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THE OPTIMAL CONTROL OF A GENERAL TANDEM QUEUE

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We consider a scheduling problem with two interconnected queues and two flexible servers. It is assumed that all jobs are present at the beginning and that there are no further arrivals to the system at any time. For each job, there are waiting costs per unit of time until the job leaves the system. A job of queue 1, after being served, joins queue 2 with probability p and leaves the system with probability $1 - p$. The objective is how to allocate the two servers to the queues such that the expected total holding costs until the system is empty are minimized. We give a sufficient condition such that for any number of jobs in queue 1 and queue 2, it is optimal to allocate both servers to queue 1 (resp. queue 2).

1. INTRODUCTION

We consider a scheduling problem with two interconnected queues, queue 1 and queue 2, and two parallel servers of the same kind. Each of the two servers can serve both types of job: jobs of queue 1 and jobs of queue 2. Initially, all jobs are present and no further arrivals to the system take place at any time. Further, we assume that a job of queue 1 (after being processed) joins to queue 2 with probability p and leaves the system with probability $1 - p$, whereas a job of queue 2 leaves the system (after being processed). For both servers, the service time for a

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job of queue 1 (resp. queue 2) is exponentially distributed with mean $1/\mu_1$ (resp. $1/\mu_2$). Moreover, for each job of queue 1 (resp. queue 2), there are holding costs c_1 (resp. c_2) per unit of time until the job leaves the system. We will distinguish between the *preemptive* and the *nonpreemptive* case. If there are two jobs processed and one job is finished, then, in the preemptive case, the other job is preempted and *both* servers can be allocated to the remaining jobs, whereas in the nonpreemptive case, the other job is not preempted and only *one* server can be allocated to the remaining jobs. Assuming that initially there are i (resp. j) jobs in queue 1 (resp. queue 2), our objective is to find an optimal strategy of how to allocate the two servers to the queues such that the expected total holding costs until the system is cleared are minimized. In the literature, such a system is called a *clearing system* (i.e., the target is to minimize the expected value),

$$\mathrm{E}\left[\int_0^\infty (c_1 Q_1(t) + c_2 Q_2(t))\, dt\right],$$

where the random variable $Q_1(t)$ (resp. $Q_2(t)$) denotes the number of jobs at time t in queue 1 (resp. queue 2) and $Q_1(0) = i$ ($Q_2(0) = j$).

The starting point of our considerations were the papers of Ahn, Duenyas, and Zhang [1] and of Pandelis and Teneketzis [4]. The structure of the optimal control for the case $p = 1$ is solved partially (only the pure strategies) by Ahn et al. in [1] and completely by the authors in [6]. In [2], Chang, Nelson, and Pinedo consider the case $p = 0$ for m parallel servers. They show that the optimal policy among all preemptive policies is of threshold type. In [4], Pandelis and Teneketzis consider the discounted holding cost problem with m parallel servers and more general service times (not necessarily exponentially distributed). They give a sufficient condition on the parameters c_1, c_2, and p, such that it is always optimal to process jobs of queue 1 (if possible).

The article is organized as follows. In Section 2 we introduce the so-called dynamic equations and define the optimal control matrix. The main results are given in Section 3. In Section 4 we prove some structural lemmas and give the proofs of the main results. Finally, in Section 5 our results are compared to similar results given in the literature.

2. MODEL

First, let us note that, if not stated otherwise, we will always assume the preemptive case. Let (i, j, X) be the state of the system, where i and j are the number of jobs in queue 1 and queue 2, respectively, and let X denote the state of the two servers. There are four possible states for X, namely 11, 22, 12, and R, where 11 means that we take two jobs from queue 1, etc. and R means that one server has finished processing a job, the other job is preempted, and one has to decide whether to choose 11, 22, or 12. Further, we define $V(i, j, X)$ as the minimal expected total holding costs until the system is cleared. The function $V(i, j, X)$ satisfies the following dynamic equations [5, §3.1]:

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$$
\begin{aligned}
V(i,j,11) &= \frac{ic_1 + jc_2}{2\mu_1} + pV(i-1,j+1,R) \\
&\quad + (1-p)V(i-1,j,R), \qquad i \geq 2, j \geq 0, \\
V(i,j,22) &= \frac{ic_1 + jc_2}{2\mu_2} + V(i,j-1,R), \qquad i \geq 0, j \geq 2, \\
V(i,j,12) &= \frac{ic_1 + jc_2}{\mu_1 + \mu_2} + \frac{\mu_1}{\mu_1 + \mu_2} \\
&\quad \times [\, pV(i-1,j+1,R) + (1-p)V(i-1,j,R)] \\
&\quad + \frac{\mu_2}{\mu_1 + \mu_2} V(i,j-1,R), \qquad i,j \geq 1,
\end{aligned}
\tag{1}
$$

and

$$
\begin{aligned}
V(i,j,R) &= \min\{V(i,j,11), V(i,j,22), V(i,j,12)\}, \qquad i,j \geq 2, \\
V(i,1,R) &= \min\{V(i,1,11), V(i,1,12)\}, \qquad i \geq 2, \\
V(1,j,R) &= \min\{V(1,j,22), V(1,j,12)\}, \qquad j \geq 2.
\end{aligned}
\tag{2}
$$

Furthermore, the following initial conditions are obviously valid:

$$
\begin{aligned}
V(1,1,R) &= V(1,1,12), \\
V(0,j,R) &= V(0,j,22), \qquad j \geq 2, \\
V(i,0,R) &= V(i,0,11), \qquad i \geq 2, \\
V(1,0,R) &= \frac{c_1}{\mu_1} + p\,\frac{c_2}{\mu_2}, \\
V(0,1,R) &= \frac{c_2}{\mu_2}.
\end{aligned}
\tag{3}
$$

In order to characterize the optimal strategy, we introduce the *control matrix* $\mathcal{C}(i,j)_{i,j=0}^{\infty}$, which gives for each $(i,j) \in \mathbb{N}_0 \times \mathbb{N}_0$ the optimal control of the queue. Thus, if there are $i \geq 0$ jobs in queue 1 and $j \geq 0$ jobs in queue 2, $i + j \geq 2$, we have at least one of the following three possibilities for the optimal control:

- Take two jobs of queue 1, denoted by $\mathcal{C}(i,j) = 11$.
- Take two jobs of queue 2, denoted by $\mathcal{C}(i,j) = 22$.
- Take one job of queue 1 and one job of queue 2, denoted by $\mathcal{C}(i,j) = 12$.

