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Admissible Regions for Higher-Order Finite Volume Method Grids

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Abstract. Admissible regions for higher-order finite volume method (FVM) grids are considered. A new Hermite quintic FVM and a new hybrid quintic FVM are constructed to solve elliptic boundary value problems, and the corresponding admissible regions are investigated. A sufficient condition for the uniform local-ellipticity of the new hybrid quintic FVM is obtained when its admissible region is known. In addition, the admissible regions for a large number of higher-order FVMs are provided. For the same class of FVM (Lagrange, Hermite or hybrid), the higher order FVM has a smaller admissible region such that stronger geometric restrictions are required to guarantee its uniform local-ellipticity.

AMS subject classifications: 65N30, 65N12

Key words: Finite volume method, admissible region, uniform local-ellipticity.

1. Introduction

Finite volume methods (FVM) are often invoked to solve partial differential equations numerically [2, 3, 17–19, 23]. The preservation of certain local conservation laws [14, 22] and flexible algorithm constructions [5, 7, 8, 12, 15, 16]) are attractive advantages, and the numerical solution of the linear system that arise have been addressed [10, 20]. Moreover, a high order FVM scheme [5–8, 19, 21, 23] is helpful in developing the *hp*-version of a FVM. High order FVM on rectangular meshes have previously been considered [4, 24–26], and here we focus on high order FVM on triangular meshes.

A FVM can be viewed as a special type of Petrov-Galerkin method where the chosen trial space is a standard finite element space. The test space is not the same as the trial space for the FVM, as the uniform local-ellipticity of the family of discrete bilinear forms may

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be destroyed to some degree. For a linear FVM, the uniform local-ellipticity of the family of the discrete bilinear forms does hold without additional requirements on the primary triangulation [19]. However, for higher-order FVM certain geometric requirements on the shapes of the triangles in the primary triangular mesh are needed to establish the uniform local-ellipticity [5–8, 19, 21, 23]. For example, in Ref. [7] it is shown that uniform local-ellipticity holds if $\theta_{\min} \geq 9.98^\circ$ for the quadratic FVM scheme proposed in Ref. [21], where $\theta_{\min}$ denotes the minimal angle of the triangles in the triangulation.

The key point for the investigation of uniform local-ellipticity lies in establishing the admissible region, which we now illustrate. Letting Ω denote a polygonal domain in $\mathbb{R}^2$ and assuming $f \in L^2(\Omega)$, we consider the FVM on a triangular mesh to solve the Poisson equation

$$-\Delta u = f \tag{1.1}$$

in Ω for the unknown function u subject to the Dirichlet boundary condition, where Δ is the Laplacian operator. Let $\mathbb{N}_m := \{1, 2, \cdots, m\}$ for a positive integer m, and denote the three edges of a triangle K by ℓ_i, $i \in \mathbb{N}_3$, where $|\ell_i|$ is the length of the edge ℓ_i and without loss of generality we have $|\ell_1| \geq |\ell_2| \geq |\ell_3|$. Letting $T := \{K\}$ denote a triangulation of Ω, for each triangle $K \in T$ we introduce the two geometric parameters

$$r_{1,K} := |\ell_2|^2/|\ell_1|^2, \qquad r_{2,K} := |\ell_3|^2/|\ell_1|^2. \tag{1.2}$$

A family $\mathscr{T} := \{T\}$ of triangulations of Ω is said to be *regular* if there exists a positive constant $\theta_{\inf}$ such that

$$\theta_{\min,K} \geq \theta_{\inf} \ \ \forall\, K \in \bigcup_{T\in\mathscr{T}} T, \tag{1.3}$$

where $\theta_{\min,K}$ denotes the minimum angle of the triangle K.

When possibly weaker requirements for boundary triangles intersecting the boundary of Ω are ignored, a necessary and sufficient condition on uniform local-ellipticity obtained as a corollary in Ref. [8] is as follows.

Proposition 1.1. *If a family $\mathscr{T}$ of triangulations of Ω is regular, then the corresponding family of discrete bilinear forms is uniformly local-elliptic if and only if there exists a compact subset $\mathbb{G}_0$ of the admissible region such that for all $T \in \mathscr{T}$ and all $K \in T$, $(r_{1,K}, r_{2,K}) \in \mathbb{G}_0$.*

Thus the uniform local-ellipticity corresponds to geometric requirements on the triangle shapes in the primary triangular mesh, where a larger admissible region implies weaker geometric requirements (and conversely). Moreover, when the admissible region is known, convenient sufficient conditions can be derived for the uniform local-ellipticity by choosing a proper $\mathbb{G}_0$. In brief, the admissible region is vital for the establishment of the uniform local-ellipticity, which leads to the optimal error estimate for the FVM.

In this article, we construct a new Hermite quintic FVM and a new hybrid quintic FVM. For the new Hermite quintic FVM grid, the admissible region is shown to be empty so that its uniform local-ellipticity does not hold no matter how regular the triangulation may be. The admissible region for the new hybrid quintic FVM grid is determined, and a convenient sufficient condition for its uniform local-ellipticity is derived.

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From a large number of experiments, we find that for each class of FVM (Lagrange, Hermite or hybrid) the higher order FVM scheme has a smaller admissible region. As in Ref. [7], if piecewise polynomials of degree k and k' are respectively used for the trial space on the triangular grid and the test space on the dual volume grid, the corresponding scheme is called a k-k' FVM. For the Lagrange FVM, the uniform local-ellipticity of the Lagrange 1-0 FVM holds on any non-degenerate triangulation [7, 19], so its admissible region is largest. Ref. [7] gave the admissible regions for the Lagrange k-0 FVM for $k = 2, 3, \cdots, 6$, with admissible regions decreasing as k increases. For the Hermite FVM, Ref. [7] drew the non-empty admissible region of the Zienkiewicz 3-1 FVM, where the trial space is a piecewise incomplete cubic polynomial space with respect to the triangulation. Here we prove that the admissible region for the new Hermite quintic FVM is empty, where the trial space is a piecewise incomplete quintic polynomial space with respect to the triangulation. We also find that the admissible region of our new hybrid quintic FVM is much smaller than that of the hybrid cubic FVM in Ref. [8].

In Section 2, we describe the geometric shape of the admissible region for the triangulation based FVM grid to solve the elliptic boundary value problem, and present a convenient sufficient condition for the uniform local-ellipticity. In Section 3, we construct the Hermite quintic FVM and prove that its admissible region is empty; and in Section 4 we construct our new hybrid quintic FVM, and study its admissible region and sufficient condition for uniform local-ellipticity. In Section 5, we compare the admissible regions and sufficient conditions for the uniform local-ellipticity of the respective Lagrange, Hermite and hybrid FVM. The matrices for the new Hermite quintic FVM and hybrid quintic FVM are presented in the Appendix.

2. Admissible Region

We first present the FVM for solving the elliptic boundary value problem (1.1), and recall the main results on the admissible region for the FVM grid established in Ref. [8]. Let $T^* := \{K^*\}$ be a dual partition of T, which is a triangulation of Ω. The elements in T^* are called control volumes. The finite dimensional trial space $\mathbb{U}_T$ is chosen as the standard conforming finite element space with respect to T and the test space $\mathbb{V}_{T^*}$ is a space of piecewise polynomials with respect to T^*, such that $\dim \mathbb{V}_{T^*} = \dim \mathbb{U}_T$ and $\mathbb{V}_{T^*}$ includes the characteristic functionals of the control volumes in T^*. For all $K \in T$, the discrete bilinear form on K for all $w \in \mathbb{U}_T$ and $v \in \mathbb{V}_{T^*}$ is

$$a_K(w, v) := \sum_{K^* \in T^*} \left\{ \int_{K^* \cap K} \nabla w \cdot \nabla v - \int_{\partial K^* \cap \mathrm{int} K} v \nabla w \cdot \mathbf{n} \right\},$$

where ∂K^* denotes the boundary of K^*, int$K$ the interior of K, and $\mathbf{n}$ the outward unit normal vector on ∂K^*. On defining

$$a_T(\cdot, \cdot) := \sum_{K \in T} a_K(\cdot, \cdot),$$

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for Eq. (1.1) we have the FVM scheme to find $u_T \in \mathbb{U}_T$:

$$a_T(u_T, v) = (f, v), \quad \text{for all} \quad v \in \mathbb{V}_{T^*}. \tag{2.1}$$

