminus \{0\}$, *then*

$$\lim_{\epsilon\to 0} \lambda_i^\epsilon = \lambda_i.$$

Remark 1.11. One may hope that (A1) can be weakened into some integrability condition and (A2) can be dropped. Since we have one specific application in mind, where T is deterministic and hence (A1) and (A2) are trivially satisfied, we content ourselves with the present version and indicate where these assumptions enter the proof.

2. *Proof of the main result*

2.1. *Construction of* E^ϵ. Let $\mathcal{G}_d(X)$ denote the space of d-dimensional split subspaces of X, i.e. for $E \in \mathcal{G}_d(X)$ there is $F \in \mathcal{V}(X)$ with $E \oplus F = X$. Given any splitting $E \oplus F = X$ with $E \in \mathcal{G}_d(X)$, put $U_F := \{E_S \in \mathcal{G}_d(X) : E_S \oplus F = X\}$. Each $E_S \in U_F$ is represented by a family $S : E \to F$ of linear maps via

$$E_S = \operatorname{graph} S = \{x + Sx : x \in E\}$$

(cf. Abraham *et al.* [**1**, Example 3.1.8 G, p. 145 f]).

Consider the measurable subspace E defined by $E(\omega) := E_1(\omega) \oplus \cdots \oplus E_p(\omega)$ and put $d := \sum_{i=1}^p d_i = \dim E(\omega)$. For $\delta > 0$, define $L(\delta)$ to be the set of all measurable subspaces $E_S : \Omega \to \mathcal{G}_d(X)$ satisfying

$$E_S(\omega) \in U_{F(\omega)} \quad \text{and} \quad \|S(\omega)\| \le \delta.$$

For $E_S \in L(\delta)$, $n \in \mathbb{N}$, and $\epsilon > 0$, define the measurable subspace $E_S^\epsilon(n)$ by

$$E_S^\epsilon(n,\omega) := T_\epsilon(n, \theta^{-n}\omega) E_S(\theta^{-n}\omega).$$

The following two lemmas are an adaption of Young's graph-transform argument [**20**] (the phrase refers to the definition of $E_S^\epsilon(n,\omega)$ as the 'transformed graph' of the linear map $S(\theta^{-n}\omega)$).

LEMMA 2.2. *Fix* $0 < \delta < 1$. *There is* $N \in \mathbb{N}$ *and* $\tilde\epsilon > 0$ *such that for all* $0 < \epsilon < \tilde\epsilon$ *we have*

$$E_S^\epsilon(n) \in L(\delta), \quad \text{for all } n \ge N.$$

Proof. Fix λ' with $\tilde\lambda < \lambda' < \lambda_p$. We choose $N \ge N_0$ such that for $n \ge N$ we have:

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(i) $\|T(n,\omega)x\| \geq e^{n\lambda'}\|x\|$ for $x \in E(\omega) \setminus \{0\}$;
(ii) $\|T(n,\omega)x\| \leq e^{n\tilde{\lambda}}\|x\|$ for $x \in F(\omega)$;
(iii) $e^{n(\lambda'-\tilde{\lambda})} > 8\|\Pi_{E\|F}(\omega)\|$;
(iv) $e^{n(\tilde{\lambda}-\lambda')} < \frac{1}{16}\frac{1}{\|\Pi_{F\|E}(\omega)\| \cdot \|\Pi_{E\|F}(\omega)\|}$.

Here we used both (A1) and (A2). Further, put

$$C_n^{\omega}(\epsilon) := \sup_{x \in E(\omega)} \frac{\|T_\epsilon(n,\theta^{-n}\omega)x - T(n,\theta^{-n}\omega)x\|}{\|x\|}.$$

By condition (1) of Definition 1.7 and since $E(\omega)$ is finite dimensional, we have $\lim_{\epsilon\to 0} C_n^{\omega}(\epsilon) = 0$ uniformly in ω for each $n \in \mathbb{N}$.

For a fixed $n \geq N$, we will find an $\tilde{\epsilon}(n) > 0$ independent of E_S such that $E_S^{\epsilon}(n) \in L(\delta)$ for all $0 < \epsilon < \tilde{\epsilon}(n)$. From this, the claim follows immediately with $\tilde{\epsilon} = \min_{N\leq n<2N} \tilde{\epsilon}(n)$. (Indeed, for arbitrary $n \geq N$ write $n = p_n N + q_n$ with $p_n \in \mathbb{N}_0$ and $N \leq q_n < 2N$, and just observe that $T_\epsilon(n,\theta^{-n}\omega) = T_\epsilon(q_n,\theta^{-n}\omega) \circ T_\epsilon(N,\theta^{-p_n N}\omega) \circ \cdots \circ T_\epsilon(N,\theta^{-N}\omega)$.)

So now let $n \geq N$ and $E_S \in L(\delta)$ be arbitrary, but fixed. First, we have to check that $E_S^{\epsilon}(n,\omega) \in \mathcal{G}_d(X)$, i.e. we need to show that the linear map

$$\begin{aligned}\Phi^{\epsilon}(n,\omega) : E(\theta^{-n}\omega) &\to E(\omega)\\ x &\mapsto \Pi_{E\|F}(\omega) \circ T_\epsilon(n,\theta^{-n}\omega)(x + S(\theta^{-n}\omega)x)\end{aligned}$$

is one-to-one.

We have for $x \in E(\theta^{-n}\omega)$

$$\begin{aligned}&\|\Phi^{\epsilon}(n,\omega)x\|\\ &\quad= \|\Pi_{E\|F}(\omega) \circ T_\epsilon(n,\theta^{-n}\omega)(x) + \Pi_{E\|F}(\omega) \circ T_\epsilon(n,\theta^{-n}\omega)(S(\theta^{-n}\omega)x)\|\\ &\quad= \|\Pi_{E\|F}(\omega) \circ T(n,\theta^{-n}\omega)(x) + \Pi_{E\|F}(\omega) \circ (T_\epsilon(n,\theta^{-n}\omega) - T(n,\theta^{-n}\omega))(x)\\ &\qquad+ \Pi_{E\|F}(\omega) \circ T_\epsilon(n,\theta^{-n}\omega)(S(\theta^{-n}\omega)x)\|\\ &\quad\geq \|T(n,\theta^{-n}\omega)(x)\| - \|\Pi_{E\|F}(\omega)\|\\ &\qquad\cdot [\|(T_\epsilon(n,\theta^{-n}\omega) - T(n,\theta^{-n}\omega))(x) + T_\epsilon(n,\theta^{-n}\omega)(S(\theta^{-n}\omega)x)\|].\end{aligned}$$