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If we have $\mathcal{C}(i,j)=11$, this means that $V(i,j,11)\le \min\{V(i,j,12),V(i,j,22)\}$. Note that if, for example, $V(i,j,11)=V(i,j,22)$, then in state (i,j) we have two possibilities for the optimal strategy, and in this case, $\mathcal{C}(i,j)$ is not uniquely determined.

3. MAIN RESULTS

In this section we present the main results concerning the optimal control of the system. As stated in Section 1, we will give sufficient conditions on the parameters c_1, c_2, μ_1, μ_2, and p such that in each state (i,j), it is always optimal to allocate both servers to queue 1 (resp. queue 2). In order to state the theorems, we distinguish between the two cases

$$\mu_2 > \mu_1(1-p) \quad \text{and} \quad \mu_2 < \mu_1(1-p). \tag{4}$$

With respect to these cases, we define two parameter regions A and B for c_1/c_2:

$$A := \begin{cases} \left(0, \dfrac{p\mu_1+2\mu_2}{\mu_1+\mu_2}\right) & \text{if } \mu_2 > \mu_1(1-p) \\ \left(0, p+\dfrac{\mu_2}{\mu_1}\right) & \text{if } \mu_2 < \mu_1(1-p), \end{cases} \tag{5}$$

$$B := \begin{cases} \left(p+\dfrac{\mu_2}{\mu_1}, \infty\right) & \text{if } \mu_2 > \mu_1(1-p) \\ \left(\dfrac{1}{2}\left(1+\dfrac{\mu_2}{\mu_1}+\dfrac{p^2\mu_1+2p\mu_2}{\mu_1+\mu_2}\right), \infty\right) & \text{if } \mu_2 < \mu_1(1-p). \end{cases} \tag{6}$$

If $\mu_2=\mu_1(1-p)$, we have

$$\frac{p\mu_1+2\mu_2}{\mu_1+\mu_2} = p+\frac{\mu_2}{\mu_1} = \frac{1}{2}\left(1+\frac{\mu_2}{\mu_1}+\frac{p^2\mu_1+2p\mu_2}{\mu_1+\mu_2}\right) = 1.$$

Hence, in that case, we define $A := (0,1)$ and $B := (1,\infty)$. Note that the sets A and B are always disjoint (see Lemma 6).

THEOREM 1: *Let $c_1,c_2,\mu_1,\mu_2>0$, $p\in[0,1]$. For $c_1/c_2\in A$, in the preemptive model it is always optimal to allocate both servers to queue 2 (if possible) (i.e., $\mathcal{C}(i,j)=22$ for $i\ge 1$, $j\ge 2$, and $\mathcal{C}(i,1)=12$ for $i\ge 2$) (see Fig. 1a).*

THEOREM 2: *Let $c_1,c_2,\mu_1,\mu_2>0$, $p\in[0,1]$. For $c_1/c_2\in B$, in the preemptive model it is always optimal to allocate both servers to queue 1 (if possible) (i.e., $\mathcal{C}(i,j)=11$ for $i\ge 2$, $j\ge 1$, and $\mathcal{C}(1,j)=12$ for $j\ge 2$) (see Fig. 1b).*

In the case $\mu_2=\mu_1(1-p)$, there are only pure strategies; more precisely, the following statement holds.

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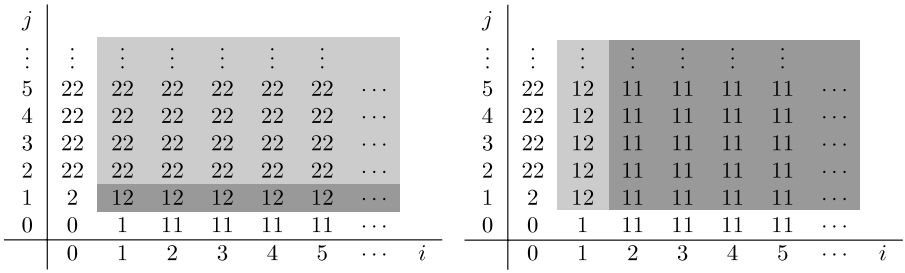


FIGURE 1. The control matrix $\mathcal{C}(i,j)_{i,j=0}^{\infty}$ for regions A and B.

COROLLARY 1: *Let* $c_1, c_2, \mu_1, \mu_2 > 0, p \in [0,1]$, *and assume that* $\mu_2 = \mu_1(1-p)$ *holds; then for* $c_1/c_2 < 1$ $(c_1/c_2 > 1)$, *it is always optimal to allocate both servers to queue 2 (queue 1) (if possible).*

Note that the case $\mu_2 = \mu_1(1-p)$ and $c_1 = c_2$ is a kind of equilibrium, in which for each state (i,j) all possible controls give the same expected holding costs. Moreover, note that $\mu_2 = \mu_1(1-p)$ is possible if and only if $p < 1$.

Theorems 1 and 2 give the optimal strategy in the *preemptive* model. Given *any* strategy, the expected holding costs in each state (i,j), until the system is empty, in the *preemptive* model are less than or equal to those in the *nonpreemptive* model because of the memoryless of the exponential distribution. Therefore, if the optimal control of the *preemptive* model can be satisfied in the *nonpreemptive* one along all paths to the state $(0,0)$, then the optimal control is the same for the *nonpreemptive model*. Since this is true in the cases of Theorems 1 and 2 (see Figs. 1a and 1b), we arrive at the following corollary for the *nonpreemptive* model.

COROLLARY 2: *Let* $c_1, c_2, \mu_1, \mu_2 > 0$, $p \in [0,1]$. *For* $c_1/c_2 \in A \cup B$, *the optimal control in the nonpreemptive model is the same as in the preemptive model.*

Remark: Now the sufficient conditions given in Theorems 1 and 2 will be explained. We will only consider the case $\mu_2 > \mu_1(1-p)$; the arguments for the other case are analogous.