To introduce the uniform local-ellipticity of the family of the discrete bilinear forms, let $\mathscr{T} := \{T\}$ denote a family of triangulations of Ω, and for $w \in \mathbb{U}_T$ and $K \in T$ the H^1 semi-norm of w on K is denoted by $|w|_{1,K}$. If there exists a positive constant σ such that for all $T \in \mathscr{T}$ and the associated T^* there are linear mappings $\Pi_{T^*} : \mathbb{U}_T \to \mathbb{V}_{T^*}$ with $\Pi_{T^*}\mathbb{U}_T = \mathbb{V}_{T^*}$ satisfying

$$a_K(w, \Pi_{T^*} w) \geq \sigma |w|^2_{1,K} \tag{2.2}$$

for all $K \in T$ and all $w \in \mathbb{U}_T$, then we say that the family $\mathscr{A}_{\mathscr{T}} := \{a_T(\cdot, \Pi_{T^*}\cdot) : T \in \mathscr{T}\}$ of the discrete bilinear forms is uniformly local-elliptic [7, 8]. The uniform local-ellipticity of $\mathscr{A}_{\mathscr{T}}$ implies that the FVM equation (2.1) has a unique solution with the optimal error estimate [8].

To introduce the admissible region for the FVM grid, we choose the isosceles right-angled triangle $\hat{K}$ with vertices $\hat{P}_1 := (0,0)$, $\hat{P}_2 := (1,0)$ and $\hat{P}_3 := (0,1)$ as a reference triangle. For each triangle K, there is an invertible affine mapping $\mathscr{F}_K$ from $\hat{K}$ to K [19], so we only need to consider the trial and test spaces on $\hat{K}$. Assume that the trial space on $\hat{K}$ is $\hat{n}$-dimensional with a basis $\Phi_{\hat{K}} := \{\hat{\phi}_i : i \in \mathbb{N}_{\hat{n}}\}$, there is a basis $\Psi_{\hat{T}^*} := \{\hat{\psi}_j : j \in \mathbb{N}_{\hat{n}}\}$ for the test space on $\hat{K}$, and a set of degrees of freedom $\hat{\Sigma} := \{\hat{\eta}_i : i \in \mathbb{N}_{\hat{n}}\}$ on $\hat{K}$ such that

$$\hat{\eta}_i(\hat{\varphi}_j) = \hat{\eta}_i(\hat{\psi}_j) = \delta_{i,j} := \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

The FVM presented in Refs. [7,8] all satisfy the above assumption, and the $\hat{\eta}_i, i \in \mathbb{N}_{\hat{n}}$ may be point evaluation functionals, differences, directional derivatives or higher-order directional derivatives. Let

$$\mathbf{e} := \left[\hat{\eta}_i(\chi_{\hat{K}}) : i \in \mathbb{N}_{\hat{n}}\right]^T \quad \text{and} \quad \mathbf{E} := \frac{1}{\mathbf{e}^T\mathbf{e}}\mathbf{e}\mathbf{e}^T,$$

where $\chi_{\hat{K}}$ is the characteristic function of $\hat{K}$ and the element of $\mathbf{e}$ is 1 or 0. For each $K \in T$, we respectively define the element stiff matrix on K and its symmetrisation as

$$\mathbf{A}_K := \left[a_K\left(\phi_{j,K}, \psi_{i,K}\right) : i, j \in \mathbb{N}_{\hat{n}}\right], \qquad \tilde{\mathbf{A}}_K := (\mathbf{A}_K + \mathbf{A}_K^T)/2,$$

where $\phi_{i,K} := \hat{\phi}_i \circ \mathscr{F}_K^{-1}$, $\psi_{i,K} := \hat{\psi}_i \circ \mathscr{F}_K^{-1}$, $i \in \mathbb{N}_{\hat{n}}$ with $\mathscr{F}_K^{-1}$ being the invertible mapping of $\mathscr{F}_K$. The uniform local-ellipticity of the family $\mathscr{A}_{\mathscr{T}}$ of the discrete bilinear forms is equivalent to the existence of a positive constant σ such that for all $T \in \mathscr{T}$, all $K \in T$ and all $\mathbf{w} \in \mathbb{R}^{\hat{n}}$ we have [8]

$$\mathbf{w}_K^T(\tilde{\mathbf{A}}_K + \mathbf{E})\mathbf{w}_K \geq \frac{\sigma}{1+\sigma}\mathbf{w}_K^T\mathbf{w}_K. \tag{2.3}$$

The geometric parameters defined in (1.2) have the relationship

$$0 < r_{2,K} \leq r_{1,K} \leq 1.$$

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For each triangle $K \in T$ with area denoted by $|K|$, we define another geometric parameter

$$\alpha_K := |\ell_1|^2/(4|K|). \tag{2.4}$$

By translating the integral $a_K(\phi_{j,K}, \psi_{i,K})$ onto the reference triangle $\hat{K}$, we get three matrices $\mathbf{A}_i$, $i = 0, 1, 2$ that are only related to the reference triangle $\hat{K}$, such that for all $K \in T$ (cf. Theorem 4.2 of [7]):

$$\tilde{\mathbf{A}}_K = \alpha_K(\mathbf{A}_0 + r_{1,K}\mathbf{A}_1 + r_{2,K}\mathbf{A}_2). \tag{2.5}$$

If $\mathscr{T}$ is regular, there is a positive constant θ_{inf} satisfying (1.3), and from the area formula for a triangle we obtain

$$1/\sqrt{3} \le \alpha_K \le 1/\tan\theta_{\text{inf}}\,. \tag{2.6}$$

Given inequality (2.3), Eq. (2.5) and inequality (2.6), for each $(r_1, r_2) \in \mathbb{R}^2$ we introduce the matrix

$$\mathbf{H}(r_1, r_2) := \mathbf{A}_0 + r_1\mathbf{A}_1 + r_2\mathbf{A}_2 + \mathbf{E}\,,$$

where the elements of $\mathbf{H}(r_1, r_2)$ are continuous functions of $(r_1, r_2) \in \mathbb{R}^2$ and

$$\mathbf{H}(r_{1,K}, r_{2,K}) = \alpha_K^{-1}\tilde{\mathbf{A}}_K + \mathbf{E}\,.$$

Let $\lambda_{\min}(\mathbf{M})$ denote the minimum eigenvalue of the matrix $\mathbf{M}$. From Theorem 5.2 of Ref. [8], if we ignore the possibly weaker requirement for boundary triangles the uniform local-ellipticity of $\mathscr{A}_{\mathscr{T}}$ is equivalent to the existence of a positive constant c such that

$$\lambda_{\min}\big(\mathbf{H}(r_{1,K}, r_{2,K})\big) \ge c\,, \quad \text{for all } T \in \mathscr{T} \text{ and all } K \in T\,. \tag{2.7}$$

Consequently, we define the admissible region for the FVM as grid

$$\mathbb{G}_{\mathbf{H}} := \big\{(r_1, r_2) \in \mathbb{R}^2 :\ \lambda_{\min}(\mathbf{H}(r_1, r_2)) > 0,\ 0 \le r_2 \le r_1 \le 1\big\}\,.$$

Proposition 1.1 shows that the admissible region is crucial in determining the uniform local-ellipticity of $\mathscr{A}_{\mathscr{T}}$, and the admissible region was characterised as follows in Ref. [8].

