Supersymmetry 4

The focus of this chapter is supersymmetry in the context of scattering amplitudes. We begin with a very brief introduction to supersymmetry; it serves to provide the minimal background material we need for our amplitude studies. We then discuss *supersymmetry Ward identities* for the amplitudes and introduce *on-shell superspace* as a tool for organizing the amplitudes into *superamplitudes*. This is particularly powerful in $\mathcal{N} = 4$ super Yang–Mills theory where it allows us to solve a supersymmetric version of the BCFW recursion relations to find all tree-level superamplitudes in this theory. We had better get started.

4.1 $\mathcal{N} = 1$ supersymmetry: chiral model

Let us begin with a simple example of supersymmetry. Consider the free Lagrangian for a Weyl fermion ψ and a complex scalar field ϕ:

$$\mathcal{L}_0 = i\,\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi - \partial_\mu \bar{\phi}\,\partial^\mu \phi\,. \tag{4.1}$$

The bar on ϕ denotes the complex conjugate. In addition to Poincaré symmetry, $\mathcal{L}_0$ also has a symmetry that mixes fermions and bosons:

$$\begin{aligned} \delta_\epsilon \phi &= \epsilon\psi\,, & \delta_\epsilon \bar{\phi} &= \epsilon^\dagger \psi^\dagger\,, \\ \delta_\epsilon \psi_a &= -i\sigma^\mu_{a\dot{b}}\,\epsilon^{\dagger\dot{b}}\partial_\mu \phi\,, & \delta_\epsilon \psi^\dagger_{\dot{a}} &= i\,\partial_\mu \bar{\phi}\,\epsilon^b \sigma^\mu_{b\dot{a}}\,. \end{aligned} \tag{4.2}$$

This is an example of a ***supersymmetry transformation***. The anti-commuting constant spinor ϵ is the supersymmetry parameter (a fermionic analogue of the infinitesimal angle θ of a rotation transformation), and $\epsilon\psi = \epsilon^a \psi_a$ and $\epsilon^\dagger \psi^\dagger = \epsilon^\dagger_{\dot{a}} \psi^{\dagger\dot{a}}$ are the 2-component spinor products with spinor indices raised/lowered with the 2-index Levi-Civita symbol, just as in (2.13).

The conserved Noether charges associated with the supersymmetry transformations are anticommuting supercharges Q and $Q^\dagger$. They enhance the Poincaré algebra to a super-Poincaré graded Lie algebra with both bosonic and fermionic generators. Thus, in addition to the translation and rotation/boost generators P_μ and $M_{\mu\nu}$ whose algebra is

$$\begin{aligned} &\big[P^\mu, P^\nu\big] = 0\,, \quad \big[P^\mu, M^{\rho\sigma}\big] = i\big(\eta^{\mu\rho} P^\sigma - \eta^{\mu\sigma} P^\rho\big)\,, \\ &\big[M^{\mu\nu}, M^{\rho\sigma}\big] = i\big(\eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\sigma} M^{\nu\rho} + \eta^{\nu\sigma} M^{\mu\rho}\big)\,, \end{aligned} \tag{4.3}$$

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the super-Poincaré algebra has

$$\begin{aligned} \big[Q_a, P^\mu\big] &= 0\,, \quad \big[Q_a, M^{\mu\nu}\big] = \tfrac{i}{4}\epsilon^{\dot a c}\big(\sigma^\mu_{a\dot a}\sigma^\nu_{c\dot c} - \sigma^\nu_{a\dot a}\sigma^\mu_{c\dot c}\big)Q^c\,, \\ \big\{Q_a, Q_b\big\} &= 0\,, \quad \big\{Q_a, Q^\dagger_{\dot b}\big\} = -2\sigma^\mu_{a\dot b}P_\mu\,, \end{aligned} \tag{4.4}$$

and similarly for $Q^\dagger$. Note that the supercharges Q and $Q^\dagger$ commute with the spacetime translation operators, but not with the rotations/boosts, and that the anticommutator of Q and $Q^\dagger$ is a spacetime translation.

If you have not previously seen supersymmetry, you should promptly go ahead and do these two exercises:

▶ **Exercise 4.1**

Check that $\mathcal{L}_0$ in (4.1) is invariant under the supersymmetry variation (4.2) up to a total derivative.

▶ **Exercise 4.2**

Calculate $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]$ by acting with it on the fields; first try on ϕ, then afterwards on ψ. You should find that in each case, the combination of two supersymmetry transformations is a spacetime translation.

The 2-component Weyl spinors can be combined into 4-component Majorana spinors

$$\Psi_{\text{M}} = \begin{pmatrix} \psi_a \\ \psi^{\dagger\dot a} \end{pmatrix} \quad \text{and} \quad \epsilon_{\text{M}} = \begin{pmatrix} \epsilon_a \\ \epsilon^{\dagger\dot a} \end{pmatrix}, \tag{4.5}$$

and the supersymmetry transformations are then defined with suitable L- and R-projections $P_{L,R}$ from (A.9). We write the free field expansions as

$$\begin{aligned} \phi(x) &= \int \widetilde{dp}\,\Big[a_-(p)\,e^{ip.x} + a_+^\dagger(p)\,e^{-ip.x}\Big] \\ \psi_a(x) &= P_L\Psi_{\text{M}}(x) = \sum_{s=\pm}\int \widetilde{dp}\,\Big[b_s(p)\,P_L u_s(p)\,e^{ip.x} + b_s^\dagger(p)\,P_L v_s(p)\,e^{-ip.x}\Big], \end{aligned} \tag{4.6}$$

and similarly for $\bar\phi(x)$ and $\psi^{\dagger\dot a} = P_R\Psi_{\text{M}}$. The Lorentz-invariant measure $\widetilde{dp}$ is defined below (2.5).

Upon canonical quantization, the coefficients satisfy the algebra of bosonic/fermionic creation-annihilation operators,

$$\big[a_\pm(p), a_\pm^\dagger(p')\big] = (2\pi)^3\,2E_p\,\delta^3(\vec p - \vec p\,')\,, \qquad \big\{b_\pm(p), b_\pm^\dagger(p')\big\} = (2\pi)^3\,2E_p\,\delta^3(\vec p - \vec p\,')\,, \tag{4.7}$$

with all other (anti)commutators vanishing.

For the fermions, the $\pm$-subscripts on the operators indicate the helicity $h = \pm\frac{1}{2}$ of the associated 1-particle states. As a matter of later convenience, we have also labeled the two sets of annihilation/creation operators associated with the complex field ϕ with

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$\pm$-subscripts. The corresponding particles are of course scalars with $h = 0$, but the label indicates which spinor helicity state the scalar state is matched to via supersymmetry. Let us see how that works.

The supersymmetry transformations (4.2) transform the fields ψ and ϕ into each other, and therefore the associated annihilation/creation operators are also related. The relationship is straightforward to extract from the free field expansions (4.6). Recalling from Section 2.2 that $P_L v_s(p)$ is equal to $|p]$ for $s = +$ and vanishes for $s = -$ (and similarly for P_R), one finds

$$\begin{aligned} \delta_\epsilon a_-(p) &= [\epsilon\, p]\, b_-(p)\,, \quad \delta_\epsilon b_-(p) = \langle \epsilon\, p\rangle\, a_-(p)\,, \\ \delta_\epsilon a_+(p) &= \langle \epsilon\, p\rangle\, b_+(p)\,, \quad \delta_\epsilon b_+(p) = [\epsilon\, p]\, a_+(p)\,. \end{aligned} \tag{4.8}$$

We have introduced anti-commuting bra-kets $\langle\epsilon|_{\dot a} = \epsilon^\dagger_{\dot a}$ and $|\epsilon]_a = \epsilon_a$ for the supersymmetry parameter. Using $-|p]\langle p| = p_{a\dot b}$, it is easy to see that $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\mathcal{O}(p) = a^\mu p_\mu \mathcal{O}(p)$ for $\mathcal{O}(p)$ any one of the creation/annihilation operators. Hence, the commutator of two supersymmetry transformations is a translation, as in the algebra (4.4). The translation parameter a^μ can be written in terms of Majorana spinors as $a^\mu = \epsilon_{\text{M},2}\gamma^\mu\epsilon_{\text{M},1}$.

The supersymmetry generators $Q_{\text{M}} = \begin{pmatrix} |Q]_a \\ |Q^\dagger\rangle^{\dot a} \end{pmatrix}$ can be deduced from

$$\delta_\epsilon \mathcal{O} = \big[\bar\epsilon_{\text{M}} Q_{\text{M}}, \mathcal{O}\big] = \big[[\epsilon Q] + \langle \epsilon Q\rangle, \mathcal{O}\big]\,. \tag{4.9}$$

Using this, one finds (and you should check it) that

$$\begin{aligned} |Q]_a &= \int \widetilde{dp}\ |p]_a \left(a_+(p)\, b^\dagger_+(p) - b_-(p)\, a^\dagger_-(p)\right), \\ |Q^\dagger\rangle^{\dot a} &= \int \widetilde{dp}\ |p\rangle^{\dot a} \left(a_-(p)\, b^\dagger_-(p) - b_+(p)\, a^\dagger_+(p)\right), \end{aligned} \tag{4.10}$$

reproduces (4.8).

▶ **Exercise 4.3**

Show that $\{|Q]_a, \langle Q^\dagger|_{\dot b}\}$ equals $p_{a\dot b}$ times a sum of number operators.

The action of the supercharges (4.10) on the annihilation operators is

$$\begin{aligned} &[Q\,, a_-(p)] = |p]\, b_-(p)\,, && [Q^\dagger\,, b_-(p)] = |p\rangle\, a_-(p)\,, \\ &[Q\,, b_-(p)] = 0\,, && [Q^\dagger\,, a_-(p)] = 0\,, \\ &[Q, b_+(p)] = |p]\, a_+(p)\,, && [Q^\dagger, a_+(p)] = |p\rangle\, b_+(p)\,, \\ &[Q, a_+(p)] = 0\,, && [Q^\dagger, b_+(p)] = 0\,, \end{aligned} \tag{4.11}$$

where $[\,.\,,\,.\,]$ is a graded bracket that is an anticommutator when both arguments are Grassmann and otherwise a commutator. The 2-component spinor-indices are suppressed. A similar set of relations holds for the creation operators.

The action of the supersymmetry generators on the creation/annihilation operators implies that the associated bosonic and fermionic 1-particle states are related – they are said

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to belong to the same ***supermultiplet***. Note that Q lowers the helicity by $\frac{1}{2}$ and that $Q^\dagger$ raises it by $\frac{1}{2}$. Because the supersymmetry generators commute with the Hamiltonian, P^0, and the space-translation generators, P^i, it follows that states in the same supermultiplet must have the same mass.

In the chiral model (4.1), the mass of the bosons and fermions is of course zero. It follows from our discussion that the spectrum of the chiral model splits into a "negative helicity sector" and a "positive helicity sector"; CPT symmetry requires us to have both.

In a supermultiplet, the number of on-shell bosonic states equals the number of on-shell fermionic states. This is manifest in the example of the chiral model: the two real scalar states encoded in the complex scalar field are supersymmetry partners of the two fermionic states with $h = \pm 1/2$.