(a) If

$$\mu_2 > \mu_1(1-p) \quad \text{and} \quad \frac{c_1}{c_2} > p + \frac{\mu_2}{\mu_1}, \tag{7}$$

then, by Theorem 1, it is always optimal to allocate both servers to queue 1. Since (7) is equivalent to

$$\frac{1}{\mu_1} + \frac{p}{\mu_2} > \frac{1}{\mu_2} \quad \text{and} \quad \frac{c_1}{\mu_2} > c_2\left(\frac{1}{\mu_1} + \frac{p}{\mu_2}\right),$$

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these conditions can be interpreted as follows: If the LEPT (largest expected processing time) rule and the $c\mu$ rule suggest the same strategy (allocation to queue 1), then this strategy is optimal.

(b) If

$$\mu_2 > \mu_1(1-p) \quad \text{and} \quad \frac{c_1}{c_2} < p + \frac{\mu_2}{\mu_1}, \tag{8}$$

then the LEPT rule and the $c\mu$ rule suggest *different* strategies. Thus, a somewhat "stronger" condition for c_1/c_2 given by

$$\frac{c_1}{c_2} < \frac{p\mu_1 + 2\mu_2}{\mu_1 + \mu_2} \quad \left(< p + \frac{\mu_2}{\mu_1} \right)$$

is needed to guarantee that it is always optimal to allocate both servers to queue 2. Note that the last condition arises from comparing the two possible strategies in the state $X = (1,2)$.

4. PROOF OF THE MAIN RESULTS

To proceed toward the proofs of our main theorems, we have to compute some costs and characterize the structure of the control matrix. The following lemma states that for every state (i,j), $i,j \geq 2$, it is sufficient to consider only the two strategies $X = 11$ and $X = 22$.

LEMMA 1: *For $i,j \geq 2$,*

$$V(i,j,12) \geq \min\{V(i,j,11), V(i,j,22)\}. \tag{9}$$

PROOF: Let $i,j \geq 2$. Then the dynamic equations (1) give

$$V(i,j,12) = \frac{\mu_1}{\mu_1 + \mu_2} V(i,j,11) + \frac{\mu_2}{\mu_1 + \mu_2} V(i,j,22).$$

Thus, $V(i,j,12)$ is a convex combination of $V(i,j,11)$ and $V(i,j,22)$, which gives (9). ■

LEMMA 2:

(a) For $j \geq 1$,

$$V(0,j,22) = \frac{c_2(2 + j + j^2)}{4\mu_2}. \tag{10}$$

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(b) *Let $j \geq 1$. If $\mathcal{C}(1,k) = 12$ for $k \in \{1,2,\ldots,j-1\}$, then $V(1,j,12)$ is given explicitly by*

$$V(1,j,12) = \frac{1}{4\mu_1^2\mu_2}\Bigg[4c_1\mu_1\mu_2 + c_2\Bigg((2+j+j^2)\mu_1^2 + 2j\mu_1\mu_2 - 2\mu_2^2 + 2\left(\frac{\mu_2}{\mu_1+\mu_2}\right)^j(\mu_1+\mu_2)(\mu_2-\mu_1(1-p)) + p(\mu_1^2(2+2j) - 2\mu_1\mu_2)\Bigg)\Bigg]. \quad \textbf{(11)}$$

PROOF: Straightforward computation using the dynamic equations (1). ■

LEMMA 3: *Let $i \geq 1, j \geq 2$ and $c_1/c_2 < p + \mu_2/\mu_1$. If $\mathcal{C}(i,j) = 22$ and $\mathcal{C}(i,j+1) = 22$, then $\mathcal{C}(i+1,j) = 22$.*

PROOF: Let $\mathcal{C}(i,j) = \mathcal{C}(i,j+1) = 22$; then by the dynamic equations (1) and (2),

$$\begin{aligned}
&V(i+1,j,22) - V(i+1,j,11)\\
&= \frac{(i+1)c_1 + jc_2}{2\mu_2} + V(i+1,j-1,R) - \frac{(i+1)c_1 + jc_2}{2\mu_1} - pV(i,j+1,22)\\
&\quad - (1-p)V(i,j,22)\\
&\leq \frac{(i+1)c_1 + jc_2}{2\mu_2} + V(i+1,j-1,11) - \frac{(i+1)c_1 + jc_2}{2\mu_1} - pV(i,j+1,22)\\
&\quad - (1-p)V(i,j,22)\\
&= \frac{(i+1)c_1 + jc_2}{2\mu_2} + \frac{(i+1)c_1 + (j-1)c_2}{2\mu_1} + pV(i,j,22)\\
&\quad + (1-p)V(i,j-1,R) - \frac{(i+1)c_1 + jc_2}{2\mu_1} - p\,\frac{ic_1 + (j+1)c_2}{2\mu_2}\\
&\quad - pV(i,j,22) - (1-p)\,\frac{ic_1 + jc_2}{2\mu_2} - (1-p)V(i,j-1,R)\\
&= \frac{c_1 - pc_2}{2\mu_2} - \frac{c_2}{2\mu_1} = \frac{c_2}{2\mu_2}\left(\frac{c_1}{c_2} - p - \frac{\mu_2}{\mu_1}\right) < 0.
\end{aligned}$$

■

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LEMMA 4: *Let $i \geq 2$, $j \geq 1$ and $c_1/c_2 > p + \mu_2/\mu_1$. If $\mathcal{C}(i,j) = 11$, then $\mathcal{C}(i,j+1) = 11$.*