Proposition 2.1. *If the admissible region $\mathbb{G}_{\mathbf{H}}$ is non-empty, then $\mathbb{G}_{\mathbf{H}}$ is a convex set surrounded by the lines $r_1 = r_2$, $r_1 = 1$ and a curve $\Gamma_{\mathbf{H}} := \{(r_1, r_2) \in \mathbb{R}^2 :\ \lambda_{\min}(\mathbf{H}(r_1, r_2)) = 0,\ 0 \le r_2 \le r_1 \le 1\}$, being close on $r_1 = r_2$, $r_1 = 1$, and open on $\Gamma_{\mathbf{H}}$.*

Thus Proposition 2.1 describes the shape of the admissible region, and if $(1, 1) \notin \mathbb{G}_{\mathbf{H}}$ then $\mathbb{G}_{\mathbf{H}}$ is empty.

Under Proposition 1.1, by choosing different compact subsets of the admissible region we can get many types of sufficient conditions for the uniform local-ellipticity of the family of the discrete bilinear forms. We next present one type of convenient sufficient conditions for the uniform local-ellipticity developed in Ref. [7]. Recalling that $\theta_{\min,K}$ denotes the smallest angle in a triangle K such that

$$\cos\theta_{\min,K} = \frac{1 + r_{1,K} - r_{2,K}}{2r_{1,K}^{1/2}}\,,$$

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for $(r_1, r_2) \in \mathbb{R}^2$ we define

$$\theta_{\min}(r_1, r_2) := \arccos \frac{1 + r_1 - r_2}{2(r_1)^{1/2}}.$$

For each triangle K with parameters $r_{1,K}$ and $r_{2,K}$ we have $\theta_{\min}(r_{1,K}, r_{2,K}) = \theta_{\min,K}$ such that

$$\frac{\partial \theta_{\min}(r_1, r_2)}{\partial r_2} > 0\,,$$

so on denoting the maximum value

$$\theta_0 := \max_{(r_1,r_2)\in\Gamma_{\mathbf{H}}} \theta_{\min}(r_1, r_2)$$

for any positive constant ϵ_1 we have that

$$\mathbb{G}_0 := \left\{(r_{1,K}, r_{2,K}) : \theta_{\min}(r_{1,K}, r_{2,K}) \geq \epsilon_1 + \theta_0\right\}$$

is a compact subset of $\mathbb{G}_{\mathbf{H}}$ — a sufficient condition for uniform local-ellipticity according to Proposition 1.1.

Rather than attempting to evaluate θ_0 directly as there are infinitely many $(r_1, r_2) \in \Gamma_{\mathbf{H}}$, we compute an upper bound of θ_0 by using an inscribed polygonal line to approximate $\Gamma_{\mathbf{H}}$ so only finitely many values are involved. The error between this bound and θ_0 of course depends on how well this inscribed polygonal line approximates $\Gamma_{\mathbf{H}}$, and the approximation can be improved by selecting more points on $\Gamma_{\mathbf{H}}$ as follows [8]. For any $S := (s_1, s_2)$ and $T := (t_1, t_2)$ such that $t_1 > s_1$, we introduce

$$l_1 := 1 - \frac{t_2 - s_2}{t_1 - s_1}\,, \qquad l_2 := 1 + s_1 - s_2 - l_1 s_1\,, \qquad h(t) := \frac{l_1 t + l_2}{2t^{1/2}}\,.$$

Define $\eta_1 := h(t_1)$ $\eta_2 := \min\{\eta_1, h(s_1)\}$ and $\eta_3 := \min\{\eta_2, h(l_2/l_1)\}$, and compute

$$\eta(S, T) := \begin{cases} \cos^{-1}\eta_1\,, & l_1 = 0\,, \\ \cos^{-1}\eta_2\,, & l_1 \neq 0 \text{ and } \frac{l_2}{l_1} \notin [s_1, t_1]\,, \\ \cos^{-1}\eta_3\,, & \text{otherwise}\,. \end{cases}$$

Let $\overline{ST}$ denote the closed line segment between S and T. We note that

$$\eta(S, T) = \max_{(r_1,r_2)\in\overline{ST}} \theta_{\min}(r_1, r_2)\,.$$

For a given integer N, we select $N + 1$ points $R_l := (r_{1,l}, r_{2,l})$, $l = 0, 1, \cdots, N$ in $\Gamma_{\mathbf{H}}$, where

$$r_{1,0} := r_{1,\mathbf{H}}, \quad r_{2,0} := r_{1,\mathbf{H}} \quad \text{and} \quad r_{1,l} := r_{1,\mathbf{H}} + \frac{l(1 - r_{1,\mathbf{H}})}{N}, \quad r_{2,l} := r_{2,\mathbf{H}}(r_{1,l}), \quad l \in \mathbb{N}_N\,,$$

where $r_{2,\mathbf{H}}(r_1), r_1 \in (r_{1,\mathbf{H}}, 1]$ is the expression of the curve $\Gamma_{\mathbf{H}}$.

The next proposition presents a convenient sufficient condition for the uniform local-ellipticity of the family of the discrete bilinear forms established in Ref. [8].

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Proposition 2.2. *If there exists a positive constant ϵ_1 such that for all $T \in \mathscr{T}$ and all $K \in T$*

$$\theta_{\min}(r_{1,K}, r_{2,K}) \geq \epsilon_1 + \max_{0 \leq l \leq N-1} \eta(R_l, R_{l+1}),$$

then the family $\mathscr{A}_{\mathscr{T}}$ of the discrete bilinear forms is uniformly local-elliptic.

3. Hermite Quintic FVM

In this section, we present a Hermite quintic FVM and discuss its admissible region. According to Refs. [7, 8], to describe a FVM it is suffices to present the FE triple element $(\hat{K}, \hat{\Sigma}, \mathbb{U}_{\hat{K}})$ and the triple element $(\hat{T}^*, \hat{\Sigma}, \mathbb{V}_{\hat{T}^*})$ on the reference triangle $\hat{K}$ where $\mathbb{U}_{\hat{K}}$ is the trial space on $\hat{K}$, $\hat{T}^*$ is the dual partition on $\hat{K}$ and $\mathbb{V}_{\hat{T}^*}$ is the test space on $\hat{K}$, and $\hat{\Sigma} = \{\hat{\eta}_i : i \in \mathbb{N}_{\hat{n}}\}$ is a common set of degrees of freedom for $\mathbb{U}_{\hat{K}}$ and $\mathbb{V}_{\hat{T}^*}$.

For the Hermite quintic FVM considered here, $\mathbb{U}_{\hat{K}}$ is chosen as the space of polynomials of degree less than or equal to 5 on $\hat{K}$, with element w satisfying

$$4\{\partial_{\xi_i} w(P_i) + \partial_{\xi_i} w(P_{i+1})\} - 8\partial_{\xi_i} w(Q_i) + \big(D\partial_{\xi_i} w(P_i) - D\partial_{\xi_i} w(P_{i+1})\big) \cdot (P_{i+1} - P_i) = 0, \quad i \in \mathbb{N}_3, \quad (3.1)$$

where $\xi_1 := (1,-2)$, $\xi_2 := (1,1)$ and $\xi_3 := (-2,1)$ — and for each $i \in \mathbb{N}_3$ the symbol ∂_{ξ_i} denotes the directional derivative along the direction ξ_i, and

$$\begin{aligned} &P_1 := (0,0), && P_2 := (1,0), && P_3 := (0,1), \\ &Q_1 := (1/2,0), && Q_2 := (1/2,1/2), && Q_3 := (0,1/2). \end{aligned}$$