By item (i) of the choice of N and since $x \in E(\theta^{-n}\omega)$ we have $\|T(n,\theta^{-n}\omega)x\| \geq e^{n\lambda'}\|x\|$. For $\epsilon < \epsilon(n)$, where $\epsilon(n)$ is from condition (2) of Definition 1.7, the term in brackets can be estimated by

$$\begin{aligned}[\ldots] &\leq C_n^{\omega}(\epsilon)\|x\| + \|T(n,\theta^{-n}\omega)(S(\theta^{-n}\omega)x)\| + e^{n\tilde{\lambda}}\|S(\theta^{-n}\omega)x\|\\ &\leq C_n^{\omega}(\epsilon)\|x\| + 2e^{n\tilde{\lambda}}\|S(\theta^{-n}\omega)x\|\end{aligned}$$

by item (ii) of the choice of N and since $S(\theta^{-n}\omega)x \in F(\theta^{-n}\omega)$. Using $\|S(\theta^{-n}\omega)\| \leq \delta < 1$ we conclude for $x \in E(\theta^{-n}\omega)$

$$\|\Phi^{\epsilon}(n,\omega)x\| \geq e^{n\lambda'}\|x\| - \|\Pi_{E\|F}(\omega)\| \cdot [(C_n^{\omega}(\epsilon) + 2e^{n\tilde{\lambda}})\|x\|].$$

By item (iii) of the choice of N, and since $\lim_{\epsilon\to 0} C_n^{\omega}(\epsilon) = 0$, this yields $\|\Phi^{\epsilon}(n,\omega)x\| > 0$ for sufficiently small ϵ, hence $\Phi^{\epsilon}(n,\omega)$ is one-to-one.

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More specifically, we can choose $\tilde{\epsilon}(n)$ such that

$$\|\Phi^\epsilon(n,\omega)\| \geq \tfrac{1}{2}e^{n\lambda'}, \quad \text{for all } \epsilon < \tilde{\epsilon}(n). \tag{$*$}$$

We will use this to check that $\|\Pi_S^\epsilon(n,\omega)\| \leq \delta/2$, where $\Pi_S^\epsilon(n,\omega)$ denotes the restriction of $\Pi_{F\|E}(\omega)$ to $E_S^\epsilon(n,\omega)$. From this, one easily concludes that the linear map $S^\epsilon(n,\omega) : E(\omega) \to F(\omega)$ representing $E_S^\epsilon(n,\omega)$ satisfies $\|S^\epsilon(n,\omega)\| \leq \delta$.

First, note that

$$\begin{aligned}\|\Pi_S^\epsilon(n,\omega)\| &= \sup_{x\in E_S^\epsilon(n,\omega)} \frac{\|\Pi_{F\|E}(\omega)x\|}{\|x\|} \\ &= \sup_{x\in E(\theta^{-n}\omega)} \frac{\|\Pi_{F\|E}(\omega)\circ T_\epsilon(n,\theta^{-n}\omega)(x+S(\theta^{-n}\omega)x)\|}{\|T_\epsilon(n,\theta^{-n}\omega)(x+S(\theta^{-n}\omega)x)\|}.\end{aligned}$$

Now, by $(*)$ we have for $x \in E(\theta^{-n}\omega)$

$$\|T_\epsilon(n,\theta^{-n}\omega)(x+S(\theta^{-n}\omega)x)\| \geq \frac{1}{\|\Pi_{E\|F}(\omega)\|}\frac{1}{2}e^{n\lambda'}\|x\|.$$

On the other hand, one obtains as above for $x \in E(\theta^{-n}\omega)$

$$\begin{aligned}&\|\Pi_{F\|E}(\omega)\circ T_\epsilon(n,\theta^{-n}\omega)(x+S(\theta^{-n}\omega)x)\| \\ &\quad= \|\Pi_{F\|E}(\omega)\circ T(n,\theta^{-n}\omega)(x) + \Pi_{F\|E}(\omega)\circ(T_\epsilon(n,\theta^{-n}\omega) - T(n,\theta^{-n}\omega))(x) \\ &\qquad+ \Pi_{F\|E}(\omega)\circ T_\epsilon(n,\theta^{-n}\omega)(S(\theta^{-n}\omega)x)\| \\ &\quad\leq \|\Pi_{F\|E}(\omega)\|\cdot[(C_n^\omega(\epsilon) + 2e^{n\tilde{\lambda}}\delta)\|x\|].\end{aligned}$$

Putting things together, thus, we have

$$\begin{aligned}\|\Pi_S^\epsilon(n,\omega)\| &\leq (\|\Pi_{E\|F}(\omega)\|2e^{-n\lambda'})(\|\Pi_{F\|E}(\omega)\|(C_n^\omega(\epsilon) + 2e^{n\tilde{\lambda}}\delta)) \\ &\leq \tfrac{1}{8}(e^{-n\tilde{\lambda}}C_n^\omega(\epsilon) + 2\delta)\end{aligned}$$

by item (iv) of the choice of N. Hence $\|\Pi_S^\epsilon(n,\omega)\| \leq \delta/2$ for ϵ small enough, which completes the proof. □

Observe that $L(\delta)$ is a closed subset of the Banach space of all measurable subspaces E_S satisfying $E_S(\omega) \oplus F(\omega) = X$ and

$$\|E_S\| := \int \|S(\omega)\|\, d\mathbb{P}(\omega) < \infty.$$

LEMMA 2.3. *Fix $0 < \delta < 1$. There is $\epsilon(\delta) > 0$ such that for all $0 < \epsilon < \epsilon(\delta)$ the limit*

$$\lim_{n\to\infty} E_S^\epsilon(n) =: E^\epsilon$$

exists and is independent of $E_S \in L(\delta)$.

Moreover, $E^\epsilon \in L(\delta)$ and $T_\epsilon(\omega)E^\epsilon(\omega) = E^\epsilon(\theta\omega)$.

Proof. Fix λ' with $\tilde{\lambda} < \lambda' < \lambda_p$ and choose N as in the proof of Lemma 2.2. We will show that the map $\Gamma_\epsilon^n : L(\delta) \to L(\delta)$ defined by

$$(\Gamma_\epsilon^n E_S)(\omega) := E_S^\epsilon(n,\omega)$$

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is a contraction for fixed $n \geq N$ and $\epsilon < \tilde{\epsilon}(n)$, where $\tilde{\epsilon}(n)$ is from $(*)$ of the proof of Lemma 2.2. From this, the claim follows immediately with $\epsilon(\delta) = \min_{N \leq n < 2N} \tilde{\epsilon}(n)$ by the Banach contraction principle.

Let $E_{S_1}, E_{S_2} \in L(\delta)$ be given. The proof of Lemma 2.2 yields that the linear maps $S_i^\epsilon(n,\omega) : E(\omega) \to F(\omega)$ representing $E_{S_i}^\epsilon(n,\omega)$ are given by

$$S_i^\epsilon(n,\omega)x = \Pi_{F\|E}(\omega) \circ T_\epsilon(n, \theta^{-n}\omega)(\Phi_i^\epsilon(n,\omega)^{-1}x + S_i(\theta^{-n}\omega)(\Phi_i^\epsilon(n,\omega)^{-1}x)).$$

We have to estimate

$$\begin{aligned} \|S_1^\epsilon(n,\omega) - S_2^\epsilon(n,\omega)\| &= \sup_{x \in E(\omega)} \frac{\|(S_1^\epsilon(n,\omega) - S_2^\epsilon(n,\omega))(x)\|}{\|x\|} \\ &= \sup_{x \in E(\theta^{-n}\omega)} \frac{\|(S_1^\epsilon(n,\omega) - S_2^\epsilon(n,\omega))(\Phi_1^\epsilon(n,\omega)x)\|}{\|\Phi_1^\epsilon(n,\omega)x\|}. \qquad (**) \end{aligned}$$