Interactions. Next, we would like to introduce interactions – after all, this is all about scattering amplitudes so we need something to happen! We want to study interactions that preserve supersymmetry. For our chiral model, one can introduce a "superpotential" interaction of the form

$$\mathcal{L}_I = \tfrac{1}{2} g\, \phi\, \psi\psi + \tfrac{1}{2} g^*\, \bar{\phi}\, \psi^\dagger \psi^\dagger - \tfrac{1}{4} |g|^2\, |\phi|^4 \,. \tag{4.12}$$

A small modification of the supersymmetry transformations (4.2) is now needed in the transformation rule of the fermion field:

$$\begin{aligned} \delta_\epsilon \phi &= \epsilon \psi \,, & \delta_\epsilon \bar{\phi} &= \epsilon^\dagger \psi^\dagger \,, \\ \delta_\epsilon \psi_a &= -i \sigma^\mu_{a\dot{b}}\, \epsilon^{\dagger \dot{b}} \partial_\mu \phi + \tfrac{1}{2} g^* \bar{\phi}^2 \epsilon_a \,, & \delta_\epsilon \psi^\dagger_{\dot{a}} &= i\, \partial_\mu \bar{\phi}\, \epsilon^b \sigma^\mu_{b\dot{a}} + \tfrac{1}{2} g \phi^2\, \epsilon^\dagger_{\dot{a}} \,. \end{aligned} \tag{4.13}$$

Note that the coupling of the 4-scalar interaction in (4.12) is fixed in terms of the Yukawa coupling g; this is required by supersymmetry.

▶ **Exercise 4.4**

Show that (4.13) is a symmetry of $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$.

A linear version of the supersymmetry transformations can be given using an auxiliary field. Supersymmetric actions can be expressed compactly and conveniently using off-shell superspace formalism. You can find much more about this in textbooks such as Wess and Bagger [37].

Extended supersymmetry

So far we have focused on a very simple case of a "chiral supermultiplet" in which a spin-0 particle is partnered with a spin-$\frac{1}{2}$ particle. One can repeat the analysis for any $\mathcal{N} = 1$ supersymmetric model with particles of spin $(s, s + \frac{1}{2})$. The $\mathcal{N}$ counts the number of supersymmetry generators Q^A and $Q^\dagger_A$ with $A = 1, 2, \ldots, \mathcal{N}$. When $\mathcal{N} > 1$, the model is said to have ***extended supersymmetry***. The super-Poincaré algebra with extended supersymmetry

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is very similar to (4.4), but with

$$\left\{Q_a^A,\, Q^\dagger_{B\dot{b}}\right\} = -2\sigma^\mu_{a\dot{b}} P_\mu\, \delta^A{}_B\,. \tag{4.14}$$

In some cases of extended supersymmetry, a "central charge" Z^{AB} can appear in the anti-commutator $\{Q_a^A,\, Q_b^B\} = Z^{AB}\epsilon_{ab}$. In almost all applications of extended supersymmetry in this book, there will be no central charge.

The extended supersymmetry algebra (without central charges) has a global symmetry that rotates the supersymmetry charges: this is called an ***R-symmetry***. An R-symmetry is characterized by the property that it does not commute with the supersymmetry transformations; this contrasts "ordinary" flavor-type global symmetries that do commute with supersymmetry. With $\mathcal{N}$-fold extended supersymmetry, the R-symmetry is $SU(\mathcal{N})$; this full R-symmetry may or may not be realized for a given model.

As examples of $\mathcal{N} = 1$ supermultiplets, $\mathcal{N} = 1$ super-QED has a photon and a photino with helicities ± 1 and $\pm\frac{1}{2}$. $\mathcal{N} = 1$ ***super Yang–Mills theory (SYM)*** with a gluon ($h = \pm 1$) and its fermionic partner, the gluino ($h = \pm\frac{1}{2}$), will be discussed in Section 4.3.

In extended supersymmetry, there are $2^{\mathcal{N}}$ states in the massless supermultiplets (and the same in the CPT conjugates). For example, for $\mathcal{N} = 2$ SYM, one supermultiplet consists of a helicity -1 gluon, two gluinos with helicity $-\frac{1}{2}$, and a scalar with helicity 0. Thus the $\mathcal{N} = 2$ gluon supermultiplet has two bosonic states and two fermionic states. The CPT conjugate multiplet contains the same types of states but with opposite signs for the helicities.

The spectrum of states is generated by acting with the supercharges on the "highest weight state"; in the case of massless particles in four dimensions, this is the highest helicity state. To avoid states with spin greater than 1, the maximal amount of supersymmetry in fourdimensions is $\mathcal{N} = 4$. This large symmetry-requirement places such strong constraints on the theory that it is unique (up to choice of gauge group): this is $\mathcal{N} = 4$ super Yang–Mills theory. We are going to study the supersymmetry constraints on the $\mathcal{N} = 4$ SYM amplitudes in much further detail in Section 4.4. For now, let us content ourselves with $\mathcal{N} = 1$ supersymmetry and study the consequences it has on the scattering amplitudes in our chiral scalar model.

4.2 Amplitudes and supersymmetry Ward identities

Consider the simple supersymmetric chiral scalar model whose Lagrangian $\mathcal{L}$ is the sum of the kinetic terms (4.1) and the interaction terms (4.12). The 4-point tree amplitudes were essentially already presented in Exercise 2.8. To adapt the results (2.46), we just need to take the coupling of the 4-scalar interaction to be $\lambda = |g|^2$. The 4-point tree amplitudes are then:

$$A_4^{\text{tree}}\big(\phi\phi\bar{\phi}\bar{\phi}\big) = -|g|^2\,,\quad A_4^{\text{tree}}\big(\phi\, f^-\, f^+\bar{\phi}\big) = -|g|^2\,\frac{\langle 24\rangle}{\langle 34\rangle}\,,\quad A_4^{\text{tree}}\big(f^-\, f^-\, f^+\, f^+\big) = |g|^2\,\frac{\langle 12\rangle}{\langle 34\rangle}\,. \tag{4.15}$$

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By inspection of (4.15), we observe that

$$A_4^{\text{tree}}\big(\phi\, f^-\, f^+\bar{\phi}\big) = \frac{\langle 24\rangle}{\langle 34\rangle}\, A_4^{\text{tree}}\big(\phi\phi\bar{\phi}\bar{\phi}\big)\,, \tag{4.16}$$

$$A_4^{\text{tree}}\big(f^-\, f^-\, f^+\, f^+\big) = -\frac{\langle 12\rangle}{\langle 24\rangle}\, A_4^{\text{tree}}\big(\phi\, f^-\, f^+\bar{\phi}\big)\,. \tag{4.17}$$

These relations hold not just for the tree-level amplitudes, as we have seen just now; supersymmetry ensures that (4.16) and (4.17) hold at all orders in the perturbation expansion. We will now see how that comes about.

We can think of an n-point amplitude with all-outgoing particles as the S-matrix element $\langle 0|\mathcal{O}_1(p_1)\ldots\mathcal{O}_n(p_n)|0\rangle$ in which n annihilation operators $\mathcal{O}_i(p_i)$, $i = 1,\ldots,n$, act to the left on the out-vacuum. For example, $A_4\big(\phi\, f^-\, f^+\bar{\phi}\big) = \langle 0|a_-(p_1)b_-(p_2)b_+(p_3)a_+(p_4)|0\rangle$. Suppose the vacuum is supersymmetric: $Q|0\rangle = 0 = Q^\dagger|0\rangle$. Then for any set of n annihilation (or creation) operators, we have

$$\begin{aligned} 0 &= \langle 0|\big[Q^\dagger, \mathcal{O}_1(p_1)\ldots\mathcal{O}_n(p_n)\big]|0\rangle \\ &= \sum_{i=1}^{n}(-1)^{\sum_{j<i}|\mathcal{O}_j|}\,\langle 0|\mathcal{O}_1(p_1)\cdots\big[Q^\dagger, \mathcal{O}_i(p_i)\big]\cdots\mathcal{O}_n(p_n)|0\rangle\,, \end{aligned} \tag{4.18}$$

and similarly for Q. Here the sign-factor takes into account that a minus sign is picked up every time $Q^\dagger$ passes by a fermionic operator: so $|\mathcal{O}|$ is 0 when the operator $\mathcal{O}$ is bosonic and 1 if it is fermionic. Now using the action of the supersymmetry generators (4.11) on the free asymptotic states, the equation (4.18) describes a linear relation among scattering amplitudes whose external states are related by supersymmetry. Such relations are called ***supersymmetry Ward identities*** and they are valid at any order in perturbation theory. The supersymmetry Ward identities are best illustrated in explicit examples. Thus, to start with, consider:

$$\begin{aligned} 0 &= \langle 0|\big[Q^\dagger, a_-(p_1)b_-(p_2)a_+(p_3)a_+(p_4)\big]|0\rangle \\ &= |2\rangle\,\langle a_-(p_1)a_-(p_2)a_+(p_3)a_+(p_4)\rangle - |3\rangle\,\langle a_-(p_1)b_-(p_2)b_+(p_3)a_+(p_4)\rangle \\ &\quad -|4\rangle\,\langle a_-(p_1)b_-(p_2)a_+(p_3)b_+(p_4)\rangle\,. \end{aligned} \tag{4.19}$$

We have used (4.11) that $Q^\dagger$ annihilates $a_-(p)$. Translating (4.19) to amplitudes, we have the supersymmetry Ward identity

$$0 = |2\rangle\, A_4\big(\phi\phi\bar{\phi}\bar{\phi}\big) - |3\rangle\, A_4\big(\phi\, f^-\, f^+\bar{\phi}\big) - |4\rangle\, A_4\big(\phi\, f^-\bar{\phi}\, f^+\big)\,. \tag{4.20}$$

Each identity (4.18) encodes two relations, since $Q^\dagger$ (Q) has two components. This is also visible in our example (4.20). We project out the two independent relations by dotting in a suitable choice of bra-spinor $\langle r|$. Picking $\langle r| = \langle 4|$, we find

$$0 = \langle 42\rangle\, A_4\big(\phi\phi\bar{\phi}\bar{\phi}\big) - \langle 43\rangle\, A_4\big(\phi\, f^-\, f^+\bar{\phi}\big)\,. \tag{4.21}$$

At tree-level, this is precisely the relation noted in (4.16). We emphasize that supersymmetry ensures that (4.21) holds true at any loop order.

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A second relation is extracted from (4.20) by choosing $\langle r| = \langle 2|$:

$$A_4\big(\phi\, f^-\bar{\phi} f^+\big) = -\frac{\langle 23\rangle}{\langle 24\rangle}\, A_4\big(\phi\, f^- f^+\bar{\phi}\big)\,. \tag{4.22}$$

Note how the factor $\frac{\langle 23\rangle}{\langle 24\rangle}$ compensates the different little group scaling of the two amplitudes.

▶ **Exercise 4.5**

Plug the two supersymmetry Ward identities (4.21) and (4.22) into (4.20) to show that there is no further independent information available in (4.20).

▶ **Exercise 4.6**

Derive (4.17) as a supersymmetry Ward identity.

▶ **Exercise 4.7**

Find a Q-supersymmetry Ward identity that gives $A_4(\phi\, f^- f^+\bar{\phi}) = -\frac{[13]}{[12]}\, A_4\big(\phi\phi\bar{\phi}\bar{\phi}\big)$. Show that this relation is equivalent to (4.16).

Let us take a brief look at the supersymmetry Ward identities at higher points, for example for amplitudes with 6-particles. Starting with $0 = \langle 0|[Q^\dagger, a_{1-}a_{2-}b_{3-}a_{4+}a_{5+}a_{6+}]|0\rangle$ (using a short-hand notation to indicate the momentum with a subscript) we find, after dotting in $\langle r|$,

$$\begin{aligned} 0 = {} & \langle r3\rangle\, A_6\big(\phi\phi\phi\bar{\phi}\bar{\phi}\bar{\phi}\big) \;-\; \langle r4\rangle\, A_6\big(\phi\phi\, f^- f^+\bar{\phi}\bar{\phi}\big) \\ & -\langle r5\rangle\, A_6\big(\phi\phi\, f^-\bar{\phi}\, f^+\bar{\phi}\big) \;-\; \langle r6\rangle\, A_6\big(\phi\phi\, f^-\bar{\phi}\bar{\phi}\, f^+\big)\,. \end{aligned} \tag{4.23}$$

There are two relations, but four "unknowns" (the amplitudes), so this time the supersymmetry Ward identity does not give simple proportionality relations among pairs of amplitudes. Instead, one gets a web of linear relations. This is typical for "non-MHV type" amplitudes; we come back to this point briefly in the next two sections.