PROOF: Let $\mathcal{C}(i,j) = 11$; then by the dynamic equations (1) and (2),

$$\begin{aligned}
&V(i,j+1,11) - V(i,j+1,22)\\
&\quad = \frac{ic_1 + (j+1)c_2}{2\mu_1} + pV(i-1,j+2,R) + (1-p)V(i-1,j+1,R)\\
&\qquad - \frac{ic_1 + (j+1)c_2}{2\mu_2} - V(i,j,R)\\
&\quad \leq \frac{ic_1 + (j+1)c_2}{2\mu_1} - \frac{ic_1 + (j+1)c_2}{2\mu_2} + pV(i-1,j+2,22)\\
&\qquad + (1-p)V(i-1,j+1,22) - V(i,j,11)\\
&\quad = \frac{ic_1 + (j+1)c_2}{2\mu_1} - \frac{ic_1 + (j+1)c_2}{2\mu_2} + p\,\frac{(i-1)c_1 + (j+2)c_2}{2\mu_2}\\
&\qquad + pV(i-1,j+1,R) + (1-p)\,\frac{(i-1)c_1 + (j+1)c_2}{2\mu_2}\\
&\qquad + (1-p)V(i-1,j,R)\\
&\qquad - \frac{ic_1 + jc_2}{2\mu_1} - pV(i-1,j+1,R) - (1-p)V(i-1,j,R)\\
&\quad = \frac{c_2}{2\mu_1} - \frac{ic_1 + (j+1)c_2}{2\mu_2} + \frac{(i-1)c_1 + (j+1)c_2}{2\mu_2} + p\,\frac{c_2}{2\mu_2}\\
&\quad = \frac{c_2}{2\mu_1} - \frac{c_1}{2\mu_2} + p\,\frac{c_2}{2\mu_2} = \frac{c_2}{2\mu_2}\left(p + \frac{\mu_2}{\mu_1} - \frac{c_1}{c_2}\right) < 0.
\end{aligned}$$

■

LEMMA 5: *For $\mu_1, \mu_2 > 0$, $p \in [0,1]$, we have*

$$\frac{p\mu_1 + 2\mu_2}{\mu_1 + \mu_2} \leq \frac{\mu_1 + \mu_2}{\mu_1(2-p)}. \tag{12}$$

PROOF: By simple computation, we obtain that (12) is equivalent to

$$2p\mu_1^2 + 4\mu_1\mu_2 - p^2\mu_1^2 - 2p\mu_1\mu_2 \leq \mu_1^2 + 2\mu_1\mu_2 + \mu_2^2,$$

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which is equivalent to

$$0 \le \mu_1^2(1-2p+p^2) - 2(1-p)\mu_1\mu_2 + \mu_2^2 = (\mu_1(1-p)-\mu_2)^2,$$

which gives the assertion. ■

Straightforward computation gives the following lemma (see Fig. 2, in sect. 5).

LEMMA 6: *If* $\mu_2 > \mu_1(1-p)$, *then*

$$\frac{p\mu_1+2\mu_2}{\mu_1+\mu_2} < \frac{1}{2}\left(1+\frac{\mu_2}{\mu_1}+\frac{p^2\mu_1+2p\mu_2}{\mu_1+\mu_2}\right) < p+\frac{\mu_2}{\mu_1};$$

if $\mu_2 < \mu_1(1-p)$, *then*

$$p+\frac{\mu_2}{\mu_1} < \frac{p\mu_1+2\mu_2}{\mu_1+\mu_2} < \frac{1}{2}\left(1+\frac{\mu_2}{\mu_1}+\frac{p^2\mu_1+2p\mu_2}{\mu_1+\mu_2}\right).$$

LEMMA 7: *For* $i=1$ *and* $j=2$, *we have*

$$\mathcal{C}(1,2) = \begin{cases} 12 & \text{if } \dfrac{c_1}{c_2} > \dfrac{p\mu_1+2\mu_2}{\mu_1+\mu_2} \\ 22 & \text{if } \dfrac{c_1}{c_2} < \dfrac{p\mu_1+2\mu_2}{\mu_1+\mu_2}. \end{cases}$$

For $c_1/c_2 = (p\mu_1+2\mu_2)/(\mu_1+\mu_2)$, we have $V(1,2,12) = V(1,2,22)$.

PROOF: By Lemma 2(b),

$$\begin{aligned} V(1,2,12) - V(1,2,22) &= V(1,2,12) - \frac{c_1+2c_2}{2\mu_2} - V(1,1,12) \\ &= \frac{c_2}{2\mu_2}\left(\frac{\mu_1 p+2\mu_2}{\mu_1+\mu_2} - \frac{c_1}{c_2}\right), \end{aligned}$$

which gives the assertion. ■

LEMMA 8: *For* $i=2$ *and* $j=1$, *we have*

$$\mathcal{C}(2,1) = \begin{cases} 11 & \text{if } \dfrac{c_1}{c_2} > \dfrac{1}{2}\left(1+\dfrac{\mu_2}{\mu_1}+\dfrac{p^2\mu_1+2p\mu_2}{\mu_1+\mu_2}\right) \\ 12 & \text{if } \dfrac{c_1}{c_2} < \dfrac{1}{2}\left(1+\dfrac{\mu_2}{\mu_1}+\dfrac{p^2\mu_1+2p\mu_2}{\mu_1+\mu_2}\right). \end{cases}$$

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For

$$c_1/c_2 = \frac{1}{2}\left(1 + \frac{\mu_2}{\mu_1} + \frac{p^2\mu_1 + 2p\mu_2}{\mu_1 + \mu_2}\right),$$

we have $V(2,1,11) = V(2,1,12)$.

PROOF: By the dynamic equations (1),

$$\begin{aligned} &V(2,1,11) - V(2,1,12) \\ &\quad = \frac{2c_1 + c_2}{2\mu_1} + pV(1,2,R) + (1-p)V(1,1,12) - \frac{2c_1 + c_2}{\mu_1 + \mu_2} \\ &\qquad - \frac{\mu_1}{\mu_1 + \mu_2}\left(pV(1,2,R) + (1-p)V(1,1,12)\right) - \frac{\mu_2}{\mu_1 + \mu_2} V(2,0,11) \\ &\quad = \frac{2c_1 + c_2}{2\mu_1} - \frac{2c_1 + c_2}{\mu_1 + \mu_2} + \frac{\mu_2}{\mu_1 + \mu_2}\left(pV(1,2,R) + (1-p)V(1,1,12)\right. \\ &\qquad\qquad \left. - V(2,0,11)\right) =: (*) \end{aligned}$$

Now we distinguish between two cases.