Obviously, $\dim \mathbb{U}_{\hat{K}} = 18$. We also define the set $\hat{\Sigma}$ of degrees of freedom. Thus for a sufficiently smooth function w and directions $\varsigma_1 := (\varsigma_{1,1}, \varsigma_{1,2})$ and $\varsigma_2 := (\varsigma_{2,1}, \varsigma_{2,2})$ we introduce the first and second order directional derivatives

$$Dw(x) \cdot \varsigma_1 := \sum_{i=1}^{2} \frac{\partial w(x)}{\partial x_i} \varsigma_{1,i} \quad \text{and} \quad D^2 w(x) \cdot (\varsigma_1, \varsigma_2) := \sum_{i,j=1}^{2} \frac{\partial^2 w(x)}{\partial x_i \partial x_j} \varsigma_{1,i} \varsigma_{2,j},$$

and choose six directions

$$\begin{aligned} &\zeta_1 := (1,0), && \zeta_2 := (0,1), && \zeta_3 := (-1,1), \\ &\zeta_4 := (-1,0), && \zeta_5 := (0,-1), && \zeta_6 := (1,-1), \end{aligned}$$

to produce the set of degrees of freedom $\hat{\Sigma} := \{\hat{\eta}_l : l \in \mathbb{N}_{18}\}$ where

$$\begin{aligned} &\hat{\eta}_i(w) := w(P_i),\ i \in \mathbb{N}_3, \quad \hat{\eta}_{3+i}(w) := Dw(P_j) \cdot \zeta_i, \quad i \in \mathbb{N}_6,\ j = \lceil i/2 \rceil, \\ &\hat{\eta}_{10}(w) := D^2 w(P_1) \cdot (\zeta_1, \zeta_1), \quad \hat{\eta}_{11}(w) := D^2 w(P_1) \cdot (\zeta_1, \zeta_2), \quad \hat{\eta}_{12}(w) := D^2 w(P_1) \cdot (\zeta_2, \zeta_2), \\ &\hat{\eta}_{13}(w) := D^2 w(P_2) \cdot (\zeta_3, \zeta_3), \quad \hat{\eta}_{14}(w) := D^2 w(P_2) \cdot (\zeta_3, \zeta_4), \quad \hat{\eta}_{15}(w) := D^2 w(P_2) \cdot (\zeta_4, \zeta_4), \\ &\hat{\eta}_{16}(w) := D^2 w(P_3) \cdot (\zeta_5, \zeta_5), \quad \hat{\eta}_{17}(w) := D^2 w(P_3) \cdot (\zeta_5, \zeta_6), \quad \hat{\eta}_{18}(w) := D^2 w(P_3) \cdot (\zeta_6, \zeta_6). \end{aligned}$$

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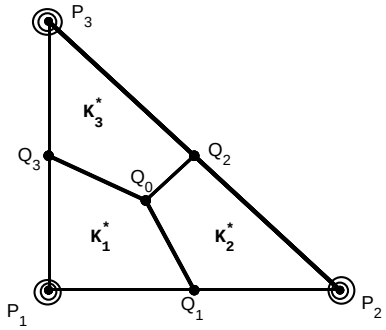


Figure 1: The freedom functionals and dual partition of the Hermite quintic FVM.

In Fig. 1, $\hat{\eta}_1$, $\hat{\eta}_2$ and $\hat{\eta}_3$ are shown as bullets at the nodes, $\hat{\eta}_4,\cdots,\hat{\eta}_9$ as inside circles around the nodes, and $\hat{\eta}_{10},\cdots,\hat{\eta}_{18}$ as outside circles around the nodes. It is straightforward to verify that there is a basis denoted by $\Phi_{\hat{K}} := \{\hat{\varphi}_i, i \in \mathbb{N}_{18}\}$ for $\mathbb{U}_{\hat{K}}$ such that $\hat{\eta}_i(\hat{\varphi}_j) = \delta_{i,j}$, $i,j \in \mathbb{N}_{18}$.

For each $K \in T$, using the affine mapping from the reference triangle $\hat{K}$ to K we derive the overall trial space $\mathbb{U}_T$ for the Hermite quintic FVM, where $\mathbb{U}_T$ is C^0-continuous but not C^1-continuous. The classical Bell finite element space is C^1-continuous [11], but the restrictive conditions (3.1) constitute the significant difference between $\mathbb{U}_T$ and the Bell finite element space. We employ the affine invariant directions ξ_i, $i \in \mathbb{N}_3$ in the conditions (3.1), instead of the normal derivatives on the edges $P_iP_{i+1}, i \in \mathbb{N}_3$ used for the Bell finite element. This produces a jump of the derived functions in $\mathbb{U}_T$ on the common edge of two adjacent triangles, so $\mathbb{U}_T$ is not C^1-continuous.

For $m > 2$ and m points $Q_j \in \mathbb{R}^2$, $j \in \mathbb{N}_m$, we use $\Theta\{Q_1,Q_2,\cdots,Q_m\}$ for the polygon with the vertices Q_i, $i \in \mathbb{N}_m$ and Q_1 being connected consecutively. Letting $Q_0 = (1/3,1/3)$, we define the volume partition $\hat{\mathcal{T}}^* := \{K_l^* : l \in \mathbb{N}_3\}$ of $\hat{K}$ by setting

$$K_1^* := \Theta\{P_1,Q_1,Q_0,Q_3\}, \qquad K_2^* := \Theta\{P_2,Q_2,Q_0,Q_1\}, \qquad K_3^* := \Theta\{P_3,Q_3,Q_0,Q_2\}.$$

These control volumes are as shown in Fig. 1. For $x := (x_1,x_2) \in \hat{K}$, let

$$q_1(x) := x_1, \qquad q_2(x) := x_2, \qquad q_3(x) := 1 - x_1 - x_2,$$

and for K_i^*, $i \in \mathbb{N}_3$ let χ_i denote the characteristic function on K_i^*. In the Hermite quintic FVM, we choose $\mathbb{V}_{\hat{T}^*}$ as a space of piecewise quadratic polynomials with respect to $\hat{T}^*$ on $\hat{K}$ spanned by $\Psi_{\hat{T}^*}$, where $\Psi_{\hat{T}^*}$ consists of $\hat{\psi}_i := \chi_i$, $i \in \mathbb{N}_3$ and

$$\begin{aligned}
&\hat{\psi}_4 := q_1\chi_1, && \hat{\psi}_5 := q_2\chi_1, && \hat{\psi}_6 := q_2\chi_2, && \hat{\psi}_7 := q_3\chi_2,\\
&\hat{\psi}_8 := q_3\chi_3, && \hat{\psi}_9 := q_1\chi_3, && \hat{\psi}_{10} := \frac{1}{2}q_1^2\chi_1, && \hat{\psi}_{11} := q_1q_2\chi_1,\\
&\hat{\psi}_{12} := \frac{1}{2}q_2^2\chi_1, && \hat{\psi}_{13} := \frac{1}{2}q_2^2\chi_2, && \hat{\psi}_{14} := q_2q_3\chi_2, && \hat{\psi}_{15} := \frac{1}{2}q_3^2\chi_2,\\
&\hat{\psi}_{16} := \frac{1}{2}q_3^2\chi_3, && \hat{\psi}_{17} := q_1q_3\chi_3, && \hat{\psi}_{18} := \frac{1}{2}q_1^2\chi_3.
\end{aligned}$$

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We next discuss the admissible region for the Hermite quintic FVM grid constructed above. From the description in the last section, a principal step is to derive the matrices $\mathbf{A}_i, i \in \mathbb{N}_3$ and $\mathbf{E}$ for the Hermite quintic FVM. The matrices $\mathbf{A}_i, i = 0, 1, 2$ are presented in the Appendix, and from the definition of the matrix $\mathbf{E}$ we get

$$\mathbf{e} = [1, 1, 1, 0, \cdots, 0]^T_{1\times 18}\,, \qquad \mathbf{E} = (1/3)\mathbf{e}\mathbf{e}^T\,, \tag{3.2}$$

so the matrices $\mathbf{H}(\cdot,\cdot)$ for the Hermite quintic FVM are derived. We prove the following theorem.

Theorem 3.1. *The admissible region of the Hermite quintic FVM constructed above is empty.*

Proof. By simple calculation, the minimum eigenvalue of the matrix $\mathbf{H}(1,1)$ of the Hermite quintic FVM is -0.0003. We claim that the admissible region $\mathbb{G}_{\mathbf{H}}$ of the Hermite quintic FVM is empty. Otherwise, we have from Proposition 2.1 that $(1,1) \in \mathbb{G}_{\mathbf{H}}$, and from the definition of $\mathbb{G}_{\mathbf{H}}$ the contradiction that $\lambda_{\min}(\mathbf{H}(1,1)) > 0$. □

From Proposition 1.1 and Theorem 3.1, we get the following corollary.