To summarize, amplitudes whose external states are related by supersymmetry are linked to each other through linear relationships called ***supersymmetry Ward identities***. They were first studied in 1977 by Grisaru, Pendleton, and van Nieuwenhuizen [38] and have since then had multiple applications.

4.3 $\mathcal{N} = 1$ supersymmetry: gauge theory

We have introduced supersymmetry for the simple chiral scalar model and derived the consequences of supersymmetry for the amplitudes. Let us now adapt these results to gauge theory. The Lagrangian for pure $\mathcal{N} = 1$ super Yang–Mills theory (SYM) is

$$\mathcal{L} = -\frac{1}{4}\,\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + i\,\mathrm{Tr}\,\lambda^\dagger\bar{\sigma}^\mu D_\mu\lambda\,, \tag{4.24}$$

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where all fields are in the adjoint of the $SU(N)$ gauge group and the covariant derivative is $D_\mu\lambda = \partial_\mu\lambda - ig[A_\mu, \lambda]$. The on-shell spectrum consists of helicity $h = \pm 1$ states of the gluon $g^\pm$ and $h = \pm\frac{1}{2}$ states of the gluino $\lambda^\pm$. (For an abelian gauge group, we would call the states photon and photino.) We will not need the explicit supersymmetry transformation of the fields (they can be found for example in [39]), but will simply state the consequences on the states. The supercharge $\tilde{Q}$ raises the helicity by $\frac{1}{2}$ and Q lowers it by $\frac{1}{2}$, so if $c_\pm$ ($b_\pm$) are the annihilation operators for gluons of helicity ± 1 (gluinos of helicity $\pm\frac{1}{2}$), then

$$\begin{aligned} \big[\tilde{Q}, c_+(i)\big] &= 0\,, & \big[Q, c_+(i)\big] &= [i\,|\,b_-(i)\,, \\ \big[\tilde{Q}, b_+(i)\big] &= |i\rangle\, c_+(i)\,, & \big[Q, b_+(i)\big] &= 0\,, \\ \big[\tilde{Q}, b_-(i)\big] &= 0\,, & \big[Q, b_-(i)\big] &= [i\,|\,c_-(i)\,, \\ \big[\tilde{Q}, c_-(i)\big] &= |i\rangle\, b_-(i)\,, & \big[Q, c_-(i)\big] &= 0\,, \end{aligned} \tag{4.25}$$

for $\mathcal{N} = 1$ SYM. This is derived as (4.11) for the chiral model in Section 4.1.

Supersymmetry Ward identities for $\mathcal{N} = 1$ SYM are derived from the action of the supercharges on the states in (4.25). Since $\tilde{Q}$ annihilates $c_+(i)$, the supersymmetry Ward identity $\langle 0|[\tilde{Q}, b_{1+}c_{2+}\dots c_{n+}]|0\rangle = 0$ gives $|1\rangle A_n[g^+g^+g^+\dots g^+] = 0$. This says that the all-plus gluon amplitude vanishes at all orders in perturbation theory in $\mathcal{N} = 1$ SYM. Similarly, one can show that the gluon amplitude with exactly one negative helicity gluon also vanishes. So

$$\text{super Yang–Mills:} \quad A_n^{L\text{-loop}}[g^+g^+g^+\dots g^+] \;=\; A_n^{L\text{-loop}}[g^-g^+g^+\dots g^+] \;=\; 0\,. \tag{4.26}$$

We used only one supersymmetry generator for this argument, so the statement (4.26) is true also with extended supersymmetry (as long as supersymmetry is unbroken). Moreover, as indicated, (4.26) holds *true at all orders in the loop-expansion* because it relies only on the existence of supercharges that annihilate the vacuum.

In Section 2.7, we showed that (4.26) holds at tree-level in pure non-supersymmetric Yang–Mills,

$$\text{Yang–Mills:} \quad A_n^{\text{tree}}[g^+g^+g^+\dots g^+] \;=\; A_n^{\text{tree}}[g^-g^+g^+\dots g^+] \;=\; 0\,. \tag{4.27}$$

However, at loop-level (4.26) is *not true* without supersymmetry: for example in pure Yang–Mills theory, non-vanishing all-plus amplitudes are indeed generated at the 1-loop level. The reason the result (4.27) holds at tree-level in pure Yang–Mills theory is that the superpartners of the gluon couple quadratically to the gluon. So an amplitude whose external states are all gluons "sees" the superpartner states only via loops. Thus, in pure Yang–Mills the gluon amplitudes at tree-level have no choice but to obey the same supersymmetry Ward identity constraints as the gluon amplitudes in super Yang–Mills. The argument is even stronger: it says that the all-gluon amplitudes must be exactly the same at tree-level in super Yang–Mills theory as in pure Yang–Mills theory.

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▶ **Exercise 4.8**

The non-vanishing of the $n = 3$ anti-MHV amplitude $A_3[1^-2^+3^+]$ escapes the Ward identity that forces $A_n[g^-g^+g^+\dots g^+] = 0$ for $n > 3$. Explain how.

▶ **Exercise 4.9**

Show in $\mathcal{N} = 1$ SYM that supersymmetry Ward identities give

$$A_n\big[g^-\lambda^-\lambda^+g^+\dots g^+\big] = \frac{\langle 13\rangle}{\langle 12\rangle} A_n\big[g^-g^-g^+g^+\dots g^+\big]. \tag{4.28}$$

Relate $A_n[\lambda^-\lambda^-\lambda^+\lambda^+g^+\dots g^+]$ to the MHV gluon amplitude.

Does $\mathcal{N} = 1$ supersymmetry allow you to relate gluon amplitudes such as for example $A_n[1^-2^-3^+\dots 4^+]$ and $A_n[1^-2^+3^-\dots 4^+]$? Can it relate gluon MHV and NMHV amplitudes?

The $\mathcal{N} = 1$ supersymmetry Ward identities for NMHV gluon amplitude are more involved and do not result in simple proportionality relations among pairs of amplitudes, such as (4.28). For $n = 6$, the NMHV $\mathcal{N} = 1$ supersymmetry Ward identities have been solved explicitly [36, 38].

In our discussion of recursion relations in Section 3.3, we learned that in general we should not expect it to be possible to produce a 4-scalar amplitude recursively from 3-particle amplitudes because of the possibility of independent input from a 4-scalar contact term. However, in our supersymmetric chiral scalar model of Section 4.1, supersymmetry determines the coupling of the 4-scalar interaction in terms of the 3-point interactions, so one could hope that recursion relations work in that case. In particular, one might anticipate that all 4-point amplitudes can be determined by the 3-point ones. For this to be true, the recursion algorithm has to "know" which principle fixes the 4-scalar coupling, so supersymmetry must be built into the recursion relations. This is done most efficiently in two steps: first one introduces *on-shell superspace* and groups the amplitudes into *superamplitudes*. Secondly, one incorporates a shift of a Grassmann super-parameter of the on-shell superspace into the BCFW shift. Then one gets recursion relations for the superamplitudes. This is best illustrated for $\mathcal{N} = 4$ SYM, so that is what we will turn to next.

4.4 $\mathcal{N} = 4$ SYM: on-shell superspace and superamplitudes

The action for $\mathcal{N} = 4$ super Yang–Mills theory (SYM) can be written compactly as

$$S = \int d^4x\, \mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(D\Phi_I)^2 + \frac{i}{2}\overline{\Psi}\not{D}\Psi + \frac{g}{2}\overline{\Psi}\Gamma^I[\Phi_I, \Psi] + \frac{g^2}{4}[\Phi_I, \Phi_J]^2\right). \tag{4.29}$$

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Here $D_\mu = \partial_\mu - ig[A_\mu, \cdot\,]$ is the covariant derivative, A_μ is the vector potential field, and Φ_I are six real scalar fields with labels $I = 1, \ldots, 6$ of a global $SO(6) \sim SU(4)$ R-symmetry. All fields transform in the adjoint of the gauge group, which we take to be $SU(N)$. The commutators in (4.29) are associated with the $SU(N)$ matrix structure of the fields. The fermions are encoded in 10-dimensional Majorana–Weyl fields Ψ and the Γ^I are gamma-matrices of the 10d Clifford algebra. This description of $\mathcal{N} = 4$ SYM follows from dimensional reduction of $\mathcal{N} = 1$ SYM in 10d [40].

It is convenient to group the six real scalars Φ_I into six complex scalar fields $\varphi^{AB} = -\varphi^{BA}$, with $A, B = 1, 2, 3, 4$, that satisfy the self-duality condition $\overline{\varphi}_{AB} = \frac{1}{2}\epsilon_{ABCD}\,\varphi^{CD}$. Here ϵ_{ABCD} is the Levi-Civita symbol of $SU(4) \sim SO(6)$, and the scalars φ^{AB} transform in the fully antisymmetric 2-index representation of $SU(4)$. In this language, the 10d fermion fields give 4+4 gluino states λ^A ($h = +\frac{1}{2}$) and $\bar{\lambda}_A$ ($h = -\frac{1}{2}$) that transform in the (anti-)fundamental of $SU(4)$.

In a supersymmetric model, the value V_0 of the scalar potential V at the vacuum is an order parameter of supersymmetry breaking [37]: in flat space, $V_0 = 0$ is necessary for preserving supersymmetry while $V_0 > 0$ breaks supersymmetry. In $\mathcal{N} = 4$ SYM, the scalar potential is $V = \frac{g^2}{4}[\Phi_I, \Phi_J]^2$, so the theory has a ***moduli space*** of $\mathcal{N} = 4$ supersymmetric vacua with $[\Phi_I, \Phi_J] = 0$. At the *origin of moduli space*, where all the scalar vacuum expectation values (vevs) vanish, $\langle\varphi^{AB}\rangle = 0$, all states are massless and the theory contains no dimensionful parameters. In fact, the theory has ***conformal symmetry***: the trace of the stress-tensor is zero, up to the trace anomaly. In particular, the beta-function vanishes at all orders in perturbation theory and there is no running of the coupling. The theory is invariant under an enlarged spacetime symmetry group, namely the conformal group $SO(2, 4) \sim SU(2, 2)$. Supersymmetry enhances the conformal symmetry to ***superconformal symmetry*** $SU(2, 2|4)$. The bosonic part of this super-group consists of the conformal group $SU(2, 2)$ and the R-symmetry group $SU(4)$. The fermionic parts are the four supersymmetry generators as well as their four superconformal partners. The symmetries of $\mathcal{N} = 4$ SYM are the subject of Chapter 5.

When the scalars acquire vevs in such a way that full supersymmetry is preserved, i.e. $[\Phi_I, \Phi_J] = 0$, the theory is said to be on the Coulomb branch. The scale introduced by the scalar vevs breaks the conformal symmetry, so the theory is not superconformal away from the origin, but it still has $\mathcal{N} = 4$ supersymmetry. We will briefly discuss scattering amplitudes on the Coulomb branch of $\mathcal{N} = 4$ SYM in Exercise 12.3, but otherwise we focus entirely on the superconformal theory at the origin of moduli space: ***henceforth, when we discuss amplitudes in $\mathcal{N} = 4$ SYM this means the theory at the origin of moduli space.***

Given that the theory is conformal, we should clarify what we mean by the scattering-matrix in $\mathcal{N} = 4$ SYM. One way to think about this is to consider the theory in $4 - \epsilon$ dimensions: then the conformal symmetry is broken and the S-matrix is well-defined. This turns out to be a little inconvenient for keeping on-shell symmetries manifest, and it can therefore be better to consider the theory on the Coulomb branch and define the $\mathcal{N} = 4$ SYM S-matrix as the zero-vev limit of the Coulomb branch S-matrix. These subtleties will not affect the majority of our discussion and therefore we proceed to discuss the amplitudes of $\mathcal{N} = 4$ SYM without further hesitation.