Case 1:

$$\frac{c_1}{c_2} > \frac{p\mu_1 + 2\mu_2}{\mu_1 + \mu_2}.$$

By Lemma 7, $V(1,2,R) = V(1,2,12)$; hence, by Lemma 2(b),

$$\begin{aligned} (*) &= \frac{2c_1 + c_2}{2\mu_1} - \frac{2c_1 + c_2}{\mu_1 + \mu_2} + \frac{\mu_2}{\mu_1 + \mu_2} \\ &\qquad \times \left[pV(1,2,12) + (1-p)\left(\frac{c_1}{\mu_1} + \frac{(1+p)c_2}{\mu_2}\right) - \frac{2c_1}{\mu_1} - \frac{2pc_2}{\mu_2}\right] \\ &= \frac{-2c_1\mu_1(\mu_1 + \mu_2) + c_2((\mu_1 + \mu_2)^2 + 2p\mu_1\mu_2 + p^2\mu_1^2)}{2\mu_1(\mu_1 + \mu_2)^2} \\ &= \frac{2c_2\mu_1(\mu_1 + \mu_2)}{2\mu_1(\mu_1 + \mu_2)^2}\left[-\frac{c_1}{c_2} + \frac{1}{2}\left(1 + \frac{\mu_2}{\mu_1} + \frac{p^2\mu_1 + 2p\mu_2}{\mu_1 + \mu_2}\right)\right]. \end{aligned}$$

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Case 2:

$$\frac{c_1}{c_2} < \frac{p\mu_1 + 2\mu_2}{\mu_1 + \mu_2};$$

hence, by Lemma 6,

$$\frac{c_1}{c_2} < \frac{1}{2}\left(1 + \frac{\mu_2}{\mu_1} + \frac{p^2\mu_1 + 2p\mu_2}{\mu_1 + \mu_2}\right).$$

By Lemma 7, $V(1,2,R) = V(1,2,22)$; hence, by Lemma 2(b),

$$(*) = \frac{2c_1 + c_2}{2\mu_1} - \frac{2c_1 + c_2}{\mu_1 + \mu_2} + \frac{\mu_2}{\mu_1 + \mu_2}\left[p\,\frac{c_1 + 2c_2}{2\mu_2} + \frac{c_2}{\mu_2} - \frac{c_1}{\mu_1} - p\,\frac{c_2}{\mu_2}\right]$$

$$= \frac{c_2\mu_1(2-p)}{2\mu_1(\mu_1 + \mu_2)}\left(-\frac{c_1}{c_2} + \frac{\mu_1 + \mu_2}{\mu_1(2-p)}\right),$$

which is always positive, since

$$\frac{c_1}{c_2} < \frac{p\mu_1 + 2\mu_2}{\mu_1 + \mu_2}$$

and, by Lemma 5,

$$\frac{p\mu_1 + 2\mu_2}{\mu_1 + \mu_2} < \frac{\mu_1 + \mu_2}{\mu_1(2-p)}.$$

■

LEMMA 9: *For* $c_1/c_2 \in A$, $\mathcal{C}(1,j) = 22$, $j \geq 2$.

PROOF: Induction on j, $j \geq 2$. Let $c_1/c_2 \in A$. By (5) and Lemmas 6 and 7, $\mathcal{C}(1,2) = 22$. Now, suppose that $\mathcal{C}(1,j) = 22$ holds; then by

$$0 \geq V(1,j,22) - V(1,j,12)$$

$$= \frac{c_1 + jc_2}{2\mu_2} + V(1,j-1,R) - \frac{c_1 + jc_2}{\mu_1 + \mu_2}$$

$$- \frac{\mu_1}{\mu_1 + \mu_2}\left(pV(0,j+1,22) + (1-p)V(0,j,22)\right) - \frac{\mu_2}{\mu_1 + \mu_2}V(1,j-1,R),$$

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we get the inequality

$$\frac{\mu_1}{\mu_1+\mu_2}V(1,j-1,R) \le -\frac{c_1+jc_2}{2\mu_2}+\frac{c_1+jc_2}{\mu_1+\mu_2}+\frac{\mu_1}{\mu_1+\mu_2} \times (pV(0,j+1,22)+(1-p)V(0,j,22)). \quad \textbf{(13)}$$

Further, by the induction hypothesis $V(1,j,R)=V(1,j,22)$, we get

$$\begin{aligned} &V(1,j+1,22)-V(1,j+1,12)\\ &= \frac{c_1+(j+1)c_2}{2\mu_2}+V(1,j,22)-\frac{c_1+(j+1)c_2}{\mu_1+\mu_2}\\ &\quad -\frac{\mu_1}{\mu_1+\mu_2}(pV(0,j+2,22)+(1-p)V(0,j+1,22))-\frac{\mu_2}{\mu_1+\mu_2}V(1,j,22)\\ &= \frac{c_1+(j+1)c_2}{2\mu_2}+\frac{\mu_1}{\mu_1+\mu_2}\left(\frac{c_1+jc_2}{2\mu_2}+V(1,j-1,R)\right)-\frac{c_1+(j+1)c_2}{\mu_1+\mu_2}\\ &\quad -\frac{\mu_1}{\mu_1+\mu_2}\left[p\frac{(j+2)c_2}{2\mu_2}+pV(0,j+1,22)+(1-p)\frac{(j+1)c_2}{2\mu_2}\right.\\ &\qquad \left.+(1-p)V(0,j,22)\right] =: (*). \end{aligned}$$

Thus, by (13) and Lemma 2(a),

$$\begin{aligned} (*) &\le \frac{c_1+(j+1)c_2}{2\mu_2}+\frac{\mu_1}{\mu_1+\mu_2}\frac{c_1+jc_2}{2\mu_2}-\frac{c_1+jc_2}{2\mu_2}+\frac{c_1+jc_2}{\mu_1+\mu_2}\\ &\quad -\frac{\mu_1}{\mu_1+\mu_2}\left[p\frac{(j+2)c_2}{2\mu_2}+(1-p)\frac{(j+1)c_2}{2\mu_2}\right]\\ &= \frac{c_2}{2\mu_2}-\frac{c_2}{\mu_1+\mu_2}+\frac{\mu_1}{\mu_1+\mu_2}\frac{c_1-c_2}{2\mu_2}-\frac{pc_2}{2\mu_2}\\ &= \frac{c_2\mu_1}{2\mu_2(\mu_1+\mu_2)}\left(\frac{c_1}{c_2}-\left(p+\frac{\mu_2}{\mu_1}\right)\right), \end{aligned}$$

which is negative by (5). Hence, $\mathcal{C}(1,j+1)=22$, which concludes the induction. ■