Corollary 3.1. *The uniform local-ellipticity of the family of the discrete bilinear forms of the Hermite quintic FVM does not hold.*

Thus the uniform local-ellipticity of the family of discrete bilinear forms for the Hermite quintic FVM can be satisfied no matter how regular the family of triangulations may be.

4. The Hybrid Quintic FVM

We now describe our hybrid quintic FVM for solving Eq. (1.1), and discuss its admissible region and uniform local-ellipticity. According to the description in the last section, we only have to present triple elements $(\hat{K}, \hat{\Sigma}, \mathbb{U}_{\hat{K}})$ and $(\hat{T}^*, \hat{\Sigma}, \mathbb{V}_{\hat{T}^*})$ — and the FE triple element $(\hat{K}, \hat{\Sigma}, \mathbb{U}_{\hat{K}})$ and the dual partition $\hat{\mathscr{T}}^* := \{K_i^* : i \in \mathbb{N}_3\}$ of the reference triangle $\hat{K}$ are the same as those for the Hermite quintic FVM described in the last section, as is the basis $\Phi_{\hat{K}} = \{\hat{\varphi}_i, i \in \mathbb{N}_{18}\}$ for $\mathbb{U}_{\hat{K}}$, but we need to choose another space $\mathbb{V}_{\hat{T}^*}$ for the hybrid quintic FVM — viz. $\mathbb{V}_{\hat{T}^*} :=$ span $\Psi_{\hat{T}^*}$, with the basis $\Psi_{\hat{T}^*}$ consisting of functions

$$\hat{\psi}_i := \chi_i,\ i \in \mathbb{N}_3 \quad \text{and} \quad \hat{\psi}_i := \hat{\varphi}_i,\ i \in \mathbb{N}_{18} \setminus \mathbb{N}_3\,,$$

where χ_i is the characteristic function on $K_i^* \in \hat{\mathscr{T}}^*$.

To examine the admissible region and the uniform local-ellipticity of the hybrid quintic FVM, we first calculate the corresponding matrices $\mathbf{A}_i$, $i \in \mathbb{N}_3$ and $\mathbf{E}$. Since the set of degrees of freedom for the hybrid quintic FVM is the same as for the Hermite quintic FVM, the matrix $\mathbf{E}$ for the hybrid quintic FVM is the same as in (3.2). The matrices $\mathbf{A}_i$, $i \in \mathbb{N}_3$ for the hybrid quintic FVM are calculated in the Appendix, so we obtain the corresponding matrices $\mathbf{H}(\cdot,\cdot)$. Thus the admissible region for the hybrid quintic FVM grid can be derived, and accordingly sufficient conditions for the uniform local-ellipticity can be computed. The admissible region of the hybrid quintic FVM shown in Fig. 4 in the next section can be compared with the admissible region for an hybrid cubic FVM, and a sufficient condition for the hybrid quintic FVM is given in the following theorem.

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Theorem 4.1. *For the hybrid quintic FVM, if the smallest angle in each triangular element of the triangulation is greater than* 57.53°*, then the family* $\mathscr{A}_{\mathscr{T}}$ *of discrete bilinear forms is uniformly local-elliptic.*

Proof. Using the definition of $\mathbb{G}_{\mathbf{H}}$ and Proposition 2.1, we can get $\mathbb{G}_{\mathbf{H}}$ for the hybrid quintic FVM (cf. Fig. 4). Thus we obtain the curve $\Gamma_{\mathbf{H}}$ for the hybrid quintic FVM, and the desired result computed using Proposition 2.2 with R_l, $l = 0, 1, \cdots, 100$ on $\Gamma_{\mathbf{H}}$. □

5. Comparison of the Admissible Regions

In this section, we compare the admissible regions of the lower order and higher order Lagrange, Hermite and hybrid FVM respectively. It emerges in each type the higher order case has a smaller admissible region than the lower order, so needs stronger geometric restrictions to ensure its uniform local-ellipticity — cf. Proposition 1.1.

5.1. Lagrange FVM

Using the information presented in Ref. [7], we first draw the admissible regions for the uniform Lagrange k-0 FVMs (denoted by LFVM k) for $k = 1, 2, \cdots, 6$ in Fig. 2. Combined with the line labeled '$r_1 = r_2$' and the line $r_1 = 1$, the curves $\Gamma_{\mathbf{H}}$ labeled LFVM k, $k = 1, 2, \cdots, 6$ form the corresponding admissible regions $\mathbb{G}_{\mathbf{H}}$ according to Proposition 2.1. We see that the admissible regions become smaller as k increases, and by Proposition 1.1 the geometric requirements on the shapes of the triangles in the triangulation grow stronger. This can be seen more clearly from the sufficient conditions for the uniform local-ellipticity listed in Table 1. If the smallest angle in each triangular element of the triangulation is greater than the value listed in Table 1, then for each of the LFVM k, $k = 2, 3, \cdots, 6$ the corresponding family of discrete bilinear forms is uniformly local-elliptic and in turn the FVM has an optimal approximating solution. The sufficient conditions for the LFVM k, $k = 2, \cdots, 6$ in Table 1 can be calculated according to Proposition 2.2 with $N = 100$, and were already presented in Ref. [7]. (We do not list sufficient conditions for the LFVM 1 in Table 1, since it was verified in Ref. [7] that uniform local-ellipticity in that case holds on any non-degenerate triangulation.) A smaller value for the type I sufficient condition implies weaker geometric restrictions, and in Table 1 we note that the geometric conditions for the uniform local-ellipticity of the LFVM k, $2, 3, \cdots, 6$ become stronger at larger k.

5.2. The Hermite FVM

From Lemma 3.1, we know that the admissible region for the Hermite quintic FVM is empty. The admissible region for the Zienkiewicz 3-1 FVM shown in Fig. 3 is surrounded by the curve labeled 'ZFVM', the line labeled as '$r_1 = r_2$' and the line $r_1 = 1$. Obviously, the admissible region for the Zienkiewicz 3-1 FVM is not empty, and the geometric restrictions on the shapes of the triangles in the corresponding triangulation are much weaker than for the Hermite quintic FVM according to Proposition 1.1. This can be seen more clearly from

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Table 1: Sufficient conditions for the LFVM k.

LFVM 2	LFVM 3	LFVM 4	LFVM 5	LFVM 6
9.98°	17.26°	26.30°	35.69°	46.19°

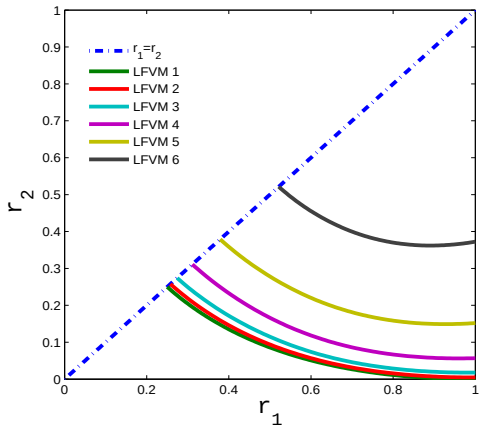


Figure 2: The admissible regions for the Lagrange FVMs.

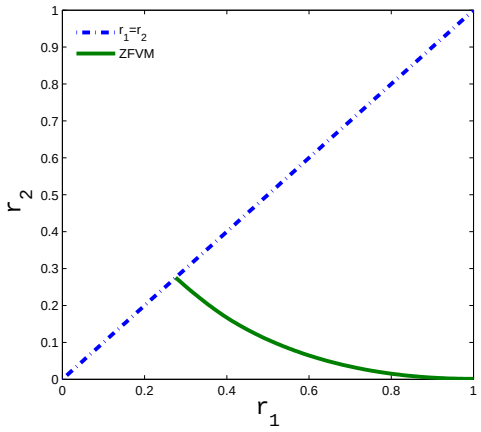


Figure 3: The admissible region for the Zienkiewicz 3-1 FVM.

the sufficient conditions for the uniform local-ellipticity. From Corollary 3.1, the uniform local-ellipticity of the Hermite quintic FVM does not hold, no matter how regular the shape of the triangles in the triangulation. For the Zienkiewicz 3-1 FVM, if the smallest angle in each triangular element of the triangulation is greater than 17.75°, then the family of the discrete bilinear forms is uniformly local-elliptic [7].