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Spectrum and supersymmetry Ward identities

The spectrum of the massless $\mathcal{N} = 4$ supermultiplet consists of 16 states:

$$\begin{array}{lccccc} & \underbrace{a}_{1\text{ gluon } g^+}, & \underbrace{a^A}_{4\text{ gluinos } \lambda^A}, & \underbrace{a^{AB}}_{6\text{ scalars } S^{AB}}, & \underbrace{a^{ABC}}_{4\text{ gluinos } \lambda^{ABC} \sim \bar{\lambda}_D}, & \underbrace{a^{1234}}_{1\text{ gluon } g^-}. \\ \text{helicity} = & 1 & \frac{1}{2} & 0 & -\frac{1}{2} & -1 \end{array} \tag{4.30}$$

The indices $A, B, \ldots = 1, 2, 3, 4$ are labels of the global $SU(4)$ R-symmetry that rotates the four sets of supersymmetry generators Q^A and $\tilde{Q}_A \equiv Q^\dagger_A$. The helicity-$h$ states transform in fully antisymmetric $2(1-h)$-index representations of $SU(4)$. Note the important property that the $\mathcal{N} = 4$ supermultiplet is CPT self-conjugate: in particular $\mathcal{N} = 4$ supersymmetry connects the gluon states of $h = \pm 1$.

The action of the supercharges on the annihilation operators is

$$\begin{aligned} \left[\tilde{Q}_A, a(i)\right] &= 0\,, & \left[Q^A, a(i)\right] &= [i|\, a^A(i)\,, \\ \left[\tilde{Q}_A, a^B(i)\right] &= |i\rangle\, \delta^B_A\, a(i)\,, & \left[Q^A, a^B(i)\right] &= [i|\, a^{AB}(i)\,, \\ \left[\tilde{Q}_A, a^{BC}(i)\right] &= |i\rangle\, 2!\, \delta^{[B}_A\, a^{C]}(i)\,, & \left[Q^A, a^{BC}(i)\right] &= [i|\, a^{ABC}(i)\,, \\ \left[\tilde{Q}_A, a^{BCD}(i)\right] &= |i\rangle\, 3!\, \delta^{[B}_A a^{CD]}(i)\,, & \left[Q^A, a^{BCD}(i)\right] &= [i|\, a^{ABCD}(i)\,, \\ \left[\tilde{Q}_A, a^{BCDE}(i)\right] &= |i\rangle\, 4!\, \delta^{[B}_A a^{CDE]}(i)\,, & \left[Q^A, a^{1234}(i)\right] &= 0\,. \end{aligned} \tag{4.31}$$

Note that $\tilde{Q}_A$ raises the helicity of all operators by $\frac{1}{2}$ and removes the index A (if it is not available to be removed, then the operator is annihilated). Q^A does the opposite. Truncating the spectrum to include just the gluons and gluinos, one finds that (4.31) reduces to the result (4.25) for $\mathcal{N} = 1$ SYM.

▶ **Exercise 4.10**

Show that the supersymmetry Ward identities give the following relationships among the color-ordered amplitudes in $\mathcal{N} = 4$ SYM:

$$\begin{aligned} 0 = &-|1\rangle A_n[\lambda^{123} g^- \lambda^4 g^+ \ldots g^+] - |2\rangle A_n[g^- \lambda^{123} \lambda^4 g^+ \ldots g^+] \\ &+ |3\rangle A_n[g^- g^- g^+ g^+ \ldots g^+] \end{aligned} \tag{4.32}$$

and extract

$$A_n[g^- \lambda^{123} \lambda^4 g^+ \ldots g^+] = \frac{\langle 13\rangle}{\langle 12\rangle} A_n[g^- g^- g^+ g^+ \ldots g^+]\,. \tag{4.33}$$

Then derive

$$A_n[g^- S^{12} S^{34} g^+ \ldots g^+] = \frac{\langle 13\rangle^2}{\langle 12\rangle^2} A_n[g^- g^- g^+ g^+ \ldots g^+]\,. \tag{4.34}$$

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Since supersymmetry connects the entire $\mathcal{N}=4$ SYM multiplet, there is an important $\mathcal{N}=4$ supersymmetry Ward identity that connects color-ordered MHV gluon amplitudes with the negative helicity gluons in different positions:

$$\mathcal{N}=4 \text{ SYM:} \quad A_n[g^+g^+\dots g_i^-\dots g_j^-\dots g^+] \;=\; \frac{\langle ij\rangle^4}{\langle 12\rangle^4} A_n[g^-g^-g^+g^+\dots g^+]\,. \tag{4.35}$$

► **Exercise 4.11**

Derive (4.35) by extending the strategy you used for (4.34).

Convince yourself that the supersymmetry Ward identities cannot mix gluon amplitudes with different K in the N^{K}MHV classification.

You were prompted in Exercise 4.9 to note that $\mathcal{N}=1$ supersymmetry would not be sufficient to derive (4.35). We need to relate the positive and negative helicity gluons and this is only possible because the $\mathcal{N}=4$ supermultiplet is CPT self-conjugate.

Gluon *tree* amplitudes do not see the supersymmetry partners, so they must comply with the supersymmetry Ward identities, in particular the tree amplitudes in pure Yang–Mills theory must obey (4.35). Indeed, this relation is automatic in the Parke–Taylor formula (2.114) for the tree-level MHV gluon amplitudes! Furthermore, the focus on the "split-helicity" MHV amplitude $A_n[1^-2^-3^+\dots n^+]$ in our recursive proof of the Parke–Taylor formula in Section 3.2 is now justified: the supersymmetry Ward identities (4.35) ensure that Parke–Taylor holds for an MHV tree gluon amplitude with the two negative helicity gluons in any position.

N^{K}MHV classification – revisited

We introduced N^{K}MHV amplitudes in Section 2.7 as the gluon amplitudes with $K+2$ negative helicity gluons. Now we can expand this to define the ***N^{K}MHV sector*** to be all amplitudes in $\mathcal{N}=4$ SYM connected to the N^{K}MHV gluon amplitude via supersymmetry. For example, all the amplitudes in (4.33), (4.34), and (4.35) are MHV. In fact, all MHV amplitudes in $\mathcal{N}=4$ SYM are proportional to $A_n[g^-g^-g^+g^+\dots g^+]$.

The global $SU(4)$ R-symmetry implies that an amplitude vanishes unless the external states combine to an $SU(4)$ singlet. This requires that the (upper) $SU(4)$ indices appear as a combination of $(K+2)$ sets of $\{1234\}$ for N^{K}MHV amplitudes. Since the supersymmetric Ward identities relate amplitudes with the same number of $SU(4)$ indices, this then provides an alternative definition of the N^{K}MHV sector.

On-shell superspace

It is highly convenient to introduce an on-shell[1] superspace in order to keep track of the states and the amplitudes. We introduce four Grassmann variables η_A labeled by the $SU(4)$

[1] There is no (known) off-shell superspace formalism for $\mathcal{N}=4$ SYM.

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index $A = 1, 2, 3, 4$.[2] This allows us to collect the 16 states into an $\mathcal{N}=4$ on-shell chiral ***superfield*** (or "superwavefunction")

$$\Omega = g^+ + \eta_A \lambda^A - \frac{1}{2!}\eta_A\eta_B S^{AB} - \frac{1}{3!}\eta_A\eta_B\eta_C \lambda^{ABC} + \eta_1\eta_2\eta_3\eta_4\, g^- \,. \tag{4.36}$$

The relative signs are chosen such that the Grassmann differential operators

$$\mathcal{N}=4 \text{ SYM:} \quad \begin{array}{c|c|c|c|c|c} \text{particle} & g^+ & \lambda^A & S^{AB} & \lambda^{ABC} & g^- = g^{1234} \\ \hline \text{operator} & 1 & \partial_i^A & \partial_i^A\partial_i^B & \partial_i^A\partial_i^B\partial_i^C & \partial_i^1\partial_i^2\partial_i^3\partial_i^4 \end{array} \tag{4.37}$$

select the associated state from $\Omega(p_i)$.

In the on-shell formalism, the supercharges are represented as

$$q^{A\,a} \equiv [p|^a \frac{\partial}{\partial \eta_A} \,, \qquad q_A^{\dagger \dot{a}} \equiv |p\rangle^{\dot{a}}\, \eta_A \,, \tag{4.38}$$

where $|p\rangle$ and $|p]$ are the spinors associated with the null momentum p of the particle.

▶ **Exercise 4.12**

Show that the supercharges satisfy the standard supersymmetry anticommutation relation (4.14), i.e. $\{q^{A\,a}, \tilde{q}_B^{\dot{b}}\} = \delta^A{}_B\, |p\rangle^{\dot{b}}[p|^a = -\delta^A{}_B\, p^{\dot{b}a}$. The supercharges (4.38) act on the spectrum by shifting states right or left in Ω. Check that this action on Ω matches (4.31).

▷ *Example.* The purpose of this example is to clarify the relation between the *on-shell* superspace introduced here and the *off-shell* superspace formalism described in textbooks on supersymmetry (e.g. [37]). In an off-shell $\mathcal{N}=1$ formalism, the superspace is $(x^\mu, \theta^a, \overline{\theta}^{\dot{a}})$ with Grassmann variables θ^a and $\overline{\theta}^{\dot{a}}$. The algebra of the supercharges is $\{\mathcal{Q}_a, \mathcal{Q}_b\} = 0$, $\{\overline{\mathcal{Q}}_{\dot{a}}, \overline{\mathcal{Q}}_{\dot{b}}\} = 0$, and $\{\mathcal{Q}_a, \overline{\mathcal{Q}}_{\dot{a}}\} = i(\sigma^\mu)_{a\dot{a}}\, \partial_\mu$. As is often convenient for studies of anti-chiral superfields, we can realize the superalgebra with an anti-chiral representation

$$\mathcal{Q}_a = \frac{\partial}{\partial\theta^a} \,, \qquad \overline{\mathcal{Q}}_{\dot{a}} = -\frac{\partial}{\partial\overline{\theta}^{\dot{a}}} + i\theta^a(\sigma^\mu)_{a\dot{a}}\, \partial_\mu \,. \tag{4.39}$$

The algebra (4.4) is represented faithfully, even when we replace $\overline{\mathcal{Q}}_{\dot{a}} \to i\theta^a(\sigma^\mu)_{a\dot{a}}\, \partial_\mu$. In momentum space, the $\mathcal{Q}\overline{\mathcal{Q}}$-anticommutator can then be written

$$\left\{ \frac{\partial}{\partial\theta^a} \,, \; -\theta^a(\sigma^\mu)_{a\dot{a}}\, p_\mu \right\} = -(\sigma^\mu)_{a\dot{a}}\, p_\mu \,. \tag{4.40}$$

[2] Originally, Ferber [41] introduced these variables as superpartners of bosonic twistor variables. Twistors are discussed in Chapter 5.