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LEMMA 10: *For $c_1/c_2 \in B$, $\mathcal{C}(1,j) = 12$, $j \geq 2$.*

PROOF: Induction on j, $j \geq 2$. Let $c_1/c_2 \in B$. By (6) and Lemmas 6 and 7, $\mathcal{C}(1,2) = 12$. Now suppose that $\mathcal{C}(1,k) = 12$, for $1 \leq k \leq j$. Using Lemma 2(b), we get

$$\begin{aligned} &V(1,j+1,12) - V(1,j+1,22) \\ &\quad = V(1,j+1,12) - \frac{c_1 + (j+1)c_2}{2\mu_2} - V(1,j,12) \\ &\quad = \frac{1}{2\mu_1\mu_2^2}\Bigg[c_2\mu_2(p\mu_1+\mu_2) \\ &\qquad + c_2\left(\frac{\mu_2}{\mu_1+\mu_2}\right)^{j+1}(\mu_1+\mu_2)(\mu_1(1-p)-\mu_2) - c_1\mu_1\mu_2\Bigg]. \end{aligned}$$

Thus, $V(1,j+1,12) - V(1,j+1,22) < 0$ is equivalent to

$$\left(\frac{\mu_2}{\mu_1+\mu_2}\right)^j(-\mu_2+\mu_1(1-p)) < \frac{c_1}{c_2}\mu_1 - (p\mu_1+\mu_2). \tag{14}$$

We consider two cases for proving (14):

If $\mu_2 > \mu_1(1-p)$, then the left-hand side of (14) is negative and, since, by (6), $c_1/c_2 > p + (\mu_2/\mu_1)$, the right-hand side of (14) is positive, which gives (14).

If $\mu_2 < \mu_1(1-p)$, then

$$\begin{aligned} \left(\frac{\mu_2}{\mu_1+\mu_2}\right)^j(-\mu_2+\mu_1(1-p)) &< \left(\frac{\mu_2}{\mu_1+\mu_2}\right)(-\mu_2+\mu_1(1-p)) \\ &< \frac{c_1}{c_2}\mu_1 - (p\mu_1+\mu_2), \end{aligned}$$

where the second inequality is equivalent to

$$\frac{c_1}{c_2} > \frac{p\mu_1+2\mu_2}{\mu_1+\mu_2},$$

which is true by Lemma 6 and since $c_1/c_2 \in B$. ■

LEMMA 11: *For $i \geq 2$, suppose that $\mathcal{C}(k,1) = 12$ and $\mathcal{C}(k,2) = 22$, $1 \leq k \leq i$, and define*

$$B_i := V(i,1,12) - V(i+1,1,12); \tag{15}$$

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then B_i is given explicitly by

$$B_i = -\left(\frac{pc_2}{\mu_2} + \frac{c_1}{2\mu_1} + \frac{c_1}{2(p\mu_1+\mu_2)}\right) - \frac{c_1(\mu_2-\mu_1(1-p))}{2\mu_1(p\mu_1+\mu_2)}\left(\frac{\mu_1(1-p)}{\mu_1+\mu_2}\right)^i - \frac{c_1(p\mu_1+\mu_2)i}{2\mu_1\mu_2}. \quad \textbf{(16)}$$

PROOF: Using the dynamic equations (2) and since $\mathcal{C}(i-2,2) = 22$, we get

$$V(i-1,1,12) = \frac{(i-1)c_1+c_2}{\mu_1+\mu_2} + \frac{\mu_1}{\mu_1+\mu_2}\left(p\frac{(i-2)c_1+2c_2}{2\mu_2} + V(i-2,1,12)\right) + \frac{\mu_2}{\mu_1+\mu_2}V(i-1,0,11),$$

thus

$$\frac{\mu_2(1-p)}{\mu_1+\mu_2}V(i-1,0,11) = -(1-p)\left(\frac{(i-1)c_1+c_2}{\mu_1+\mu_2} + \frac{p\mu_1}{\mu_1+\mu_2}\frac{(i-2)c_1+2c_2}{2\mu_2}\right) + (1-p)V(i-1,1,12) - \frac{p\mu_1(1-p)}{\mu_1+\mu_2}V(i-2,1,12). \quad \textbf{(17)}$$

By using (17) and $\mathcal{C}(i-1,2) = 22$, we get

$$\begin{aligned}
&V(i,1,12)\\
&= \frac{ic_1+c_2}{\mu_1+\mu_2} + \frac{\mu_1}{\mu_1+\mu_2}\left(p\frac{(i-1)c_1+2c_2}{2\mu_2} + V(i-1,1,12)\right) + \frac{\mu_2}{\mu_1+\mu_2}V(i,0,11)\\
&= \frac{ic_1+c_2}{\mu_1+\mu_2} + \frac{\mu_1}{\mu_1+\mu_2}p\frac{(i-1)c_1+2c_2}{2\mu_2} + \frac{\mu_2}{\mu_1+\mu_2}\frac{ic_1}{2\mu_1}\\
&\quad + \frac{\mu_2}{\mu_1+\mu_2}(1-p)V(i-1,0,11) + \frac{\mu_1+p\mu_2}{\mu_1+\mu_2}V(i-1,1,12)\\
&= \frac{ic_1+c_2}{\mu_1+\mu_2} + \frac{\mu_1}{\mu_1+\mu_2}p\frac{(i-1)c_1+2c_2}{2\mu_2} + \frac{i\mu_2c_1}{2\mu_1(\mu_1+\mu_2)}\\
&\quad - (1-p)\left(\frac{(i-1)c_1+c_2}{\mu_1+\mu_2} + \frac{p\mu_1}{\mu_1+\mu_2}\frac{(i-2)c_1+2c_2}{2\mu_2}\right)\\
&\quad - \frac{\mu_1}{\mu_1+\mu_2}(1-p)V(i-2,1,12) + \left(\frac{\mu_1+p\mu_2}{\mu_1+\mu_2} + (1-p)\right)V(i-1,1,12).
\end{aligned}$$