5.3. The Hybrid FVM

We have compared the admissible regions of the hybrid Zienkiewicz FVM and the hybrid quintic FVM. In Fig. 4, the curve labeled 'Hybrid Zienkiewicz FVM' is the lower boundary curve $\Gamma_{\mathbf{H}}$ of the admissible region of the hybrid Zienkiewicz FVM, and the curve labeled 'Hybrid Quintic FVM' is the lower boundary curve $\Gamma_{\mathbf{H}}$ of the admissible region of the hybrid quintic FVM. The admissible region of the hybrid quintic FVM is much smaller than that of the hybrid Zienkiewicz FVM, which according to Proposition 1.1 implies that the geometric restrictions for the uniform local-ellipticity of the hybrid quintic FVM is much stronger than those for the hybrid Zienkiewicz FVM. We can see this more clearly from the sufficient conditions for the uniform local-ellipticity. A sufficient condition for the hybrid Zienkiewicz FVM given in Ref. [8] is listed in Table 2, for comparison with the sufficient condition for the uniform local-ellipticity of the hybrid quintic FVM — cf. Theorem 4.1. Thus for each triangular element in the triangulation, if the smallest angle is greater than the value listed in Table 2, then the family of the corresponding bilinear forms is uniformly local-elliptic. From Table 2, we clearly see that the geometric requirements on the shapes of the triangles

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Table 2: Sufficient conditions for the hybrid Zienkiewicz and quintic FVMs.

Hybrid Zienkiewicz FVM	Hybrid Quintic FVM
37.70°	57.53°

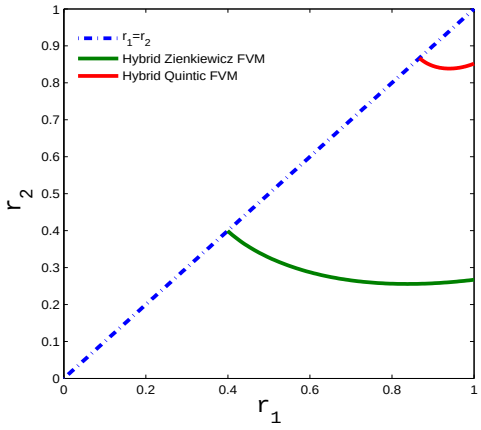


Figure 4: The admissible regions for the hybrid FVMs.

in the triangulation to guarantee the uniform local-ellipticity are much stronger for the hybrid quintic FVM than those for the hybrid Zienkiewicz FVM.

6. Conclusion

The order of the FVM has an obvious effect on the uniform local-ellipticity of its corresponding family of discrete bilinear forms. Thus for the Lagrange, Hermite and hybrid FVM considered, our numerical results show that the respective higher order FVM has a smaller admissible region — and accordingly, it needs stronger geometric restrictions to guarantee uniform local-ellipticity.

Appendix

In this Appendix, we present the matrices $\mathbf{A}_i$, $i = 0, 1, 2$ for the Hermite quintic FVM and the hybrid quintic FVM. Let us first introduce some notation. For a triangle K, we use $\nabla\mathscr{F}_K$ to denote the Jacobian matrix of the affine mapping $\mathscr{F}_K$ and $\nabla\mathscr{F}_K^{-1}$ to denote its inverse matrix, and write

$$\mathbf{M}_K := \left|\det(\nabla\mathscr{F}_K)\right| \cdot \nabla\mathscr{F}_K^{-1}(\nabla\mathscr{F}_K^{-1})^T.$$

According to Lemmas 3.1 and 3.3 of [7], we have

$$\mathbf{M}_K := \alpha_K \begin{bmatrix} 2r_{1,K} & 1-r_{1,K}-r_{2,K} \\ 1-r_{1,K}-r_{2,K} & 2r_{2,K} \end{bmatrix}. \tag{A.1}$$

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For each control volume $K_i^* \in \hat{\mathcal{T}}^*$, $i = 1,2,3$, let $l_i := \partial K_i^* \cap \text{int}\hat{K}$; and for each triangle K, on changing variables from K to $\hat{K}$ we have for each $\varphi \in \mathbb{U}_T$ and each $\psi \in \mathbb{V}_{T^*}$ that

$$a_K(\varphi,\psi) = \sum_{i=1}^{3}\left(\int_{K_i^*} (\nabla\hat{\varphi})^T \mathbf{M}_K \nabla\hat{\psi}\, d\hat{x} - \int_{l_i} \hat{\psi}(\nabla\hat{\varphi})^T \mathbf{M}_K \hat{\mathbf{n}}\, d\hat{s}\right), \tag{A.2}$$

where $\hat{\varphi} = \varphi \circ \mathscr{F}_K, \hat{\psi} = \psi \circ \mathscr{F}_K$ and $\hat{\mathbf{n}} := (n_1, n_2)^T$ is the outward unit normal on l_i. We introduce three bilinear forms on $\hat{K}$ for any $x := (x_1, x_2) \in \hat{K}$ and all $\hat{\varphi} \in \mathbb{U}_{\hat{K}}$ and $\hat{\psi} \in \mathbb{V}_{\hat{T}^*}$:

$$a_0(\hat{\varphi},\hat{\psi}) := \sum_{i=1}^{3}\left(\int_{K_i^*}\left(\frac{\partial\hat{\varphi}}{\partial x_1}\frac{\partial\hat{\psi}}{\partial x_2} + \frac{\partial\hat{\varphi}}{\partial x_2}\frac{\partial\hat{\psi}}{\partial x_1}\right)dx - \int_{l_i}\left(\frac{\partial\hat{\varphi}}{\partial x_1}\hat{\psi}n_2 + \frac{\partial\hat{\varphi}}{\partial x_2}\hat{\psi}n_1\right)ds\right),$$

$$a_1(\hat{\varphi},\hat{\psi}) := \sum_{i=1}^{3}\left(\int_{K_i^*}\left(2\frac{\partial\hat{\varphi}}{\partial x_1}\frac{\partial\hat{\psi}}{\partial x_1} - \frac{\partial\hat{\varphi}}{\partial x_1}\frac{\partial\hat{\psi}}{\partial x_2} - \frac{\partial\hat{\varphi}}{\partial x_2}\frac{\partial\hat{\psi}}{\partial x_1}\right)dx - \int_{l_i}\left(2\frac{\partial\hat{\varphi}}{\partial x_1}\hat{\psi}n_1 - \frac{\partial\hat{\varphi}}{\partial x_1}\hat{\psi}n_2 - \frac{\partial\hat{\varphi}}{\partial x_2}\hat{\psi}n_1\right)ds\right),$$

$$a_2(\hat{\varphi},\hat{\psi}) := \sum_{i=1}^{3}\left(\int_{K_i^*}\left(2\frac{\partial\hat{\varphi}}{\partial x_2}\frac{\partial\hat{\psi}}{\partial x_2} - \frac{\partial\hat{\varphi}}{\partial x_1}\frac{\partial\hat{\psi}}{\partial x_2} - \frac{\partial\hat{\varphi}}{\partial x_2}\frac{\partial\hat{\psi}}{\partial x_1}\right)dx - \int_{l_i}\left(2\frac{\partial\hat{\varphi}}{\partial x_2}\hat{\psi}n_2 - \frac{\partial\hat{\varphi}}{\partial x_1}\hat{\psi}n_2 - \frac{\partial\hat{\varphi}}{\partial x_2}\hat{\psi}n_1\right)ds\right).$$