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Let us now go on-shell by assuming that p^μ is light-like; then we can rewrite (4.40) in spinor helicity formalism as

$$\left\{\frac{\partial}{\partial\theta^a}\ ,\ \theta^a|p]_a\langle p|_{\dot a}\right\} = |p]_a\langle p|_{\dot a}\,. \tag{4.41}$$

Introduce a new Grassmann-odd variable $\eta = \theta^a|p]_a$. Then $\frac{\partial}{\partial\theta^a} = |p]_a\,\frac{\partial}{\partial\eta}$, so that (4.41) becomes

$$\left\{|p]_a\,\frac{\partial}{\partial\eta}\ ,\ \eta\langle p|_{\dot a}\right\} = |p]_a\langle p|_{\dot a} \implies \left\{[p|^a\,\frac{\partial}{\partial\eta}\ ,\ |p\rangle^{\dot a}\eta\right\} = |p\rangle^{\dot a}[p|^a\,. \tag{4.42}$$

The arguments, $[p|^a\,\frac{\partial}{\partial\eta}$ and $|p\rangle^{\dot a}\eta$, of the anticommutator are recognized as $\mathcal{N}=1$ versions of our on-shell supersymmetry generators q^a and $q^{\dagger\dot a}$ in (4.38).

We note that dotting some arbitrary reference spinors $|\tilde w\rangle$ and $[w|$ (whose brackets with the p-spinors are non-vanishing) into (4.42), we find $\{\eta\,,\,\frac{\partial}{\partial\eta}\} = 1$.

Consider the consequences of the above analysis. The superspace coordinates θ and $\overline{\theta}$ have mass-dimension $(\text{mass})^{-1/2}$ and the angle and square spinors have dimension $(\text{mass})^{1/2}$. So the on-shell superspace variables η are dimensionless. Under little group scaling, the θ and $\overline{\theta}$ are inert, and therefore we have $\eta \to t^{-1}\eta$. Consequently, the on-shell superwavefunction (4.36) scales homogeneously as $\Omega \to t^{-2}\Omega$ under a little group scaling. The state-operator map (4.37) compensates this scaling when extracting component states from Ω. $\lhd$

Superamplitudes

We can think of $\Omega_i = \Omega(p_i)$ as a superwavefunction for the ith external particle of a (color-ordered) ***superamplitude*** $\mathcal{A}_n[\Omega_1, \ldots, \Omega_n]$. It depends on the on-shell momentum p_i and a set of Grassmann variables η_{iA} for each particle $i = 1, \ldots, n$. Expanding $\mathcal{A}_n[\Omega_1, \ldots, \Omega_n]$ in the Grassmann variables, we note that the $SU(4)$-symmetry requires it to be a sum of polynomials in η_{iA} of degree $4(K+2)$. An example of a legal combination is $\eta_{i1}\eta_{i2}\eta_{i3}\eta_{i4}$; according to the map (4.37), it corresponds to particle i being a negative helicity gluon.

Using the map (4.37) to extract states from the superwavefunction of each particle $i = 1, 2, \ldots, n$, one projects out the regular amplitudes (sometimes called "component amplitudes") from the superamplitudes. For example:

$$\begin{aligned} A_n\big[1^+\ldots i^-\ldots j^-\ldots n^+\big] &= \left(\prod_{A=1}^{4}\frac{\partial}{\partial\eta_{iA}}\right)\left(\prod_{B=1}^{4}\frac{\partial}{\partial\eta_{jB}}\right)\mathcal{A}_n\big[\Omega_1, \ldots, \Omega_n\big]\Big|_{\eta_{kC}=0} \\ A_n\big[S^{12}S^{34}\,3^-4^+\ldots n^+\big] &= \left(\frac{\partial}{\partial\eta_{11}}\frac{\partial}{\partial\eta_{12}}\right)\left(\frac{\partial}{\partial\eta_{23}}\frac{\partial}{\partial\eta_{24}}\right) \\ &\quad\times\left(\prod_{A=1}^{4}\frac{\partial}{\partial\eta_{3A}}\right)\mathcal{A}_n\big[\Omega_1, \ldots, \Omega_n\big]\Big|_{\eta_{kC}=0}\,. \end{aligned} \tag{4.43}$$

Grassmann differentiations can of course equally well be expressed as Grassmann integrals.

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The order K of the Grassmann polynomial precisely corresponds to the N^KMHV sector. So we can organize the full superamplitude as

$$\mathcal{A}_n = \mathcal{A}_n^{\mathrm{MHV}} + \mathcal{A}_n^{\mathrm{NMHV}} + \mathcal{A}_n^{\mathrm{N^2MHV}} + \cdots + \mathcal{A}_n^{\text{anti-MHV}}\,, \tag{4.44}$$

where $\mathcal{A}_n^{\mathrm{MHV}}$ has Grassmann degree 8, $\mathcal{A}_n^{\mathrm{NMHV}}$ has Grassmann degree 12, etc. For example, expanding the MHV superamplitude in Grassmann monomials, we have

$$\mathcal{A}_n^{\mathrm{MHV}} = A_n\big[g^-g^-g^+\ldots g^+\big](\eta_1)^4(\eta_2)^4 + A_n\big[g^-\lambda^{123}\lambda^4\ldots g^+\big](\eta_1)^4(\eta_{21}\eta_{22}\eta_{23})(\eta_{34}) + \cdots\,, \tag{4.45}$$

where $(\eta_i)^4 \equiv \eta_{i1}\eta_{i2}\eta_{i3}\eta_{i4}$. (Quiz: why does the second term in (4.45) come with a plus?)

Next, consider the supersymmetry Ward identities. In the language of on-shell superspace, the supersymmetry Ward identities are identical to the statement that the supercharges

$$Q^A \equiv \sum_{i=1}^n q_i^A = \sum_{i=1}^n [i|\,\frac{\partial}{\partial\eta_{iA}}\,, \quad \text{and} \quad \tilde{Q}_A \equiv \sum_{i=1}^n q_A^\dagger = \sum_{i=1}^n |i\rangle\,\eta_{iA}\,, \qquad A = 1,2,3,4, \tag{4.46}$$

annihilate the superamplitude:

$$Q^A\mathcal{A}_n = 0 \quad \text{and} \quad \tilde{Q}_A\mathcal{A}_n = 0\,. \tag{4.47}$$

▶ **Exercise 4.13**

It may not be totally obvious that (4.47) encodes the supersymmetry Ward identities, so the point of this exercise is to illustrate how it works. Start by writing the (relevant terms in the) MHV superamplitude as in (4.45). The supersymmetry Ward identity (4.47) says that the coefficient of each independent Grassmann monomial in $\tilde{Q}_A\mathcal{A}_n^{\mathrm{MHV}}$ has to vanish. Pick $A = 4$ and act with $\tilde{Q}_4$ on $\mathcal{A}_n^{\mathrm{MHV}}$ to extract all terms whose Grassmann structure is $(\eta_1)^4(\eta_2)^4(\eta_{34})$. Use that to show that the "component amplitude" supersymmetry Ward identity (4.32) follows from $\tilde{Q}_4\mathcal{A}_n^{\mathrm{MHV}} = 0$.

The requirement that $\{Q^A, \tilde{Q}_B\} \sim \sum_i p_i^\mu$ annihilates the superamplitude is equivalent to the statement of momentum conservation. The associated delta function $\delta^4(\sum_{i=1}^n p_i)$ has been left implicit throughout the early chapters, but we will begin to include it explicitly in Section 5, where it plays a central role. For example, it makes momentum conservation an operational statement that the momentum operator $P^\mu = \sum_i p_i^\mu$ annihilates the amplitude: it simply acts multiplicatively and the vanishing result follows from the fact that $x\,\delta(x)$ has support nowhere.

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Note that the action of $\tilde{Q}_A$ on the superamplitude is multiplicative. We can therefore solve it easily using a Grassmann delta function $\delta^{(8)}(\tilde{Q})$ defined as

$$\delta^{(8)}(\tilde{Q}) \;=\; \frac{1}{2^4}\prod_{A=1}^{4}\tilde{Q}_{A\dot{a}}\tilde{Q}^{\dot{a}}_{A} \;=\; \frac{1}{2^4}\prod_{A=1}^{4}\sum_{i,j=1}^{n}\langle ij\rangle\eta_{iA}\eta_{jA}\,. \tag{4.48}$$

▶ **Exercise 4.14**

Show that momentum conservation ensures that Q^A annihilates $\delta^{(8)}(\tilde{Q})$.

Which property of angle spinors allows you to demonstrate explicitly that $\tilde{Q}_A\,\delta^{(8)}(\tilde{Q}) = 0$?

Half the supersymmetry constraints, namely $\tilde{Q}_A\mathcal{A}_n = 0$, are satisfied if we write the N^KMHV superamplitude as

$$\mathcal{A}_n^{\mathrm{N}^K\mathrm{MHV}} = \delta^{(8)}(\tilde{Q})\,P_n^{(4K)}\,, \tag{4.49}$$

where $P_n^{(4K)}$ is a degree $4K$ polynomial in the Grassmann variables. If $P_n^{(4K)}$ is annihilated by each Q^A, then – by Exercise 4.14 – all the supersymmetry constraints are solved. The Grassmann delta function $\delta^{(8)}(\tilde{Q})$ can be viewed as conservation of supermomentum.

The Grassmann delta function is a degree 8 polynomial in the η_{iA}s. This means that for an MHV superamplitude, $\delta^{(8)}(\tilde{Q})$ fixes the η_{iA}-dependence completely and $P_n^{(0)}$ is just a normalization that depends on the momenta. Thus, the tree-level MHV superamplitude of $\mathcal{N}=4$ SYM is

$$\mathcal{N}=4\text{ SYM tree-level:}\qquad \mathcal{A}_n^{\mathrm{MHV}}[123\ldots n] = \frac{\delta^{(8)}(\tilde{Q})}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\,. \tag{4.50}$$

This object will appear repeatedly in our study of amplitudes in $\mathcal{N}=4$ SYM, both at tree- and loop-level, so now is a good time to become familiar with it.

It is not hard to see that the MHV superamplitude (4.50) produces the Parke–Taylor gluon tree amplitudes correctly. Just use the map (4.37) to take four derivatives with respect to η_{iA} and four with respect to η_{jA} as in (4.43). Then the delta function produces the numerator-factor $\langle ij\rangle^4$ of the component amplitude $A_n[1^+\ldots i^-\ldots j^-\ldots n^+]$. Note that one component amplitude and supersymmetry uniquely fix the form of *all* MHV amplitudes in $\mathcal{N}=4$ at each order in perturbation theory.

▶ **Exercise 4.15**

Reproduce the three supersymmetry Ward identities (4.33), (4.34), and (4.35) from the MHV superamplitude $\mathcal{A}_n^{\mathrm{MHV}}$ in (4.50).

▶ **Exercise 4.16**

Use the $\mathcal{N}=4$ SYM superamplitude $\mathcal{A}_n^{\mathrm{MHV}}$ (4.50) to calculate the 4-scalar amplitude $A_4[S^{12}S^{34}S^{12}S^{34}]$. Compare your answer to the 4-scalar amplitude (3.41).

Calculate $A_4[S^{12}S^{23}S^{34}S^{41}]$ and compare with (3.45).

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In our discussion so far, we have slipped silently by the anti-MHV 3-point amplitudes $\mathcal{A}_n^{\text{anti-MHV}}$, whose supersymmetry orbit determines the anti-MHV sector with $K=-1$. Thus, the 3-point anti-MHV sector is encoded in degree-4 superamplitudes. We simply state the answer,

$$\mathcal{A}_3^{\text{anti-MHV}} = \frac{1}{[12][23][31]}\,\delta^{(4)}\big([12]\eta_3+[23]\eta_1+[31]\eta_2\big)\,, \tag{4.51}$$

with the delta function defined as

$$\delta^{(4)}\big([12]\eta_3+[23]\eta_1+[31]\eta_2\big) \equiv \prod_{A=1}^{4}\big([12]\eta_{3A}+[23]\eta_{1A}+[31]\eta_{2A}\big)\,, \tag{4.52}$$

and leave it as an exercise to show that (4.51) is annihilated by the supercharges.