We have for $x \in E(\theta^{-n}\omega)$

$$\begin{aligned} &\|(S_1^\epsilon(n,\omega) - S_2^\epsilon(n,\omega))(\Phi_1^\epsilon(n,\omega)x)\| \\ &\quad \leq \|S_1^\epsilon(n,\omega)(\Phi_1^\epsilon(n,\omega)x) - S_2^\epsilon(n,\omega)(\Phi_2^\epsilon(n,\omega)x)\| \\ &\qquad + \|S_2^\epsilon(n,\omega)(\Phi_2^\epsilon(n,\omega)x) - S_2^\epsilon(n,\omega)(\Phi_1^\epsilon(n,\omega)x)\| \\ &\quad \leq \|\Pi_{F\|E}(\omega)\|\|T_\epsilon(n,\theta^{-n}\omega)((S_1(\theta^{-n}\omega) - S_2(\theta^{-n}\omega))x)\| \\ &\qquad + \|S_2^\epsilon(n,\omega)\|\|(\Phi_2^\epsilon(n,\omega) - \Phi_1^\epsilon(n,\omega))x\| \\ &\quad \leq (\|\Pi_{F\|E}(\omega)\| + \delta\|\Pi_{E\|F}(\omega)\|)\|T_\epsilon(n,\theta^{-n}\omega)((S_1(\theta^{-n}\omega) - S_2(\theta^{-n}\omega))x)\| \end{aligned}$$

since $\Phi_2^\epsilon(n,\omega) - \Phi_1^\epsilon(n,\omega) = \Pi_{E\|F}(\omega) \circ T_\epsilon(n,\theta^{-n}\omega) \circ (S_1(\theta^{-n}\omega) - S_2(\theta^{-n}\omega))$ and $\|S_2^\epsilon(n,\omega)\| \leq \delta$ by lemma 2.2.

Using condition (2) of Definition 1.7 as before, this gives for $x \in E(\theta^{-n}\omega)$

$$\begin{aligned} &\|(S_1^\epsilon(n,\omega) - S_2^\epsilon(n,\omega))(\Phi_1^\epsilon(n,\omega)x)\| \\ &\qquad \leq (\|\Pi_{F\|E}(\omega)\| + \delta\|\Pi_{E\|F}(\omega)\|)2e^{n\tilde{\lambda}}\|(S_1(\theta^{-n}\omega) - S_2(\theta^{-n}\omega))x\|. \end{aligned}$$

Plugging this and $(*)$ into $(**)$ yields

$$\|S_1^\epsilon(n,\omega) - S_2^\epsilon(n,\omega)\| \leq (\|\Pi_{F\|E}(\omega)\| + \delta\|\Pi_{E\|F}(\omega)\|)4e^{n(\tilde{\lambda}-\lambda')}\|S_1(\theta^{-n}\omega) - S_2(\theta^{-n}\omega)\|.$$

Using the $\mathbb{P}$-invariance of θ, we thus obtain

$$\int \|S_1^\epsilon(n,\omega) - S_2^\epsilon(n,\omega)\|\, d\mathbb{P}(\omega) \leq \alpha_n \int \|S_1(\omega) - S_2(\omega)\|\, d\mathbb{P}(\omega),$$

with $\alpha_n := (\|\Pi_{F\|E}(\omega)\| + \|\Pi_{E\|F}(\omega)\|)4e^{n(\tilde{\lambda}-\lambda')}$. Adding $\alpha_n < 1$ for all $n \geq N$ to the initial requirements for the choice of N, we have indeed that Γ_ϵ^n is a contraction, which completes the proof. □

COROLLARY 2.4. *Fix $0 < \delta < 1$ and let $\epsilon(\delta)$ and E^ϵ be from Lemma 2.3. Then, for $0 < \epsilon < \epsilon(\delta)$, we have:*

(i) $(1-\delta)\|\Pi_{E\|F}(\omega)\| \leq \|\Pi_{E^\epsilon\|F}(\omega)\| \leq (1+\delta)\|\Pi_{E\|F}(\omega)\|$;

(ii) $\|\Pi_{F\|E}(\omega)\| - \delta\|\Pi_{E\|F}(\omega)\| \leq \|\Pi_{F\|E^\epsilon}(\omega)\| \leq \|\Pi_{F\|E}(\omega)\| + \delta\|\Pi_{E\|F}(\omega)\|$.

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In particular, thus,

$$\lim_{\epsilon \to 0} E^\epsilon(\omega) = E(\omega), \quad \textit{uniformly in } \omega.$$

Proof. Let $S_\epsilon(\omega) : E(\omega) \to F(\omega)$ be the linear map representing $E^\epsilon(\omega)$. Write $x \in X$ as $x = x_E + x_F$ with $x_E \in E(\omega)$ and $x_F \in F(\omega)$. We have

$$x = (x_E + S_\epsilon(\omega)x_E) + (x_F - S_\epsilon(\omega)x_E).$$

Hence

$$\Pi_{E^\epsilon \| F}(\omega) = \Pi_{E\|F}(\omega) + S_\epsilon(\omega) \circ \Pi_{E\|F}(\omega)$$

and

$$\Pi_{F\|E^\epsilon}(\omega) = \Pi_{F\|E}(\omega) - S_\epsilon(\omega) \circ \Pi_{E\|F}(\omega).$$

With this, and $\|S_\epsilon(\omega)\| \le \delta$, the claim is immediate. □

2.5. *Construction of the invariant complement.*

LEMMA 2.6. *With the notations of the preceding subsection, we have that for sufficiently small $\epsilon > 0$ there exist measurable subspaces $F^\epsilon : \Omega \to \mathcal{G}^d(X)$, where $\mathcal{G}^d(X)$ denotes the space of split subspaces of X with codimension d, such that:*
(i) $E^\epsilon(\omega) \oplus F^\epsilon(\omega) = X$;
(ii) $T_\epsilon(\omega)F^\epsilon(\omega) \subset F^\epsilon(\theta\omega)$.
Moreover,

$$\lim_{\epsilon \to 0} F^\epsilon(\omega) = F(\omega), \quad \textit{uniformly in } \omega.$$

Proof. We define $P^\epsilon(n, \omega) : X \to E^\epsilon(\omega)$ by

$$P^\epsilon(n, \omega) := T_\epsilon(n, \omega)^{-1} \circ \Pi_{E^\epsilon \| F}(\theta^n \omega) \circ T_\epsilon(n, \omega).$$

Here and in the following $T_\epsilon(n, \omega)^{-1}$ denotes the inverse of $T_\epsilon(n, \omega)$ restricted to $E^\epsilon(\omega)$. We will show for each ω separately that the limit

$$\lim_{n \to \infty} P^\epsilon(n, \omega) =: P^\epsilon(\omega) \tag{1}$$

exists as a bounded operator (defining a projection to $E^\epsilon(\omega)$).