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Proceeding analogously for $V(i+1,1,12)$, we get

$$\begin{aligned} V(i,1,12) - V(i+1,1,12) &= -\left(\frac{c_1}{\mu_1+\mu_2} + \frac{p\mu_1 c_1}{2\mu_1(\mu_1+\mu_2)} + \frac{\mu_2 c_1}{2\mu_1(\mu_1+\mu_2)}\right) + (1-p) \\ &\quad \times \left(\frac{c_1}{\mu_1+\mu_2} + \frac{p\mu_1 c_1}{2\mu_2(\mu_1+\mu_2)}\right) - \frac{\mu_1(1-p)}{\mu_1+\mu_2}(V(i-2,1,12) - V(i-1,1,12)) \\ &\quad + \left(1 + \frac{\mu_1(1-p)}{\mu_1+\mu_2}\right)(V(i-1,1,12) - V(i,1,12)). \end{aligned}$$

This leads us, using definition (15), to the difference equation

$$B_i - \left(1 + \frac{\mu_1(1-p)}{\mu_1+\mu_2}\right)B_{i-1} + \frac{\mu_1(1-p)}{\mu_1+\mu_2}B_{i-2} + \frac{c_1(p\mu_1+\mu_2)^2}{2\mu_1\mu_2(\mu_1+\mu_2)} = 0, \qquad \textbf{(18)}$$

which is solved by the common methods [3, §2.4]. The difference equation (18) has the characteristic roots 1 and $\mu_1(1-p)/(\mu_1+\mu_2)$ and the special inhomogenous solution $-c_1(p\mu_1+\mu_2)i/2\mu_1\mu_2$. Hence, we get

$$B_i = A_0 + A_1\left(\frac{\mu_1(1-p)}{\mu_1+\mu_2}\right)^i - \frac{c_1(p\mu_1+\mu_2)i}{2\mu_1\mu_2},$$

where A_0 and A_1 have to be chosen such that the initial conditions

$$B_1 = -\frac{c_1(p\mu_1^2 + 4\mu_1\mu_2 + 2\mu_2^2) + pc_2(2\mu_1^2 + 2\mu_1\mu_2)}{2\mu_1\mu_2(\mu_1+\mu_2)},$$

$$B_2 = -\frac{c_1(2p\mu_1^3 + (p^2+p+6)\mu_1^2\mu_2 + (2p+7)\mu_1\mu_2^2 + 3\mu_2^2) + 2pc_2\mu_1(\mu_1+\mu_2)^2}{2\mu_1\mu_2(\mu_1+\mu_2)^2}$$

are satisfied. After some lengthly computation, one gets

$$A_0 = -\left(\frac{pc_2}{\mu_2} + \frac{c_1}{2\mu_1} + \frac{c_1}{2(p\mu_1+\mu_2)}\right),$$

$$A_1 = -\frac{c_1(\mu_2 - \mu_1(1-p))}{2\mu_1(p\mu_1+\mu_2)},$$

which gives formula (16). ■

Proof of Theorem 1: Let $c_1/c_2 \in A$. By Lemma 9, $\mathcal{C}(1,j) = 22, j \geq 2$. Hence, by Lemmas 3 and 6, it follows that $\mathcal{C}(i,j) = 22$ for $i \geq 2, j \geq 2$. Thus, it remains to show that $\mathcal{C}(i,1) = 12$, $i \geq 2$, which is managed by induction on i. By (5) and Lemmas 6 and 8, we have $\mathcal{C}(2,1) = 12$. Now suppose that $\mathcal{C}(i,1) = 12$; then using Lemma 11,

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$$V(i+1,1,11) - V(i+1,1,12)$$
$$= \frac{(i+1)c_1 + c_2}{2\mu_1} + p\,\frac{ic_1 + 2c_2}{2\mu_2} + V(i,1,12) - V(i+1,1,12)$$
$$= \frac{c_2}{2\mu_1} - \frac{c_1}{2(p\mu_1+\mu_2)} - \frac{c_1(\mu_2 - \mu_1(1-p))}{2\mu_1(p\mu_1+\mu_2)}\left(\frac{\mu_1(1-p)}{\mu_1+\mu_2}\right)^i =: (*).$$

Now we consider two cases. If $\mu_2 > \mu_1(1-p)$, then

$$(*) \geq \frac{c_2}{2\mu_1} - \frac{c_1}{2(p\mu_1+\mu_2)} - \frac{c_1(\mu_2 - \mu_1(1-p))\mu_1(1-p)}{2\mu_1(p\mu_1+\mu_2)(\mu_1+\mu_2)}$$
$$= \frac{c_2(2-p)}{2(\mu_1+\mu_2)}\left(\frac{\mu_1+\mu_2}{\mu_1(2-p)} - \frac{c_1}{c_2}\right),$$

which is positive by Lemma 5 and $c_1/c_2 \in A$. If $\mu_2 < \mu_1(1-p)$, then

$$(*) \geq \frac{c_2}{2\mu_1} - \frac{c_1}{2(p\mu_1+\mu_2)} = \frac{c_2}{2(p\mu_1+\mu_2)}\left(p + \frac{\mu_2}{\mu_1} - \frac{c_1}{c_2}\right),$$

which is positive by Lemma 6 and $c_1/c_2 \in A$.

Hence, the induction is finished and the theorem is entirely proved. ■

Proof of Theorem 2: Let $c_1/c_2 \in B$. By Lemma 10, $\mathcal{C}(1,j) = 12, j \geq 2$. We show by induction on i that $\mathcal{C}(i,1) = 11$, $i \geq 2$; then by Lemmas 4 and 6, $\mathcal{C}(i,j) = 11$, $j \geq 2$, and the theorem is proved.