▶ **Exercise 4.17**

Show that $Q^A\mathcal{A}_n^{\text{anti-MHV}}=0$ and $\tilde{Q}_A\mathcal{A}_n^{\text{anti-MHV}}=0$.

Let us now outline three approaches to determining the superamplitudes $\mathcal{A}_n^{\text{N}^K\text{MHV}}$ beyond the MHV level.

1. ***Solution to the supersymmetry Ward identities.*** The N^KMHV superamplitudes in $\mathcal{N}=4$ SYM must obey the supersymmetry Ward identities (4.47). Writing the L-loop superamplitude $\mathcal{A}_{n,L}^{\text{N}^K\text{MHV}} = \delta^{(8)}(\tilde{Q})\,P_{n;L}^{(4K)}$, supersymmetry requires $Q^A P_{n;L}^{(4K)}=0$. In addition, we need $P_{n;L}^{(4K)}$ to obey the Ward identities of the global $SU(4)$ R-symmetry. As we have seen, these constraints are trivially satisfied at the MHV level where one L-loop component-amplitude suffices to fix $P_{n;L}^{(0)}$ and hence the full L-loop MHV n-point superamplitude.

   For non-MHV amplitudes, how many component-amplitudes does one need to fix the N^KMHV superamplitude? This question is answered by analyzing the constraints $Q^A P_{n;L}^{(4K)}=0$ and R-symmetry. It turns out that for general $K>0$, $P_{n;L}^{(4K)}$ can be written as a linear combination of L-loop "basis amplitudes" with coefficients built from the polynomials $m_{ijk,A}\equiv[ij]\eta_{kA}+[jk]\eta_{iA}+[ki]\eta_{jA}$. This is the same type of Grassmann polynomial that appears in $\mathcal{A}_3^{\text{anti-MHV}}$ given in (4.51). The solution reveals that the number of component-amplitudes sufficient to determine $\mathcal{A}_{n,L}^{\text{N}^K\text{MHV}}$ is the dimension of the irreducible representation of $SU(n-4)$ corresponding to a rectangular Young diagram with K rows and $(\mathcal{N}=)4$ columns. The independent component amplitudes are labeled by the semi-standard tableaux of this Young diagram.[3] A given particle content (gluons, fermions, scalars) of a basis amplitude corresponds to an ordered partition of the number $4K$ into n integers between 0 and 4. For each such partition, the corresponding Kostka number counts the number of independent arrangements of the $SU(4)$ R-symmetry indices. You can read more about the solution to the supersymmetry and R-symmetry Ward identities in [42, 43].

[3] By manipulating the color-structure, one can improve further on this count of "basis amplitudes" [42, 43].

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2. ***The super-MHV vertex expansion.*** The tree-level superamplitudes $\mathcal{A}_n^{\mathrm{N}^K\mathrm{MHV}}$ of $\mathcal{N}=4$ SYM can be constructed as an MHV vertex expansion in which the vertices are the MHV superamplitudes $\mathcal{A}_n^{\mathrm{MHV}}$ of (4.50). The expansion can be derived from an all-line shift, as we discussed in Section 3.4. The validity of the all-line shift for tree-level superamplitudes in $\mathcal{N}=4$ SYM was proven in [29] (see also [44]). A new feature is that one must sum over the possible intermediate states exchanged on the internal lines of the MHV vertex diagrams. This is conveniently done by integrating over the Grassmann variables $\eta_{P_I A}$ associated with internal lines; this automatically carries out a super-sum [36]. More details about such super-sums will be offered in the next section.
3. ***Super-BCFW.*** Superamplitudes can be constructed with a supersymmetric version of the BCFW shift. This construction plays a central role in many developments, so we will treat it in detail in the following section.

4.5 Super-BCFW and all tree-level amplitudes in $\mathcal{N}=4$ SYM

The BCFW shift introduced in Section 3.2 preserves the on-shell conditions $p_i^2=0$ and momentum conservation $\sum_{i=1}^n p_i=0$. However, the shift does not preserve the condition for supermomentum conservation, $\sum_{i=1}^n |i\rangle\eta_{i\,A}=0$. As a consequence, the shifted component amplitudes have large-z falloffs that depend on which types of particles are shifted, for example we have seen the difference between the $[-,+\rangle$ and $[+,-\rangle$ BCFW shifts of gluons in (3.13). This can be remedied by a small modification of the BCFW shift (3.9) that allows us to conserve supermomentum. We simply accompany the BCFW momentum shift by a shift in the Grassmann-variables [45–47]: for simplicity let us focus on the $[1,2\rangle$ ***super-BCFW*** shift (or $[1,2\rangle$ "supershift"), defined as

$$|\hat{1}]=|1]+z\,|2]\,,\qquad |\hat{2}\rangle=|2\rangle-z|1\rangle\,,\qquad \hat{\eta}_{1A}=\eta_{1A}+z\,\eta_{2A}\,. \tag{4.53}$$

No other spinors or Grassmann variables shift.

▶ **Exercise 4.18**

Show that the supermomentum is conserved under the supershift (4.53) so that $\delta^{(8)}\big(\tilde{Q}\big)$ is invariant.

It follows directly from (4.50) that the MHV superamplitudes have a $1/z$ falloff under a supershift of any adjacent lines ($1/z^2$ for non-adjacent). In fact, this falloff behavior holds true for any tree superamplitude in $\mathcal{N}=4$ SYM, as can be shown by using supersymmetry to rotate the two shifted lines to be positive helicity gluons and using the non-supersymmetric result (3.13) for the falloff under a $[+,+\rangle$-shift [19, 47].

The tree-level recursion relations that result from the super-BCFW shift (4.53) involve diagrams with two superamplitude "vertices" connected by an internal line with on-shell

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momentum $\hat{P}$. As in the non-supersymmetric case, we must sum over all possible states that can be exchanged on the internal line: in this case, this includes all 16 states of $\mathcal{N} = 4$ SYM. In terms of component amplitudes, the particle exchanged on the internal line depends on the external states: if they are all gluons, then the internal line is also a gluon and one must simply sum over the helicities. The superamplitude version of this helicity sum is

$$\left(\left[\left(\prod_{A=1}^{4}\frac{\partial}{\partial\eta_{\hat{P}A}}\right)\hat{\mathcal{A}}_{\rm L}\right]\frac{1}{P^2}\hat{\mathcal{A}}_{\rm R}+\hat{\mathcal{A}}_{\rm L}\frac{1}{P^2}\left[\left(\prod_{A=1}^{4}\frac{\partial}{\partial\eta_{\hat{P}A}}\right)\hat{\mathcal{A}}_{\rm R}\right]\right)\Bigg|_{\eta_{\hat{P}A}=0}, \tag{4.54}$$

where $\eta_{\hat{P}A}$ is the Grassmann variable associated with the internal line. In the first term of (4.54), a negative helicity gluon is projected out from the left sub-superamplitude leaving a positive helicity gluon coming out of the right sub-superamplitude. For the second term, the helicity projection is the opposite.

If a gluino can be exchanged on the internal line,[4] then we have to move one of the four Grassmann derivatives from $\hat{\mathcal{A}}_{\rm L}$ to $\hat{\mathcal{A}}_{\rm R}$ in the first term of (4.54) – in all four possible ways to sum over the four different gluinos labeled by the $SU(4)$-indices. And similarly for the second term. A scalar exchange means that two Grassmann derivatives act on $\hat{\mathcal{A}}_{\rm L}$ and the two others on $\hat{\mathcal{A}}_{\rm R}$. All in all, the entire sum over states exchanged on the internal line can be written

$$\left(\prod_{A=1}^{4}\frac{\partial}{\partial\eta_{\hat{P}A}}\right)\left[\hat{\mathcal{A}}_{\rm L}\frac{1}{P^2}\hat{\mathcal{A}}_{\rm R}\right]\Bigg|_{\eta_{\hat{P}A}=0}=\int d^4\eta_{\hat{P}}\;\hat{\mathcal{A}}_{\rm L}\frac{1}{P^2}\hat{\mathcal{A}}_{\rm R}\,. \tag{4.55}$$

Note how the product rule distributes the Grassmann derivatives $\partial/\partial\eta_{\hat{P}A}$ on the L and R superamplitudes in all possible ways to automatically carry out the state ***super-sum***. In (4.55), we have rewritten the Grassmann differentiation as a Grassmann integral. Similar super-sums are used in evaluation of unitarity cuts (see Chapter 6) of loop amplitudes where one includes integration over the Grassmann variable associated with the internal line [36, 48].

Remarkably, the super-BCFW recursion relations can be solved to give closed-form expressions for *all* tree-level superamplitudes in $\mathcal{N} = 4$ SYM [49]. We are now going to show how this works. As a warm-up, we first verify that the MHV superamplitude formula (4.50) satisfies the super-BCFW recursion relations. Then we present the most essential details of the derivation of the tree-level NMHV superamplitude. Finally we comment briefly on the results for N^KMHV.

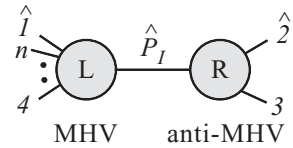
4.5.1 MHV superamplitude from super-BCFW

Consider the super-BCFW recursion relations for the MHV superamplitude. Just as in the non-supersymmetric case (3.20), there is just one non-vanishing diagram, but we must now

[4] This happens when there is an odd number of external gluinos on each side of the BCFW diagram.

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include the super-sum (4.55) over states exchanged on the internal line:

$$\mathcal{A}_n^{\rm MHV}[123\ldots n] \;=\; \raisebox{-0.5em}{
}$$

$$= \int d^4\eta_{\hat P}\, \hat{\mathcal{A}}_{n-1}[\hat 1, \hat P, 4, \ldots, n]\, \frac{1}{P^2}\, \hat{\mathcal{A}}_3[-\hat P, \hat 2, 3]$$

$$= \int d^4\eta_{\hat P}\, \frac{\delta^{(8)}\big(\sum_{i\in \rm L} |\hat i\rangle \hat\eta_i\big)}{\langle 1\hat P\rangle\langle \hat P\, 4\rangle\langle 45\rangle\cdots\langle n1\rangle}\, \frac{1}{P^2}$$
$$\times\, \frac{\delta^{(4)}\big([\hat P 2]\eta_3 + [23]\eta_{\hat P} + [3\hat P]\eta_2\big)}{[23][3\hat P][\hat P\, 2]}\,, \tag{4.56}$$

where $P = P_{23} = p_2 + p_3$ and we use (4.50) and (4.51) for the $(n-1)$-point MHV superamplitude and 3-point anti-MHV superamplitude. We also use the analytic continuation rule (3.17) for $|-P] = |P]$.

The new feature is the Grassmann integral in (4.56). The first delta function is

$$\delta^{(8)}\left(\sum_{i\in \rm L} |\hat i\rangle\hat\eta_i\right) \;=\; \delta^{(8)}\left(|1\rangle\hat\eta_1 + |\hat P\rangle\eta_{\hat P} + \sum_{i=4}^{n} |i\rangle\eta_i\right). \tag{4.57}$$

On the support of the second delta function, we can set $\eta_{\hat P} = -([\hat P 2]\eta_3 + [3\hat P]\eta_2)/[23]$ to find

$$\begin{aligned} |1\rangle\hat\eta_1 + |\hat P\rangle\eta_{\hat P} &= |1\rangle\hat\eta_1 - \frac{|\hat P\rangle}{[23]}\big([\hat P 2]\eta_3 + [3\hat P]\eta_2\big) \\ &= |1\rangle\hat\eta_1 + |3\rangle\eta_3 + |\hat 2\rangle\eta_2 \\ &= |1\rangle\eta_1 + |3\rangle\eta_3 + |2\rangle\eta_2\,. \end{aligned} \tag{4.58}$$

We used momentum conservation on the right-hand subamplitude to obtain the second line in (4.58) and the definition (4.53) of the supershift to get the third line. This way, the first delta function simply becomes $\delta^{(8)}(\sum_{i=1}^n |i\rangle\eta_i) = \delta^{(8)}(\tilde Q)$.