As the pointwise limit of strongly measurable $\mathbb{B}(X, X)$-valued random variables, P^ϵ is itself strongly measurable. Hence $F^\epsilon(\omega) := \operatorname{Ker} P^\epsilon(\omega)$ defines a measurable subspace which obviously satisfies (i). Property (ii) follows from the observation that $T_\epsilon(\omega) \circ P^\epsilon(\omega) = P^\epsilon(\theta\omega) \circ T_\epsilon(\omega)$. The convergence claim finally follows from $\lim_{\epsilon \to 0} P^\epsilon(\omega) = \Pi_{E\|F}(\omega)$.

To prove (1) we will show that

$$\limsup_{n \to \infty} \frac{1}{n} \log \| P^\epsilon(n + 1, \omega) - P^\epsilon(n, \omega) \| < 0. \tag{2}$$

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Note that $P^\epsilon(n,\omega)|E^\epsilon(\omega)$ is the identity map, so that

$$P^\epsilon(n,\omega) = \Pi_{E^\epsilon\|F}(\omega) + P^\epsilon(n,\omega)\circ\Pi_{F\|E^\epsilon}(\omega). \tag{3}$$

Further, we have

$$\begin{aligned}
&P^\epsilon(n+1,\omega)\circ\Pi_{F\|E^\epsilon}(\omega)\\
&\quad= T_\epsilon(n+1,\omega)^{-1}\circ\Pi_{E^\epsilon\|F}(\theta^{n+1}\omega)\circ T_\epsilon(n+1,\omega)\circ\Pi_{F\|E^\epsilon}(\omega)\\
&\quad= T_\epsilon(n+1,\omega)^{-1}\circ\Pi_{E^\epsilon\|F}(\theta^{n+1}\omega)\circ T_\epsilon(\theta^n\omega)\circ\Pi_{E^\epsilon\|F}(\theta^n\omega)\circ T_\epsilon(n,\omega)\circ\Pi_{F\|E^\epsilon}(\omega)\\
&\qquad+ T_\epsilon(n+1,\omega)^{-1}\circ\Pi_{E^\epsilon\|F}(\theta^{n+1}\omega)\circ T_\epsilon(\theta^n\omega)\\
&\qquad\circ\Pi_{F\|E^\epsilon}(\theta^n\omega)\circ T_\epsilon(n,\omega)\circ\Pi_{F\|E^\epsilon}(\omega).
\end{aligned}$$

Observe that the first summand above equals $P^\epsilon(n,\omega)\circ\Pi_{F\|E^\epsilon}(\omega)$, so that, in view of (3), this yields

$$\begin{aligned}
P^\epsilon(n+1,\omega) - P^\epsilon(n,\omega) = T_\epsilon(n+1,\omega)^{-1}\circ\Pi_{E^\epsilon\|F}(\theta^{n+1}\omega)\circ T_\epsilon(\theta^n\omega)\circ\Pi_{F\|E^\epsilon}(\theta^n\omega)\\
\circ T_\epsilon(n,\omega)\circ\Pi_{F\|E^\epsilon}(\omega).
\end{aligned}$$

Note that:

- $\limsup_{n\to\infty} n^{-1}\log\|\Pi_{E^\epsilon\|F}(\theta^n\omega)\| = 0$ by Corollary 2.4(i) and assumption (A1) (here $\log^+\|\Pi_{E\|F}(\cdot)\| \in \mathbb{L}^1(\mathbb{P})$ would suffice);
- $\lim_{n\to\infty} n^{-1}\log\|T_\epsilon(\theta^n\omega)\| = 0$ since $\log^+\|T_\epsilon(\cdot)\| \in \mathbb{L}^1(\mathbb{P})$.

If we put

$$\Lambda_1 := \limsup_{n\to\infty}\frac{1}{n}\log\|T_\epsilon(n,\omega)^{-1}\|$$

and

$$\Lambda_2 := \limsup_{n\to\infty}\frac{1}{n}\log\|\Pi_{F\|E^\epsilon}(\theta^n\omega)\circ T_\epsilon(n,\omega)\circ\Pi_{F\|E^\epsilon}(\omega)\|,$$

we thus have

$$\limsup_{n\to\infty}\frac{1}{n}\log\|P^\epsilon(n+1,\omega) - P^\epsilon(n,\omega)\| \le \Lambda_1 + \Lambda_2. \tag{4}$$

Estimation of Λ_1. Let $S^\epsilon(\omega): E(\omega)\to F(\omega)$ be the linear map representing $E^\epsilon(\omega)$. Note that $\Pi_{F\|E}(\omega)|E^\epsilon(\omega) = S^\epsilon(\omega)\circ\Pi_{E\|F}(\omega)$.

Fix λ' and $\hat\lambda$ with $\tilde\lambda < \hat\lambda < \lambda' < \lambda_p$. Choose $N \ge N_0$ such that for $n \ge N$ we have:

(i) $\|T(n,\omega)x\| \ge e^{n\lambda'}\|x\|$ for $x \in E(\omega)\setminus\{0\}$;

(ii) $e^{n\lambda'} - e^{n\hat\lambda} \ge e^{n\hat\lambda}$.

Fix $n \ge N$. For $\epsilon < \epsilon(n)$ we have for $x \in E(\omega)$

$$\begin{aligned}
\|T_\epsilon(n,\omega)x\| &\ge \|T(n,\omega)x\| - e^{n\tilde\lambda}\|x\|\\
&\ge \|T(n,\omega)\Pi_{E\|F}(\omega)x\| - (\|T(n,\omega)\Pi_{F\|E}(\omega)x\| + e^{n\tilde\lambda}\|x\|)\\
&\ge e^{n\lambda'}\|\Pi_{E\|F}(\omega)x\| - (\|T(n,\omega)\Pi_{F\|E}(\omega)x\| + e^{n\tilde\lambda}\|x\|)\\
&\ge e^{n\lambda'}\|x\| - ((e^{n\lambda'} + \|T(n,\omega)\|)\|\Pi_{F\|E}(\omega)x\| + e^{n\tilde\lambda}\|x\|).
\end{aligned}$$

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For the last line we simply used $\|\Pi_{E\|F}(\omega)x\| \geq \|x\| - \|\Pi_{F\|E}(\omega)x\|$. Now for $x \in E^\epsilon(\omega)$ we have $\|\Pi_{F\|E}(\omega)x\| \leq \|S^\epsilon(\omega)\|\|\Pi_{E\|F}(\omega)\|\|x\|$. Since $\lim_{\epsilon\to 0}\|S^\epsilon(\omega)\| = 0$ we can choose $\tilde{\epsilon}(n) \leq \epsilon(n)$ such that for all $\epsilon \leq \tilde{\epsilon}(n)$ the above reads

$$\|T_\epsilon(n,\omega)x\| \geq (e^{n\lambda'} - e^{n\hat{\lambda}})\|x\| \geq e^{n\hat{\lambda}}\|x\|, \quad \text{for } x \in E^\epsilon(\omega),$$

which finally yields

$$\|T_\epsilon(n,\omega)^{-1}\| \leq e^{-n\hat{\lambda}}.$$

For arbitrary $n \geq N$ write $n = p_nN + q_n$ with $p_n \in \mathbb{N}_0$ and $N \leq q_n < 2N$. With $T_\epsilon(n,\omega) = T_\epsilon(q_n, \theta^{p_nN}\omega) \circ T_\epsilon(p_nN,\omega)$ we have, for $\epsilon < \min_{N\leq n<2N}\epsilon(n)$,

$$\|T_\epsilon(n,\omega)^{-1}\| \leq \|T_\epsilon(p_nN,\omega)^{-1}\|\|T_\epsilon(q_n,\theta^{p_nN}\omega)^{-1}\| \leq \left(\prod_{i=0}^{p_n-1} e^{-N\hat{\lambda}}\right)e^{-q_n\hat{\lambda}}.$$

Hence, $\log\|T_\epsilon(n,\omega)^{-1}\| \leq -p_nN\hat{\lambda} - q_n\hat{\lambda} = -n\hat{\lambda}$, i.e. $\Lambda_1 \leq -\hat{\lambda}$.