From Eqs. (A.1) and (A.2), for all $\varphi \in \mathbb{U}_T$ and each $\psi \in \mathbb{V}_{T^*}$ we have

$$a_K(\varphi,\psi) = \alpha_K\left(a_0(\hat{\varphi},\hat{\psi}) + r_{1,K}a_1(\hat{\varphi},\hat{\psi}) + r_{2,K}a_2(\hat{\varphi},\hat{\psi})\right), \tag{A.3}$$

where $\hat{\varphi} = \varphi \circ \mathscr{F}_K$ and $\hat{\psi} = \psi \circ \mathscr{F}_K$. For the bases of the trial and test spaces on $\hat{K}$, we define three matrices

$$\mathbf{B}_0 := \left[a_0(\hat{\varphi}_j,\hat{\psi}_i)\colon i,j \in \mathbb{N}_{18}\right], \qquad \mathbf{B}_1 := \left[a_1(\hat{\varphi}_j,\hat{\psi}_i)\colon i,j \in \mathbb{N}_{18}\right],$$
$$\mathbf{B}_2 := \left[a_2(\hat{\varphi}_j,\hat{\psi}_i)\colon i,j \in \mathbb{N}_{18}\right],$$

so from Eqs. (2.5) and (A.3) we obtain

$$\mathbf{A}_0 = (\mathbf{B}_0 + \mathbf{B}_0^T)/2, \qquad \mathbf{A}_1 = (\mathbf{B}_1 + \mathbf{B}_1^T)/2, \qquad \mathbf{A}_2 = (\mathbf{B}_2 + \mathbf{B}_2^T)/2\,. \tag{A.4}$$

For an $m \times m$ matrix $\mathbf{A}$, we use $\mathscr{P}(\mathbf{A}, \{i_1, i_2, \cdots, i_m\})$ to denote the permutation of $\mathbf{A}$ with the permutation $(1,2,\cdots,m) \to (i_1, i_2, \cdots, i_m)$, so the entry (k,l) of $\mathscr{P}(\mathbf{A}, \{i_1, i_2, \cdots, i_m\})$ is equal to the entry (i_k, i_l) of $\mathbf{A}$ and the matrices $\mathbf{A}_i$, $i = 2,3$ are permutations of $\mathbf{A}_0$. Let $\mathscr{F}_{\hat{K}}$ be the affine mapping from $\hat{K}$ to itself with $\mathscr{F}_{\hat{K}}(\hat{P}_1) = \hat{P}_2$, $\mathscr{F}_{\hat{K}}(\hat{P}_2) = \hat{P}_3$ and $\mathscr{F}_{\hat{K}}(\hat{P}_3) = \hat{P}_1$. For each $m \in \mathbb{N}_{18}$, let $\hat{\varphi}_{m,\mathscr{F}} := \hat{\varphi}_m \circ \mathscr{F}_{\hat{K}}$ and $\hat{\psi}_{m,\mathscr{F}} := \hat{\psi}_m \circ \mathscr{F}_{\hat{K}}$. By a change of variables, we find for all $m, l \in \mathbb{N}_{18}$ that

$$a_0(\hat{\varphi}_m,\hat{\psi}_l) = a_1(\hat{\varphi}_{m,\mathscr{F}},\hat{\psi}_{l,\mathscr{F}}). \tag{A.5}$$

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Since the bases $\hat{\varphi}_m$, $m \in \mathbb{N}_{18}$ and $\hat{\psi}_l$, $l \in \mathbb{N}_{18}$ are affine invariant related to $\mathscr{F}_{\hat{K}}$ and they share the same freedom set, we conclude that there is a permutation matrix $\mathbf{Q}$ such that

$$\begin{aligned}\left[\hat{\varphi}_{1,\mathscr{F}}, \hat{\varphi}_{2,\mathscr{F}}, \cdots, \hat{\varphi}_{18,\mathscr{F}}\right] &= [\hat{\varphi}_1, \hat{\varphi}_2, \cdots, \hat{\varphi}_{18}]\mathbf{Q}\,,\\ \left[\hat{\psi}_{1,\mathscr{F}}, \hat{\psi}_{2,\mathscr{F}}, \cdots, \hat{\psi}_{18,\mathscr{F}}\right] &= \left[\hat{\psi}_1, \hat{\psi}_2, \cdots, \hat{\psi}_{18}\right]\mathbf{Q}\,.\end{aligned} \tag{A.6}$$

From Eqs. A.4), (A.5) and (A.6), we get

$$\mathbf{Q}^T\mathbf{A}_0\mathbf{Q} = \mathbf{A}_1$$

— i.e. $\mathbf{A}_1$ is a permutation of $\mathbf{A}_0$. Similarly, we can show that $\mathbf{A}_2$ is a permutation of $\mathbf{A}_0$.

Let $\mathbf{C}_1$ be a $m \times n_1$ matrix and $\mathbf{C}_2$ be a $m \times n_2$ matrix, where m, n_1 and n_2 are three positive integers. We use $\mathbf{C} := [\mathbf{C}_1\ \mathbf{C}_2]$ to denote a new $m \times (n_1 + n_2)$ matrix, where the first n_1 columns are the columns of $\mathbf{C}_1$ and the last columns n_2 are the columns of $\mathbf{C}_2$. Substituting the bases $\Phi_{\hat{K}} = \{\hat{\varphi}_i, i \in \mathbb{N}_{18}\}$ and $\Psi_{\hat{T}^*} := \{\hat{\psi}_i, i \in \mathbb{N}_{18}\}$ of the Hermite quintic FVM derived in Section 3 into (A.4), we get the matrices $\mathbf{A}_i$, $i = 0, 1, 2$ for the Hermite quintic FVM:

$$\mathbf{A}_0 = \frac{1}{78382080}\left[\mathbf{A}_{0,1}\ \mathbf{A}_{0,2}\right],$$

$$\mathbf{A}_1 = \mathscr{P}\left(\mathbf{A}_0, \{3,1,2,8,9,4,5,6,7,16,17,18,10,11,12,13,14,15\}\right),$$

$$\mathbf{A}_2 = \mathscr{P}\left(\mathbf{A}_0, \{2,3,1,6,7,8,9,4,5,13,14,15,16,17,18,10,11,12\}\right),$$

where

$$\mathbf{A}_{0,1} := \begin{bmatrix}
121201920 & -60600960 & -60600960 & 15513120 & 15513120 & -17055360 & -5896800 & -5896800\\
-60600960 & 10704960 & 49896000 & -15894144 & 381024 & 6513696 & -1161216 & 7058016\\
-60600960 & 49896000 & 10704960 & 381024 & -15894144 & 10541664 & 7058016 & -1161216\\
19771920 & -17574480 & -2197440 & 4178664 & 5680080 & -4581360 & -1511496 & 1396080\\
19771920 & -2197440 & -17574480 & 5680080 & 4178664 & -2494800 & 1396080 & -1511496\\
-23637600 & 10866240 & 12771360 & -5841360 & -2130912 & 2848608 & -4518864 & 588672\\
-1638000 & -4964400 & 6602400 & -166824 & 2350656 & -5651856 & 10273032 & 3346560\\
-1638000 & 6602400 & -4964400 & 2350656 & -166824 & -45360 & 3346560 & 10273032\\
-23637600 & 12771360 & 10866240 & -2130912 & -5841360 & 3102624 & 588672 & -4518864\\
3208440 & -3034680 & -173760 & 689988 & 805872 & -718992 & -267636 & 188016\\
669120 & -334560 & -334560 & 774096 & 774096 & -606816 & 439536 & 439536\\
3208440 & -173760 & -3034680 & 805872 & 689988 & -274896 & 188016 & -267636\\
-3575040 & 1625520 & 1949520 & -840264 & -467448 & 594936 & -884424 & -114168\\
-3089280 & 1544640 & 1544640 & -707664 & 44016 & -816336 & 1588656 & 836976\\
858360 & -1535640 & 677280 & 278292 & 487056 & -825696 & 1160124 & 379824\\
858360 & 677280 & -1535640 & 487056 & 278292 & -41184 & 379824 & 1160124\\
-3089280 & 1544640 & 1544640 & 44016 & -707664 & -64656 & 836976 & 1588656\\
-3575040 & 1949520 & 1625520 & -467448 & -840264 & 558216 & -114168 & -884424
\end{bmatrix},$$