Now the only $\eta_{\hat P}$-dependence is in $\delta^{(4)}(\ldots)$ and the Grassmann integral is easy to carry out:

$$\int d^4\eta_{\hat P}\, \delta^{(8)}\left(\sum_{i\in \rm L} |\hat i\rangle\hat\eta_i\right)\, \delta^{(4)}\big([23]\eta_{\hat P} + \cdots\big) = [23]^4\, \delta^{(8)}\big(\tilde Q\big)\,.$$

Coming back to (4.56), we then have

$$\mathcal{A}_n^{\rm MHV}[123\ldots n] \;=\; \frac{\delta^{(8)}\big(\tilde Q\big)}{\langle 1\hat P\rangle\langle \hat P\, 4\rangle\langle 45\rangle\cdots\langle n1\rangle}\, \frac{1}{P^2}\, \frac{[23]^4}{[3\hat P][\hat P\, 2][23]}\,. \tag{4.59}$$

Compare this with (3.21) and you will see that the two expressions are the same, except that $\delta^{(8)}(\tilde Q)$ has replaced $\langle 1\hat P\rangle^4$ and $[23]^4$ has replaced $[3\hat P]^4$. Using the identities (3.22)

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and (3.23), we promptly recover the desired result

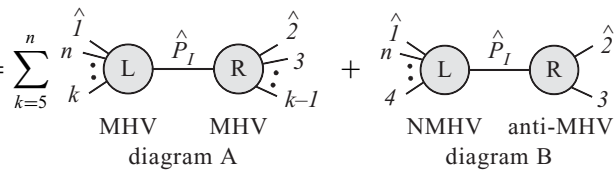
$$\mathcal{A}_n^{\text{MHV}}[123\ldots n] = \frac{\delta^{(8)}(\tilde{Q})}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\,. \tag{4.60}$$

Thus we have shown that the $\mathcal{N}=4$ SYM tree-level MHV superamplitude (4.60) satisfies the super-BCFW recursion relations. Next, we use super-BCFW to derive an important result for NMHV superamplitudes in $\mathcal{N}=4$ SYM.

4.5.2 NMHV superamplitude and beyond

The super-BCFW recursion relation for the NMHV superamplitude involves two types of diagrams

$$\mathcal{A}_n^{\text{NMHV}}[12\ldots n] = \sum_{k=5}^{n}$$



. (4.61)

The diagrams of type A involve two MHV sub-superamplitudes. Diagram B is present only for $n \geq 6$ because the L subamplitude is NMHV and therefore needs at least five legs; indeed, back in Section 3.2, we only had an MHV×MHV diagram in the calculation (3.27) of $A_5[1^-2^-3^-4^+5^+]$.

Diagram B provides the setting for an inductive proof of the NMHV superamplitude formula we are seeking, while the diagrams of type A give an "inhomogeneous" contribution. We begin with a detailed evaluation of the type A diagrams. There will be a lot of detailed calculations in this example calculation, so if you just want the result, you are free to skip ahead to the answer in (4.82). Right after the example, we summarize the full result for the NMHV superamplitude.

▷ *Example.* Calculation of diagram A. The first step is simply to plug in the MHV superamplitudes:

$$\begin{aligned}\text{Diagram A} = \int d^4\eta_{\hat{P}}\, &\frac{\delta^{(8)}(\text{L})}{\langle 1\hat{P}\rangle\langle\hat{P}\,k\rangle\langle k,k+1\rangle\ldots\langle n1\rangle}\,\frac{1}{P^2}\\ &\times\frac{\delta^{(8)}(\text{R})}{\langle\hat{P}\,\hat{2}\rangle\langle\hat{2}3\rangle\langle 34\rangle\ldots\langle k-1,\hat{P}\rangle}\,.\end{aligned} \tag{4.62}$$

Here $P = p_2 + p_3 + \cdots + p_{k-1}$ and (with the help of the rule (3.17)) we have

$$\begin{aligned}\delta^{(8)}(\text{L}) &= \delta^{(8)}\left(-|\hat{P}\rangle\eta_{\hat{P}} + |1\rangle\hat{\eta}_1 + \sum_{r=k}^{n}|r\rangle\eta_r\right),\\ \delta^{(8)}(\text{R}) &= \delta^{(8)}\left(|\hat{P}\rangle\eta_{\hat{P}} + |\hat{2}\rangle\eta_2 + \sum_{r=3}^{k-1}|r\rangle\eta_r\right).\end{aligned} \tag{4.63}$$

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On the support of $\delta^{(8)}(\mathrm{R})$, we can write $\delta^{(8)}(\mathrm{L}) = \delta^{(8)}(\mathrm{L}+\mathrm{R}) = \delta^{(8)}(\sum_{i=1}^{n}|i\rangle\eta_i) = \delta^{(8)}(\tilde{Q})$, which is independent of $\eta_{\hat{P}}$ and expresses the conservation of supermomentum for the n external states. Now only $\delta^{(8)}(\mathrm{R})$ remains under the state-sum integral $\int d^4\eta_{\hat{P}}$; since $\delta^{(8)}$ enforces two conditions we can project out two separate $\delta^{(4)}$s:

$$\delta^{(8)}\big(\mathrm{R}\big) = \frac{1}{\langle 1\hat{P}\rangle^4}\,\delta^{(4)}\left(\langle 1\hat{P}\rangle\eta_{\hat{P}} + \langle 1\hat{2}\rangle\eta_2 + \sum_{r=3}^{k-1}\langle 1r\rangle\eta_r\right)\,\delta^{(4)}\left(\langle\hat{P}\,\hat{2}\rangle\eta_2 + \sum_{r=3}^{k-1}\langle\hat{P}\,r\rangle\eta_r\right). \tag{4.64}$$

There is $\eta_{\hat{P}}$-dependence only in the first of these two $\delta^{(4)}$s, so it is straightforward to perform the Grassmann integral: it produces a factor $\langle 1\hat{P}\rangle^4$ which cancels the normalization factor included in (4.64). All in all, we have then shown that

$$\int d^4\eta_{\hat{P}}\;\delta^{(8)}\big(\mathrm{L}\big)\,\delta^{(8)}\big(\mathrm{R}\big) = \delta^{(8)}\big(\tilde{Q}\big)\,\delta^{(4)}\left(\langle\hat{P}\,\hat{2}\rangle\eta_2 + \sum_{r=3}^{k-1}\langle\hat{P}\,r\rangle\eta_r\right). \tag{4.65}$$

The factor $\delta^{(8)}(\tilde{Q})$ can be used to pull out an overall factor $\mathcal{A}_n^{\mathrm{MHV}}$ from diagram A in (4.62) and we can then write

$$\text{Diagram A} = \mathcal{A}_n^{\mathrm{MHV}}\,\frac{\langle 12\rangle\langle 23\rangle\langle k-1,\,k\rangle\,\delta^{(4)}\Big(\langle\hat{P}\,\hat{2}\rangle\eta_2 + \sum_{r=3}^{k-1}\langle\hat{P}\,r\rangle\eta_r\Big)}{\langle k\hat{P}\rangle\langle k-1,\,\hat{P}\rangle\langle\hat{2}\hat{P}\rangle\langle\hat{2}3\rangle\langle 1\hat{P}\rangle\;P^2}\,. \tag{4.66}$$

Next, we turn our attention to the brackets in (4.66) that involve the shifted spinors $|\hat{P}\rangle$ and $|\hat{2}\rangle$. As in the non-supersymmetric examples of Section 3.2, we manipulate these brackets by multiplying the numerator and denominator of (4.66) by $[\hat{P}\,2]^4\langle 21\rangle^4$. We are going to encounter sums of momenta that appear in the ordering fixed by color-structure, so for convenience we introduce the shorthand notation

$$y_{ij} \equiv p_i + p_{i+1} + \cdots + p_{j-1}\,. \tag{4.67}$$

Further, we declare that y_{ji} with $j > i$ equals $-y_{ij}$; when we think of the p_i as cyclically ordered, this simply expresses momentum conservation. We will use the variables y_{ij} to write the result for diagram A in a form that may look slightly mysterious for now, but it has some very important features that are discussed in Chapter 5.

One type of bracket in (4.66) is $\langle r\,\hat{P}\rangle$ with $r \neq 1, \hat{2}$. We manipulate this as follows:

$$\begin{aligned}\langle r\,\hat{P}\rangle[\hat{P}\,2]\langle 21\rangle &= -\langle r|\big(\hat{2}+3+\cdots+(k-1)\big)|2]\langle 21\rangle = -\langle r|y_{3k}|2]\langle 21\rangle \\ &= \langle r|y_{3k}\cdot y_{23}|1\rangle = \langle r|y_{3k}\cdot y_{13}|1\rangle = -\langle 1|y_{13}\cdot y_{3k}|r\rangle\,.\end{aligned} \tag{4.68}$$

The double angle-bracket is defined as $\langle i|p\cdot q|j\rangle \equiv \langle i|_{\dot{a}}\,p^{\dot{a}b}q_{b\dot{c}}\,|j\rangle^{\dot{c}}$. The result (4.68) applies to the brackets $\langle r\,\hat{P}\rangle$, $\langle k\hat{P}\rangle$, and $\langle k-1,\,\hat{P}\rangle$ in (4.66).

▶ Exercise 4.19

To keep you actively engaged, involved, and awake, here is an exercise: show that

$$\langle i|K.K|j\rangle = -K^2\langle ij\rangle \tag{4.69}$$

for any momentum K (light-like or not) and any spinors $\langle i|$ and $|j\rangle$.