Estimation of Λ_2. By Corollary 2.4(ii) we find a constant K such that $\|\Pi_{F\|E^\epsilon}(\omega)\| \leq K$ for all $\omega \in \Omega$. Here, we made quite an essential use of assumption (A1). Choose $N \geq N_0$ such that for $n \geq N$ we have:

(i) $\|T(n,\omega)x\| \leq e^{n\tilde{\lambda}}\|x\|$ for $x \in F(\omega)$,

(ii) $e^{n(\hat{\lambda}-\tilde{\lambda})} > 2K$,

where $\hat{\lambda}$ is the same as in the estimation of Λ_1.

Fix $n \geq N$. For $\epsilon < \epsilon(n)$ we obtain straightforwardly

$$\|T_\epsilon(n,\omega)\Pi_{F\|E^\epsilon}(\omega)\| \leq 2Ke^{n\tilde{\lambda}}.$$

For arbitrary $n \geq N$ write $n = p_nN + q_n$ as above and use the T_ϵ-invariance of E^ϵ to obtain

$$\begin{aligned}
&\Pi_{F\|E^\epsilon}(\theta^n\omega) \circ T_\epsilon(n,\omega) \circ \Pi_{F\|E^\epsilon}(\omega)\\
&\quad = \Pi_{F\|E^\epsilon}(\theta^n\omega) \circ T_\epsilon(q_n,\theta^{p_nN}\omega) \circ \Pi_{F\|E^\epsilon}(\theta^{p_nN}\omega)\\
&\qquad \circ T_\epsilon(N,\theta^{(p_n-1)N}\omega) \circ \Pi_{F\|E^\epsilon}(\theta^{(p_n-1)N}\omega) \circ \cdots \circ T_\epsilon(N,\omega) \circ \Pi_{F\|E^\epsilon}(\omega).
\end{aligned}$$

For $\epsilon < \min_{N\leq n<2N}\epsilon(n)$, thus,

$$\begin{aligned}
&\|\Pi_{F\|E^\epsilon}(\theta^n\omega) \circ T_\epsilon(n,\omega) \circ \Pi_{F\|E^\epsilon}(\omega)\|\\
&\quad \leq \|\Pi_{F\|E^\epsilon}(\theta^n\omega)\|\|T_\epsilon(q_n,\theta^{p_nN}\omega) \circ \Pi_{F\|E^\epsilon}(\theta^{p_nN}\omega)\|\\
&\qquad \times \prod_{i=0}^{p_n-1}\|T_\epsilon(N,\theta^{iN}\omega) \circ \Pi_{F\|E^\epsilon}(\theta^{iN}\omega)\|\\
&\quad \leq 2K^2e^{q_n\tilde{\lambda}}\prod_{i=0}^{p_n-1}(2Ke^{N\tilde{\lambda}}).
\end{aligned}$$

Hence

$$\Lambda_2 \leq \frac{1}{N}(\log 2K + N\tilde{\lambda}) < \hat{\lambda}.$$

Since $\Lambda_1 \leq -\hat{\lambda}$ and in view of (4) we obtain (2), which completes the proof. □

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2.7. *Construction of the quasi-Oseledets splitting.* For $1 \le k < p$, consider the measurable subspaces V_k and F_k defined by $V_k(\omega) := E_1(\omega) \oplus \cdots \oplus E_k(\omega)$ and $F_k(\omega) := E_{k+1}(\omega) \oplus \cdots \oplus E_p(\omega) \oplus F(\omega)$. Put $d_k := \sum_{i=1}^{k} d_i = \dim V_k(\omega)$. For the sake of notational consistency, put $V_p := E$, $F_p := F$, and $d_p := d$. The preceding constructions can be carried out with E and F replaced by V_k and F_k, respectively. So we have the following provisional result.

PROPOSITION 2.8. *Assume $\epsilon > 0$ to be sufficiently small. For each $k \in \mathbb{N}$ with $1 \le k \le p$ there exist measurable subspaces $V_k^\epsilon : \Omega \to \mathcal{G}_{d_k}(X)$ and $F_k^\epsilon : \Omega \to \mathcal{G}^{d_k}(X)$ such that:*
(1) $V_k^\epsilon(\omega) \oplus F_k^\epsilon(\omega) = X$;
(2) $T_\epsilon(\omega)F_k^\epsilon(\omega) \subset F_k^\epsilon(\theta\omega)$ *and* $T_\epsilon(\omega)V_k^\epsilon(\omega) = V_k^\epsilon(\theta\omega)$.
Moreover,

$$\lim_{\epsilon\to 0} V_k^\epsilon(\omega) = V_k(\omega), \quad \lim_{\epsilon\to 0} F_k^\epsilon(\omega) = F_k(\omega), \quad \textit{uniformly in } \omega.$$

2.9. *Proof of part (1) of Theorem 1.1.* Put $E_1^\epsilon := V_1^\epsilon$. For $1 < k \le p$, we construct E_k^ϵ inductively as follows.

Define a measurable subspace $\tilde{E}_k^\epsilon : \Omega \to \mathcal{G}_{d_k}(X)$ by

$$\tilde{E}_k^\epsilon(\omega) := V_{k-1}^\epsilon(\omega) \oplus E_k(\omega).$$

Obviously, $\tilde{E}_k^\epsilon(\omega)$ is complementary to $F_k(\omega)$ and represented by the linear map $\tilde{S}_k^\epsilon(\omega) := S_{k-1}^\epsilon(\omega) \circ \Pi_{V_{k-1}\| E_k}(\omega)$, where $S_{k-1}^\epsilon(\omega) : V_{k-1}(\omega) \to F_{k-1}(\omega)$ represents $V_{k-1}^\epsilon(\omega)$. Hence $\|\tilde{S}_k^\epsilon(\omega)\| \le \delta \|\Pi_{V_{k-1}\| E_k}(\omega)\|$, so by (A1) we may start the limit procedure (cf. Lemma 2.3) to construct V_k^ϵ with $\tilde{E}_k^\epsilon$.

By Lemma 2.2 and since V_{k-1}^ϵ is T_ϵ-invariant, we have

$$\tilde{E}_k^\epsilon(n,\omega) := T_\epsilon(n,\theta^{-n}\omega)\tilde{E}_k^\epsilon(\theta^{-n}\omega) = V_{k-1}^\epsilon(\omega) \oplus T_\epsilon(n,\theta^{-n}\omega)E_k(\theta^{-n}\omega)$$

for sufficiently large n.