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and

$$\mathbf{A}_{0,2} := \begin{bmatrix}
-17055360 & 1406160 & -604800 & 1406160 & -846720 & -2600640 & 438480 & 438480 & -2600640 & -846720 \\
10541664 & -1469664 & 302400 & 63504 & 269136 & 1300320 & -677376 & 238896 & 1300320 & 577584 \\
6513696 & 63504 & 302400 & -1469664 & 577584 & 1300320 & 238896 & -677376 & 1300320 & 269136 \\
-2494800 & 344148 & 284256 & 408408 & -225288 & -635040 & 130620 & 256872 & -254016 & -185976 \\
-4581360 & 408408 & 284256 & 344148 & -185976 & -254016 & 256872 & 130620 & -635040 & -225288 \\
3102624 & -435288 & -411264 & -46704 & 288792 & -332640 & -357000 & 16464 & 48384 & 264600 \\
-45360 & -83748 & 586656 & 83328 & -633528 & 1046304 & 479892 & 122304 & 665280 & -237384 \\
-5651856 & 83328 & 586656 & -83748 & -237384 & 665280 & 122304 & 479892 & 1046304 & -633528 \\
2848608 & -46704 & -411264 & -435288 & 264600 & 48384 & 16464 & -357000 & -332640 & 288792 \\
-274896 & 53006 & 66192 & 48632 & -34152 & -95088 & 19234 & 29656 & -19488 & -24648 \\
-606816 & 73576 & 79968 & 73576 & -45696 & -52080 & 69896 & 69896 & -52080 & -45696 \\
-718992 & 48632 & 66192 & 53006 & -24648 & -19488 & 29656 & 19234 & -95088 & -34152 \\
558216 & -54364 & -93072 & -13396 & 75468 & -113736 & -50804 & -4076 & -38136 & 61788 \\
-64656 & -62504 & 32592 & 7336 & -136056 & 233184 & 31592 & 25256 & 169680 & -66216 \\
-41184 & 13118 & 111552 & 16664 & -73104 & 77280 & 92386 & 19288 & 65184 & -28032 \\
-825696 & 16664 & 111552 & 13118 & -28032 & 65184 & 19288 & 92386 & 77280 & -73104 \\
-816336 & 7336 & 32592 & -62504 & -66216 & 169680 & 25256 & 31592 & 233184 & -136056 \\
594936 & -13396 & -93072 & -54364 & 61788 & -38136 & -4076 & -50804 & -113736 & 75468
\end{bmatrix}.$$

We next present the matrices for the hybrid quintic FVM. Thus on substituting the bases $\Phi_{\hat{K}} = \{\hat{\varphi}_i, i \in \mathbb{N}_{18}\}$ and $\Psi_{\hat{T}^*} := \{\hat{\psi}_i, i \in \mathbb{N}_{18}\}$ derived in Section 4 for the hybrid quintic FVM into Eqs. (A.4), we get the matrices $\mathbf{A}_i$, $i = 0, 1, 2$ for the hybrid quintic FVM

$$\mathbf{A}_0 = \frac{1}{3628800}\left[\mathbf{A}_{0,1}\ \mathbf{A}_{0,2}\right],$$

$$\mathbf{A}_1 = \mathscr{P}\left(\mathbf{A}_0, \{3,1,2,8,9,4,5,6,7,16,17,18,10,11,12,13,14,15\}\right),$$

$$\mathbf{A}_2 = \mathscr{P}\left(\mathbf{A}_0, \{2,3,1,6,7,8,9,4,5,13,14,15,16,17,18,10,11,12\}\right),$$

where

$$\mathbf{A}_{0,1} := \begin{bmatrix}
5611200 & -2805600 & -2805600 & 718200 & 718200 & -789600 & -273000 & -273000 & -789600 \\
-2805600 & 495600 & 2310000 & -735840 & 17640 & 301560 & -53760 & 326760 & 488040 \\
-2805600 & 2310000 & 495600 & 17640 & -735840 & 488040 & 326760 & -53760 & 301560 \\
496800 & -503280 & 6480 & 93744 & 152496 & -155736 & 29376 & 82944 & -79704 \\
496800 & 6480 & -503280 & 152496 & 93744 & -79704 & 82944 & 29376 & -155736 \\
-756000 & 360720 & 395280 & -155736 & -79704 & 119664 & -176904 & -28296 & 100656 \\
64800 & -252720 & 187920 & 29376 & 82944 & -176904 & 288144 & 122256 & -28296 \\
64800 & 187920 & -252720 & 82944 & 29376 & -28296 & 122256 & 288144 & -176904 \\
-756000 & 395280 & 360720 & -79704 & -155736 & 100656 & -28296 & -176904 & 119664 \\
43200 & -44280 & 1080 & 6984 & 9936 & -10476 & 576 & 3384 & -2844 \\
-14400 & 7200 & 7200 & 9360 & 9360 & -12960 & 16560 & 16560 & -12960 \\
43200 & 1080 & -44280 & 9936 & 6984 & -2844 & 3384 & 576 & -10476 \\
-50400 & 24120 & 26280 & -10476 & -8244 & 13104 & -19044 & -9756 & 10656 \\
-57600 & 28800 & 28800 & -7920 & 720 & -15120 & 29520 & 20880 & -6480 \\
28800 & -34920 & 6120 & 6696 & 8064 & -11124 & 11664 & 4536 & -1476 \\
28800 & 6120 & -34920 & 8064 & 6696 & -1476 & 4536 & 11664 & -11124 \\
-57600 & 28800 & 28800 & 720 & -7920 & -6480 & 20880 & 29520 & -15120 \\
-50400 & 26280 & 24120 & -8244 & -10476 & 10656 & -9756 & -19044 & 13104
\end{bmatrix},$$

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and

$$\mathbf{A}_{0,2} := \begin{bmatrix}
65100 & -28000 & 65100 & -39200 & -120400 & 20300 & 20300 & -120400 & -39200 \\
-68040 & 14000 & 2940 & 12460 & 60200 & -31360 & 11060 & 60200 & 26740 \\
2940 & 14000 & -68040 & 26740 & 60200 & 11060 & -31360 & 60200 & 12460 \\
6984 & 9360 & 9936 & -10476 & -7920 & 6696 & 8064 & 720 & -8244 \\
9936 & 9360 & 6984 & -8244 & 720 & 8064 & 6696 & -7920 & -10476 \\
-10476 & -12960 & -2844 & 13104 & -15120 & -11124 & -1476 & -6480 & 10656 \\
576 & 16560 & 3384 & -19044 & 29520 & 11664 & 4536 & 20880 & -9756 \\
3384 & 16560 & 576 & -9756 & 20880 & 4536 & 11664 & 29520 & -19044 \\
-2844 & -12960 & -10476 & 10656 & -6480 & -1476 & -11124 & -15120 & 13104 \\
444 & 840 & 516 & -606 & -600 & 396 & 324 & 120 & -354 \\
840 & 1440 & 840 & -1440 & 720 & 1320 & 1320 & 720 & -1440 \\
516 & 840 & 444 & -354 & 120 & 324 & 396 & -600 & -606 \\
-606 & -1440 & -354 & 1764 & -2520 & -834 & -366 & -1800 & 1476 \\
-600 & 720 & 120 & -2520 & 4320 & 600 & 600 & 3600 & -1800 \\
396 & 1320 & 324 & -834 & 600 & 924 & 276 & 600 & -366 \\
324 & 1320 & 396 & -366 & 600 & 276 & 924 & 600 & -834 \\
120 & 720 & -600 & -1800 & 3600 & 600 & 600 & 4320 & -2520 \\
-354 & -1440 & -606 & 1476 & -1800 & -366 & -834 & -2520 & 1764
\end{bmatrix}.$$

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