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The two brackets $\langle 1\hat{P}\rangle$ and $\langle\hat{2}\hat{P}\rangle$ from the denominator of (4.62) are dealt with as follows:

$$\langle 1\hat{P}\rangle[\hat{P}\,2] = -\langle 1|y_{3k}|2] \qquad \text{and} \qquad \langle\hat{2}\hat{P}\rangle[\hat{P}\,2] = -2\hat{p}_2\cdot\hat{P} = -\hat{P}^2 + y_{3k}^2 = y_{3k}^2\,. \tag{4.70}$$

In the last equality, we used the fact that the BCFW diagram is evaluated on the value of z such that $\hat{P}^2 = 0$. It is useful to note what this value of z is:

$$0 = \hat{P}^2 = \langle\hat{2}|\hat{P}|2] + y_{3k}^2 = y_{2k}^2 - z\langle 1|y_{3k}|2] \qquad \Longrightarrow \qquad z = \frac{y_{2k}^2}{\langle 1|y_{3k}|2]}\,. \tag{4.71}$$

We use this z to evaluate $\langle\hat{2}3\rangle$:

$$\begin{aligned}\langle\hat{2}3\rangle &= \langle 23\rangle - \frac{y_{2k}^2}{\langle 1|y_{3k}|2]}\langle 13\rangle = \frac{\langle 1|y_{3k}|2]\langle 23\rangle - y_{2k}^2\langle 13\rangle}{\langle 1|y_{3k}|2]}\\ &= \frac{\langle 1|y_{1k}|2]\langle 23\rangle + \langle 1|y_{2k}\cdot y_{2k}|3\rangle}{\langle 1|y_{3k}|2]} = \frac{-\langle 1|y_{1k}\cdot p_2|3\rangle + \langle 1|y_{1k}\cdot y_{2k}|3\rangle}{\langle 1|y_{3k}|2]}\\ &= \frac{\langle 1|y_{1k}\cdot(-p_2+y_{2k})|3\rangle}{\langle 1|y_{3k}|2]} = \frac{\langle 1|y_{1k}\cdot y_{3k}|3\rangle}{\langle 1|y_{3k}|2]}\,.\end{aligned} \tag{4.72}$$

In the third equality, we use the result of Exercise 4.19. Combining (4.70) and (4.72), we can write

$$\langle\hat{2}3\rangle\langle 1\hat{P}\rangle[\hat{P}\,2] = \langle 1|y_{1k}\cdot y_{k3}|3\rangle\,. \tag{4.73}$$

We have now evaluated all four angle brackets involving $|\hat{P}\rangle$ and $|\hat{2}\rangle$ in (4.66). There is, however, another manipulation that we would like to do, namely one involving the propagator $1/P^2 = 1/y_{2k}^2$. It goes like this:

$$P^2\langle 12\rangle = y_{2k}^2\langle 12\rangle = -\langle 1|y_{2k}\cdot y_{2k}|2\rangle = -\langle 1|y_{1k}\cdot y_{3k}|2\rangle = \langle 1|y_{1k}\cdot y_{k3}|2\rangle\,. \tag{4.74}$$

And then it is time to put everything together. We can now write diagram A from (4.66) as

$$\text{Diagram A} = \mathcal{A}_n^{\text{MHV}}\frac{\langle 23\rangle\langle k-1,k\rangle\,[\hat{P}\,2]^4\langle 21\rangle^4\;\delta^{(4)}\Big(\langle\hat{P}\,\hat{2}\rangle\eta_2 + \sum_{r=3}^{k-1}\langle\hat{P}\,r\rangle\eta_r\Big)}{y_{3k}^2\langle 1|y_{13}\cdot y_{3k}|k\rangle\langle 1|y_{13}\cdot y_{3k}|k-1\rangle\langle 1|y_{1k}\cdot y_{k3}|3\rangle\langle 1|y_{1k}\cdot y_{k3}|2\rangle}\,. \tag{4.75}$$

Let us examine the $\delta^{(4)}$ in the numerator. We absorb the factors $[\hat{P}\,2]^4\langle 21\rangle^4$ into the delta function whose argument then becomes (suppressing $SU(4)$-indices)

$$\Xi \equiv \langle 12\rangle[2\hat{P}]\langle\hat{P}\,\hat{2}\rangle\eta_2 + \sum_{r=3}^{k-1}\langle 12\rangle[2\hat{P}]\langle\hat{P}\,r\rangle\eta_r = \langle 12\rangle y_{3k}^2\,\eta_2 + \sum_{r=3}^{k-1}\langle 1|y_{13}.y_{3k}|r\rangle\,\eta_r\,. \tag{4.76}$$

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We are going to do a little work on this expression in order to introduce another piece of shorthand notation, namely fermionic companions of the y_{ij}s defined in (4.67):

$$|\theta_{ij,A}\rangle \equiv \sum_{r=i}^{j-1} |r\rangle\, \eta_{rA}\,. \tag{4.77}$$

Then $|\theta_{ji}\rangle = -|\theta_{ij}\rangle$ encodes supermomentum conservation.

We start by rewriting each of the terms in (4.76):

$$\begin{aligned}\langle 12\rangle\, y_{3k}^2\, \eta_2 &= -\langle 1|y_{3k}\cdot y_{3k}|2\rangle\, \eta_2 = -\langle 1|y_{1k}\cdot y_{3k}|2\rangle\, \eta_2 + \langle 1|y_{23}\cdot y_{3k}|2\rangle\, \eta_2 \\ &= -\langle 1|y_{1k}\cdot y_{3k}|\theta_{13}\rangle + \langle 1|y_{1k}\cdot y_{3k}|1\rangle\, \eta_1 + \langle 1|y_{13}\cdot y_{3k}|2\rangle\, \eta_2 \end{aligned} \tag{4.78}$$

and

$$\sum_{r=3}^{k-1}\langle 1|y_{13}\cdot y_{3k}|r\rangle\, \eta_r = \langle 1|y_{13}\cdot y_{3k}|\theta_{1k}\rangle - \langle 1|y_{13}\cdot y_{3k}|1\rangle\, \eta_1 - \langle 1|y_{13}\cdot y_{3k}|2\rangle\, \eta_2\,. \tag{4.79}$$

Adding (4.78) and (4.79) to get Ξ, the two extra η_2-terms cancel and the η_1-terms combine to

$$\begin{aligned}\langle 1|y_{1k}\cdot y_{3k}|1\rangle\, \eta_1 - \langle 1|y_{13}\cdot y_{3k}|1\rangle\, \eta_1 &= \langle 1|(y_{1k}-y_{13})\cdot y_{3k}|1\rangle\, \eta_1 \\ &= \langle 1|y_{3k}\cdot y_{3k}|1\rangle\, \eta_1 = 0 \end{aligned} \tag{4.80}$$

by (4.69). Thus we have

$$\Xi = -\langle 1|y_{1k}\cdot y_{k3}|\theta_{31}\rangle - \langle 1|y_{13}.y_{3k}|\theta_{k1}\rangle\,. \tag{4.81}$$

Our work brings us to the following form of a diagram of type A:

$$\text{Diagram A} = \mathcal{A}_n^{\text{MHV}}\, \frac{\langle 23\rangle\langle k-1,k\rangle\ \delta^{(4)}\Big(\langle 1|y_{1k}\cdot y_{k3}|\theta_{31}\rangle + \langle 1|y_{13}\cdot y_{3k}|\theta_{k1}\rangle\Big)}{y_{3k}^2\,\langle 1|y_{13}\cdot y_{3k}|k\rangle\langle 1|y_{13}\cdot y_{3k}|k-1\rangle\langle 1|y_{1k}\cdot y_{k3}|3\rangle\langle 1|y_{1k}\cdot y_{k3}|2\rangle}\,. \tag{4.82}$$

This completes our calculation of diagram A in the super-BCFW recursion relation (4.61). Next, we discuss what the full NMHV superamplitude looks like. ◁

The result (4.82) for diagram A is often written $\mathcal{A}_n^{\text{MHV}}\, R_{13k}$, where the so-called ***R-invariants***[5] are defined as

$$R_{1jk} = \frac{\langle j-1,j\rangle\langle k-1,k\rangle\ \delta^{(4)}\big(\Xi_{1jk}\big)}{y_{jk}^2\,\langle 1|y_{1j}\cdot y_{jk}|k\rangle\langle 1|y_{1j}\cdot y_{jk}|k-1\rangle\langle 1|y_{1k}\cdot y_{kj}|j\rangle\langle 1|y_{1k}\cdot y_{kj}|j-1\rangle}\,, \tag{4.83}$$

where

$$\Xi_{1jk,A} = \langle 1|y_{1k}\cdot y_{kj}|\theta_{j1,A}\rangle + \langle 1|y_{1j}\cdot y_{jk}|\theta_{k1,A}\rangle\,. \tag{4.84}$$

[5] We discuss in Chapter 5 under which symmetries R_{ijk} is invariant.

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For completeness, we repeat here the definitions (4.67) and (4.77) of the variables $y_{ij} = -y_{ji}$ and $\theta_{ij,A} = -\theta_{ji,A}$:

$$y_{ij} \equiv p_i + p_{i+1} + \cdots + p_{j-1} \quad \text{and} \quad |\theta_{ij,A}\rangle \equiv \sum_{r=i}^{j-1} |r\rangle\, \eta_{rA}\,. \tag{4.85}$$

Note the structure of (4.83) and (4.84): neighbor indices match up with each other. This is an important feature.

For $n = 5$, diagram B vanishes and diagram A with $k = 5$ is the complete result:

$$\mathcal{A}_5^{\text{NMHV}} = \mathcal{A}_5^{\text{MHV}}\, R_{135}\,. \tag{4.86}$$

For $n > 5$, the diagrams of type A contribute $\mathcal{A}_n^{\text{MHV}} \sum_{k=5}^{n} R_{13k}$. Diagram B recurses this form (we will not present the details, see [49]) and the result is a sum of R-invariants:

$$\text{Diagram B} = \mathcal{A}_n^{\text{MHV}} \sum_{j=4}^{n-2} \sum_{k=j+2}^{n} R_{1jk}\,. \tag{4.87}$$

Hence, the entire NMHV superamplitude can be expressed in terms of the R-invariants as

$$\mathcal{A}_n^{\text{NMHV}} = \mathcal{A}_n^{\text{MHV}} \sum_{j=3}^{n-2} \sum_{k=j+2}^{n} R_{1jk}\,. \tag{4.88}$$

For example,

$$\mathcal{A}_6^{\text{NMHV}} = \mathcal{A}_6^{\text{MHV}} \big(R_{135} + R_{136} + R_{146}\big)\,, \tag{4.89}$$

and

$$\mathcal{A}_7^{\text{NMHV}} = \mathcal{A}_7^{\text{MHV}} \big(R_{135} + R_{136} + R_{137} + R_{146} + R_{147} + R_{157}\big)\,. \tag{4.90}$$

A generic Feynman diagram calculation of the 7-gluon NMHV component amplitude requires 154 diagrams. It is a dramatic simplification that reduces the sum of Feynman diagrams down to the six terms in (4.90). Moreover, note that it is not just the Feynman diagrams for gluon scattering that are encoded in the sums of R_{ijk}s, rather the relatively simple expression (4.88) encodes all tree-level NMHV scattering amplitudes of the entire $\mathcal{N} = 4$ SYM spectrum.

The remarkably simple result (4.88) for the all-n tree-level NMHV superamplitudes in $\mathcal{N} = 4$ SYM was first found in studies of loop amplitudes [50]. Later, the NMHV formula was constructed as we did here using super-BCFW [49].

One final comment about the R-invariants (4.83) is that we defined them here "anchored" at momentum line 1; this came about because we used a $[1, 2\rangle$-supershift. However, by cyclic symmetry all pairs of adjacent lines in the superamplitude are on equal footing, so an $[i, i+1\rangle$-supershift would have resulted in an NMHV formula anchored at i, giving R-invariants R_{ijk}. These are defined by replacing all momentum labels 1 in (4.83) by i. In

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the literature you will often find the expressions for the NMHV superamplitude given with n as the anchor; i.e. $\mathcal{A}_n^{\text{NMHV}} = \mathcal{A}_n^{\text{MHV}} \sum_{j=2}^{n-3} \sum_{k=j+2}^{n-1} R_{njk}$.

▶ Exercise 4.20

Write down the NMHV superamplitude formula that results from a $[2, 3\rangle$-supershift. Then project out the gluon amplitude $A_6[1^+2^-3^+4^-5^+6^-]$. Can you match your result to the 3-term expression in Exercise 3.9? Can you guess what expression you get from a $[3, 2\rangle$-supershift after projecting out $A_6[1^+2^-3^+4^-5^+6^-]$? The answer is given in Chapter 10.

Formulas for tree-level N^KMHV superamplitudes have also been derived – they take the form of sums of generalized R-invariants [49]. We refer you to the original paper [49], the discussions in [46, 47, 51], and the review [52] for further details.

The superamplitudes of $\mathcal{N} = 4$ SYM are invariant under a large symmetry group. In particular, there are both "ordinary" and "hidden" symmetries, and they are realized for the tree-level superamplitudes. The structure of the R-invariants is essential for this. We will discuss these symmetries next in Chapter 5.

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