But $\lim_{n\to\infty} \tilde{E}_k^\epsilon(n,\omega) = V_k^\epsilon(\omega)$ for sufficiently small ϵ by Lemma 2.3, so the limit

$$\lim_{n\to\infty} T_\epsilon(n,\theta^{-n}\omega)E_k(\theta^{-n}\omega) =: E_k^\epsilon(\omega)$$

exists and satisfies the statements of Theorem 1.10(1). □

2.10. *Proof of part (2) of Theorem 1.1.* Let $S_k^\epsilon(\omega) : E_k(\omega) \to W_k(\omega)$ be the linear map representing $E_k^\epsilon(\omega)$, where, to recall, $W_k(\omega) := \bigoplus_{i=1,\, i\ne k}^{p} E_i(\omega) \oplus F(\omega)$.

For $x \in E_k^\epsilon(\omega)$ we have:

$$x = \Pi_{E_k\| W_k}(\omega)x + S_k^\epsilon(\omega) \circ \Pi_{E_k\| W_k}(\omega)x.$$

Fix $\delta > 0$. Choose $N \ge N_0$ such that for $n \ge N$ we have:
(i) $\|T(n,\omega)x\| \le e^{n(\lambda_k+(\delta/3))}\|x\|$ for $x \in E_k(\omega) \setminus \{0\}$,
(ii) $e^{n\tilde{\lambda}} + e^{n(\lambda_k+(\delta/3))}\|\Pi_{E_k\| W_k}(\omega)\| \le e^{n(\lambda_k+(\delta/2)}$,

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where again we have used (A1).

For $n \geq N$ choose $\tilde{\epsilon}(n) \leq \epsilon(n)$ such that

$$e^{n(\lambda_k+(\delta/2))} + \|T(n,\omega)\|\|S_k^\epsilon(\omega)\|\|\Pi_{E_k\|W_k}(\omega)\| \leq e^{n(\lambda_k+\delta)}.$$

This is possible since $\lim_{\epsilon\to 0}\|S_k^\epsilon(\omega)\| = 0$ uniformly in ω. (Recall that N_0, $\tilde{\lambda}$, and $\epsilon(n)$ are from condition (2) of Definition 1.7.)

With this, we have for fixed $n \geq N$ and $\epsilon \leq \tilde{\epsilon}(n)$

$$\begin{aligned}\|T_\epsilon(n,\omega)x\| &\leq e^{n\tilde{\lambda}}\|x\| + \|T(n,\omega)x\| \\ &\leq e^{n\tilde{\lambda}}\|x\| + \|T(n,\omega)\circ\Pi_{E_k\|W_k}(\omega)x\| + \|T(n,\omega)\circ S_k^\epsilon(\omega)\circ\Pi_{E_k\|W_k}(\omega)x\| \\ &\leq e^{n\tilde{\lambda}} + e^{n(\lambda_k+(\delta 3))}\|\Pi_{E_k\|W_k}(\omega)\| + \|T(n,\omega)\|\|S_k^\epsilon(\omega)\|\|\Pi_{E_k\|W_k}(\omega)\| \\ &\leq e^{n(\lambda_k+\delta)}, \quad \text{for } x \in E_k^\epsilon(\omega) \text{ with } \|x\| = 1\end{aligned}$$

by the choices just made.

Proceeding as in the estimation of Λ_2 in the proof of Lemma 2.6, we obtain that for all $\epsilon \leq \epsilon(\delta) := \min_{N\leq n<2N}\tilde{\epsilon}(n)$ we have

$$\lim_{n\to\infty}\frac{1}{n}\log\|T_\epsilon(n,\omega)x\| \leq \lambda_k + \delta, \quad \text{for } x \in E_k^\epsilon(\omega)\setminus\{0\}.$$

With the notation of Theorem 1.10(2), hence

$$\limsup_{\epsilon\to 0}\lambda_i^\epsilon \leq \lambda_i.$$

To prove $\liminf_{\epsilon\to 0}\lambda_i^\epsilon \geq \lambda_i$ we can use exactly the same argument as in the estimation of Λ_1 in the proof of Lemma 2.6. (Just replace $S^\epsilon(\omega) : E(\omega) \to F(\omega)$ by $S_k^\epsilon(\omega) : E_k(\omega) \to W_k(\omega)$ and λ_p by λ_k, and put $\hat{\lambda} = \lambda_k - \delta$ and $\lambda' = \lambda_k - \delta/2$.) This yields

$$\lim_{n\to\infty}\frac{1}{n}\log\|T_\epsilon(n,\theta^{-n}\omega)^{-1}x\| \leq -(\lambda_k - \delta), \quad \text{for } x \in E_k^\epsilon(\omega)\setminus\{0\},$$

hence

$$\lim_{n\to-\infty}\frac{1}{n}\log\|T_\epsilon(n,\omega)x\| \geq \lambda_k - \delta, \quad \text{for } x \in E_k^\epsilon(\omega)\setminus\{0\},$$

completing the proof of Theorem 1.10. □

3. *Application to the spectrum of randomly perturbed expanding maps*

3.1. *Random transfer operators.* Let $\mathrm{End}^k(M)$ denote the space of all $\mathcal{C}^k$ endomorphisms on a compact d-dimensional smooth Riemannian manifold M ($k \geq 2$). We endow $\mathrm{End}^k(M)$ with the measurable structure coming from the usual $\mathcal{C}^k$ topology. As before, let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be an ergodic and invertible dynamical system.

We refer to each measurable map $\varphi : \Omega \to \mathrm{End}^k(M)$ as a *random dynamical system* (abbreviated RDS henceforth), while we actually have in mind the dynamics produced by

$$\varphi(n,\omega) := \varphi(\vartheta^{n-1}\omega)\circ\cdots\circ\varphi(\omega), \quad \text{for } n > 0, \quad \text{and} \quad \varphi(0,\omega) = \mathrm{id}.$$

We say that an RDS φ is *expanding* if there is a real number $\gamma > 1$ and a Riemannian metric $\|\cdot\|$ on M such that for any $x \in M$ and $v \in T_xM$ we have $\|(D_x\varphi(\omega))(v)\| \geq \gamma\|v\|$.

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We put $X_r = \mathcal{C}^r(M, \mathbb{C})$, the Banach space of complex $\mathcal{C}^r$ functions endowed with the $\mathcal{C}^r$ norm $\|\cdot\|_r$. Here r is of the form $r = (m, \alpha)$ with $m \in \mathbb{N}_0$ and $0 \le \alpha < 1$, i.e. the mth derivative of $f \in X_r$ is α-Hölder.

For any measurable $g : \Omega \to \mathcal{C}^k(M, \mathbb{R})$ and $r = (m, \alpha)$ with $m + \alpha \le k$ we define the *random transfer operator* $\mathcal{L} = \mathcal{L}_g : \Omega \to \mathbb{B}(X_r, X_r)$ by

$$(\mathcal{L}(\omega) f)(x) = \sum_{y \in \varphi(\omega)^{-1}\{x\}} e^{g(\omega, y)} f(y).$$

It can be seen that $\mathcal{L}$ is strongly measurable. (Transfer operators depend even continuously on the dynamics and the weight function with respect to the strong operator topology.) The following is a reformulation of a result by Kifer [**11**]. It says that $\mathcal{L}$ admits a spectral theory for $r = (0, \alpha)$.

LEMMA 3.2. *Assume $\omega \mapsto \log^+ \|\mathcal{L}(\omega)\|_0$ is integrable. Then, there is a θ-invariant set $\Omega_0 \in \mathcal{F}$ of full $\mathbb{P}$-measure on which the following holds.*

(1) *There exist a unique $h : \Omega \to \mathcal{C}^{(0,\alpha)}(M, \mathbb{R}^+)$ with $\omega \mapsto \|\log h(\omega)\|_0$ integrable and a random positive scalar λ with $\log \lambda \in \mathbb{L}^1(\mathbb{P})$ such that*

$$\mathcal{L}(\omega) h(\omega) = \lambda(\omega) h(\theta\omega).$$

Further, there is a unique invariant splitting

$$X_{(0,\alpha)} = W(\omega) \oplus \mathbb{C} h(\omega),$$

that is, $\mathcal{L}(\omega) W(\omega) \subset W(\theta\omega)$.

(2) *We have*

$$f \notin W(\omega) \Rightarrow \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}(n, \omega) f\|_0 = \int \log \lambda \, d\mathbb{P} =: \Lambda$$

$$f \in W(\omega) \Rightarrow \limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}(n, \omega) f\|_0 \le K < \Lambda$$

for some uniform constant K.

Proof. h and λ can be taken directly from Kifer [**11**, Proposition 2.5], which in addition yields uniquely determined random probability measures ν_ω satisfying $\mathcal{L}^*(\omega)\nu_{\theta\omega} = \lambda(\omega)\nu_\omega$ and

$$\limsup_{n\to\infty} \frac{1}{n} \log \left\| \frac{1}{\lambda_n(\omega)} \mathcal{L}(n, \omega) f - h(\theta^n\omega) \int f \, d\nu_\omega \right\|_0 \le \text{constant} < 0$$

for all $f \in X_{(0,\alpha)}$.

Here $\lambda_n(\omega) := \lambda(\theta^{n-1}\omega) \cdot \cdots \cdot \lambda(\omega)$.

With

$$W(\omega) := \left\{ f \in X_{(0,\alpha)} : \int f \, d\nu_\omega = 0 \right\},$$

then the lemma is immediate. □

The objects above have the following dynamical meaning: Λ is the relative pressure of φ at g, and $h(\omega)$ is the ν_ω-density of a random measure μ_ω (i.e. $d\mu_\omega = h(\omega)\, d\nu_\omega$), which is the disintegration of the unique equilibrium state μ of g (cf. Kifer [**11**] or Bogenschütz [**7**] for an explanation of these notions). We are now going to take a closer look at the mixing properties of this invariant measure μ.

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3.3. *Decay of random correlation.* For $f_1, f_2 \in \mathcal{C}^k(M, \mathbb{C})$, $n \in \mathbb{N}$, and $\omega \in \Omega_0$ define the *random correlation function* to be

$$C_n^\omega(f_1, f_2) := \left| \int (f_1 \circ \varphi(n, \omega)) \cdot f_2 \, d\mu_\omega - \int f_1 \, d\mu_{\theta^n\omega} \int f_2 \, d\mu_\omega \right|$$

and consider $\tau_\omega(f_1, f_2) := \limsup_{n\to\infty} n^{-1} \log C_n^\omega(f_1, f_2)$. We are interested in the number $\tau_\omega := \sup\{\tau_\omega(f_1, f_2) : f_1, f_2 \in \mathcal{C}^k(M, \mathbb{C})\}$ which we will call the *rate of decay of random correlation*.

COROLLARY 3.4. *μ is* exponentially mixing, *by which we mean that τ_ω is bounded from above by a negative constant independent of ω.*

Proof. We have $\mathcal{L}(n, \omega)h(\omega) = \lambda_n(\omega)h(\theta^n\omega)$ and $\mathcal{L}^*(n, \omega)\nu_{\theta^n\omega} = \lambda_n(\omega)\nu_\omega$. Using this and the fact that $\mathcal{L}(n, \omega)((f_1 \circ \varphi(n, \omega)) \cdot f_2) = \mathcal{L}(n, \omega)(f_2) \cdot f_1$ we obtain

$$\int (f_1 \circ \varphi(n, \omega)) \cdot f_2 \, d\mu_\omega = \int \frac{1}{\lambda_n(\omega)} f_1 \cdot \mathcal{L}(n, \omega)(f_2 \cdot h(\omega)) \, d\nu_{\theta^n\omega}.$$

Now a standard calculation yields

$$C_n^\omega(f_1, f_2) \le \frac{1}{\lambda_n(\omega)} \|f_1\|_0 \left\| \mathcal{L}(n, \omega)(f_2 \cdot h(\omega) - h(\omega) \int f_2 \cdot h(\omega) \, d\nu_\omega) \right\|_0.$$

Note that $f_2 \cdot h(\omega) - h(\omega) \int f_2 \cdot h(\omega) \, d\nu_\omega \in W(\omega)$ so Lemma 3.2 yields

$$\tau_\omega(f_1, f_2) = \limsup_{n\to\infty} \frac{1}{n} \log C_n^\omega(f_1, f_2) \le K - \Lambda < 0. \qquad \square$$

Remark 3.5. Assume there exists a second Lyapunov exponent, i.e. there are invariant subspaces $V_2(\omega) \subset W(\omega)$ and a number $\Lambda_2 < \Lambda$ such that

$$f \in W(\omega) \setminus V_2(\omega) \Rightarrow \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}(n, \omega) f\|_0 = \Lambda_2$$
$$f \in V_2(\omega) \Rightarrow \limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}(n, \omega) f\|_0 < \Lambda_2.$$

Then we can conclude from the above that the rate of decay of random correlation of μ is bounded from above by $\Lambda_2 - \Lambda$. The question whether or under which conditions we have equality here (as is the case in the deterministic situation) is not quite clear in general.

Assume $\dim V_2(\omega) = 1$ and perform a measurable selection to obtain a measurable $h_2 : \Omega \to X_r$ with $h_2(\omega) \in V_2(\omega)$ and $\|h_2(\omega)\|_0 = 1$. Then the equation $\mathcal{L}(\omega)h_2(\omega) = \lambda_2(\omega)h_2(\theta\omega)$ yields a $\mathbb{C}$-valued random variable λ_2. We may call h_2 a 'generalized eigenfunction' to the 'generalized eigenvalue' λ_2. Note that $\Lambda_2 = \mathbb{E}(\log|\lambda_2|)$. Recall that $h(\omega) > 0$. With $f_2 := h_2/h$ we can calculate

$$C_n^\omega(f_1, f_2) = \left| \frac{1}{\lambda_n(\omega)} \right| \cdot |\lambda_n^{(2)}(\omega)| \cdot \left| \int f_1 \cdot h_2(\theta^n\omega) \, d\nu_{\theta^n\omega} \right|,$$

where $\lambda_n^{(2)}(\omega) := \lambda_2(\theta^{n-1}\omega) \cdot \cdots \cdot \lambda_2(\omega)$. Hence we end up with the task of finding f_1 such that $\limsup_{n\to\infty} n^{-1} \log|\int f_1 \cdot h_2(\theta^n\omega) \, d\nu_{\theta^n\omega}| = 0$ in order to obtain $\tau_\omega = \Lambda_2 - \Lambda$.

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In fact, we may sometimes verify this condition in the small random perturbation case, as we shall see below.

If $\dim V_2(\omega) > 1$ we have the additional problem that the above construction of a generalized eigenvalue fails. Note, however, that $\dim V_2(\omega) = 1$ may be considered as some kind of 'generic' situation (cf. Arnold and Cong [**3**] for a precise statement in the finite-dimensional framework).

3.6. *Stochastic stability results.* Fix an expanding map $\varphi_0 \in \mathrm{End}^k(M)$ and a function $g_0 \in \mathcal{C}^k(M, \mathbb{R})$. The associated transfer operator $\mathcal{L}_0$ admits a spectral theory in the spirit of Example 1.4 (cf. Ruelle [**15**]). This is our unperturbed situation.

Now consider families $\{\varphi_\epsilon : \Omega \to B_\epsilon(\varphi_0)\}$ and $\{g_\epsilon : \Omega \to B_\epsilon(g_0)\}$, where $B_\epsilon(\varphi_0)$ and $B_\epsilon(g_0)$ denote the ϵ-balls around φ_0 and g_0 in the $\mathcal{C}^k$ metric on $\mathrm{End}^k(M)$ and $\mathcal{C}^k(M, \mathbb{R})$, respectively. To these objects, we can, for small enough $\epsilon > 0$, associate random transfer operators $\mathcal{L}_\epsilon$, which we let act on X_k.

Obviously, $\mathcal{L}_\epsilon(\omega)$ does not converge to $\mathcal{L}_0$ in norm. (Transfer operators do not depend continuously on the dynamics with respect to the norm topology—except for analytic situations.) However, $\{\mathcal{L}_\epsilon\}$ forms an asymptotically small random perturbation for any split index κ with $e^{\Lambda_0} > \kappa > (1/\gamma^k)e^{\Lambda_0}$ in the sense of Definition 1.7 (cf. Baladi *et al.* [**5**, Lemma A.1]). Here γ is the expansion constant of φ_0 and Λ_0 is the pressure of g_0 with respect to φ_0.

A recent result of Campbell and Latushkin [**9**] gives an exact formula for the essential spectral radius $\rho_{\mathrm{ess}}(\mathcal{L}_0)$ of $\mathcal{L}_0$. It seems possible to use this, combined with some extension of the stability results of Baladi and Young [**6**], to obtain that $\{\mathcal{L}_\epsilon\}$ forms an asymptotically small random perturbation of $\mathcal{L}_0$ for any split index greater than $\rho_{\mathrm{ess}}(\mathcal{L}_0)$.

We apply Theorem 1.10 to obtain the following.

THEOREM 3.7. *Let Λ_ϵ denote the relative pressure and μ_ϵ the equilibrium state of g_ϵ with respect to φ_ϵ, and let Λ_0 and μ_0 be the corresponding objects of g_0 and φ_0.*

(1) *(Bogenschütz [**8**, Theorem 3.6])*
   (a) $\lim_{\epsilon\to 0} \Lambda_\epsilon = \Lambda_0$
   (b) $\lim_{\epsilon\to 0} \mu_\epsilon = \mathbb{P} \otimes \mu_0$,
   where convergence takes place in the narrow topology on random measures.

(2) *Let h_ϵ be from Lemma 3.2 applied to $\mathcal{L}_\epsilon$ and let h_0 be the normalized positive eigenfunction of $\mathcal{L}_0$ to the eigenvalue $\lambda_1 = e^{\Lambda_0}$. Then, for small enough $\epsilon > 0$, h_ϵ takes values in $\mathcal{C}^k(M, \mathbb{R}^+)$, and*

$$\lim_{\epsilon\to 0} h_\epsilon(\omega) = h_0, \quad \textit{uniformly in } \omega,$$

   where convergence takes place in the $\mathcal{C}^k$ topology.

(3) *If $\mathcal{L}_0$ possesses a 'second eigenvalue' with modulus $\lambda_2 > (1/\gamma^k)e^{\Lambda_0}$, then $\mathcal{L}_\epsilon$ possesses a second Lyapunov exponent (provided ϵ is sufficiently small).*
   In particular, thus,

$$\limsup_{\epsilon\to 0} \tau_\omega^\epsilon \le \log \tau_0,$$

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where τ_ω^ϵ denotes the rate of decay of random correlation for the system $(\varphi_\epsilon, \mu_\epsilon)$, and $\tau_0 = \lambda_2/\lambda_1$ is the corresponding object for the system (φ_0, μ_0).

Remark 3.8. It seems reasonable to conjecture the much more satisfactory statement $\lim_{\epsilon\to 0} \tau_\omega^\epsilon = \log \tau_0$. For that purpose, at least at the moment, we need to ensure that the second Lyapunov exponent Λ_2^ϵ of $\mathcal{L}_\epsilon$ has multiplicity one. We can do this by assuming that $\bigoplus_{\gamma\in\Sigma(\mathcal{L}_0):|\gamma|=\lambda_2} E(\gamma)$ has dimension one. Then, in fact, we have $\lim_{\epsilon\to 0}(\Lambda_2^\epsilon - \Lambda) = \log \tau_0$, and the equality $\tau_\omega^\epsilon = \Lambda_2^\epsilon - \Lambda$ follows from the fact that $\lim_{\epsilon\to 0} \tilde{h}_\epsilon(\omega) = \tilde{h}$ uniformly in ω, where $\tilde{h}$ denotes the eigenfunction of $\mathcal{L}_0$ for the second eigenvalue and $\tilde{h}_\epsilon$ is constructed as h_2 in Remark 3.5.

Remark 3.9. To fully exploit Theorem 1.10 to the present situation, one may assume that $\mathcal{L}_0$ possesses further isolated eigenvalues with absolute values $e^{\Lambda_0} = \lambda_1 > \lambda_2 > \cdots > \lambda_p > (1/\gamma^k)e^{\Lambda_0}$, and write $E_i := \bigoplus_{\gamma\in\Sigma(\mathcal{L}_0):|\gamma|=\lambda_i} E(\gamma)$ and $d_i := \dim E_i$. Then one has that $\mathcal{L}_\epsilon$ admits a spectral theory with quasi-Oseledets splitting $\mathbb{C}h_\epsilon =: E_1^\epsilon, E_2^\epsilon, \ldots, E_p^\epsilon, F^\epsilon$ satisfying $\dim E_i^\epsilon = d_i$ (provided ϵ is small enough). Further,

$$\lim_{\epsilon\to 0} E_i^\epsilon(\omega) = E_i(\omega), \quad \lim_{\epsilon\to 0} F^\epsilon(\omega) = F(\omega),$$

and if λ_i^ϵ is one of the (at most d_i different possible) Lyapunov exponents realized in E_i^ϵ, then

$$\lim_{\epsilon\to 0} \lambda_i^\epsilon = \log \lambda_i.$$

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