By Lemma 8, we have $\mathcal{C}(2,1) = 11$. Moreover,

$$V(3,1,12) - V(3,1,11)$$
$$= \frac{1}{2\mu_1(\mu_1+\mu_2)^2}\,(3c_1\mu_1(\mu_1+\mu_2)^2$$
$$- c_2(2\mu_2^3 + \mu_1\mu_2^2(5+4p) + \mu_1^2\mu_2(2+p)^2$$
$$+ \mu_1^3(1+3p^2-p^3))).$$

So, $\mathcal{C}(3,1) = 11$ is equivalent to

$$\frac{c_1}{c_2} > \frac{1}{3}\left(1 + \frac{2\mu_2}{\mu_1} + \frac{p(p\mu_1+4\mu_2)}{\mu_1+\mu_2} + \frac{p^2(\mu_1^2(2-p^3))}{(\mu_1+\mu_2)^2}\right) =: G.$$

Since in the case $\mu_2 > \mu_1(1-p)$, $G < p + \mu_2/\mu_1$ and in the case $\mu_2 < \mu_1(1-p)$,

$$G < \frac{1}{2}\left(1 + \frac{\mu_2}{\mu_1} + \frac{p^2\mu_1 + 2p\mu_2}{\mu_1+\mu_2}\right),$$

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we have $\mathcal{C}(3,1) = 11$. Now suppose that $\mathcal{C}(i-1,1) = \mathcal{C}(i,1) = 11$ for some $i \geq 3$. Thus, by Lemmas 4 and 6, $\mathcal{C}(i,2) = \mathcal{C}(i-1,2) = \mathcal{C}(i-1,3) = 11$. Since $V(i,1,12) - V(i,1,11) \geq 0$, we get the inequality

$$\frac{\mu_2}{\mu_1+\mu_2} V(i,0,11) \geq \frac{ic_1+c_2}{2\mu_1} - \frac{ic_1+c_2}{\mu_1+\mu_2} + \frac{\mu_2}{\mu_1+\mu_2} \times (pV(i-1,2,11) + (1-p)V(i-1,1,11)). \quad \textbf{(19)}$$

Then, using inequality (19),

$$\begin{aligned}
&V(i+1,1,12) - V(i+1,1,11)\\
&= \frac{(i+1)c_1+c_2}{\mu_1+\mu_2} - \frac{(i+1)c_1+c_2}{2\mu_1}\\
&\quad + \frac{\mu_2}{\mu_1+\mu_2}(V(i+1,0,11) - pV(i,2,11) - (1-p)V(i,1,11))\\
&= \frac{(i+1)c_1+c_2}{\mu_1+\mu_2} - \frac{(i+1)c_1+c_2}{2\mu_1} + \frac{\mu_2}{\mu_1+\mu_2}\frac{(i+1)c_1}{2\mu_1}\\
&\quad + \frac{\mu_2}{\mu_1+\mu_2}((1-p)V(i,0,11) + pV(i,1,11) - pV(i,2,11) - (1-p)V(i,1,11))\\
&\geq \frac{(i+1)c_1\mu_1 + c_2(\mu_1-\mu_2)}{2\mu_1(\mu_1+\mu_2)} - \frac{\mu_2}{\mu_1+\mu_2}(pV(i,2,11) + (1-2p)V(i,1,11))\\
&\quad + (1-p)\left[\frac{ic_1+c_2}{2\mu_1} - \frac{ic_1+c_2}{\mu_1+\mu_2} + \frac{\mu_2}{\mu_1+\mu_2}(pV(i-1,2,11) + (1-p)V(i-1,1,11))\right]\\
&= \frac{c_1(\mu_1+i\mu_2) + (c_2+ic_1)(\mu_1-\mu_2)p}{2\mu_1(\mu_1+\mu_2)}\\
&\quad + \frac{\mu_2}{\mu_1+\mu_2}\Bigg(-p\frac{ic_1+2c_2}{2\mu_1} - (1-2p)\frac{ic_1+c_2}{2\mu_1} - p^2V(i-1,3,11)\\
&\qquad\qquad - p(1-p)V(i-1,2,11) - p(1-2p)V(i-1,2,11)\\
&\qquad\qquad - (1-2p)(1-p)V(i-1,1,11)\\
&\qquad\qquad + p(1-p)V(i-1,2,11) + (1-p)^2V(i-1,1,11)\Bigg)\\
&=: (*)
\end{aligned}$$

Since $\mathcal{C}(i-1,2) = \mathcal{C}(i-1,3) = 11$, we have $V(i-1,3,22) \geq V(i-1,3,11)$ and $V(i-1,2,22) \geq V(i-1,2,11)$; thus, by $(1-2p)(1-p) = (1-p)^2 - p(1-p)$, we get

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$$
\begin{aligned}
(*) &\geq \frac{c_1(\mu_1 + i\mu_1 p) - c_2(\mu_2 - p\mu_1 + p\mu_2)}{2\mu_1(\mu_1 + \mu_2)} \\
&\quad + \frac{\mu_2}{\mu_1 + \mu_2}\,(-p^2 V(i-1,3,22) - p(1-p)V(i-1,2,22) \\
&\qquad\qquad - p(1-2p)V(i-1,2,11) - (1-2p)(1-p)V(i-1,1,11) \\
&\qquad\qquad + p(1-p)V(i-1,2,11) + (1-p)^2 V(i-1,1,11)) \\
&= \frac{c_1(\mu_1 + i\mu_1 p) - c_2(\mu_2 - p\mu_1 + p\mu_2)}{2\mu_1(\mu_1 + \mu_2)} \\
&\quad + \frac{\mu_2}{\mu_1 + \mu_2}\left(-p^2\,\frac{(i-1)c_1 + 3c_2}{2\mu_2} - p^2 V(i-1,2,11)\right. \\
&\qquad\qquad - p(1-p)\,\frac{(i-1)c_1 + 2c_2}{2\mu_2} - p(1-p)V(i-1,1,11) \\
&\qquad\qquad \left. + p^2 V(i-1,2,11) + p(1-p)V(i-1,1,11)\right) \\
&= \frac{(1+p)(c_1\mu_1 - c_2(\mu_2 + p\mu_1))}{2\mu_1(\mu_1 + \mu_2)} = \frac{(1+p)c_2\mu_1}{2\mu_1(\mu_1 + \mu_2)}\left(\frac{c_1}{c_2} - \left(p + \frac{\mu_2}{\mu_1}\right)\right) > 0.
\end{aligned}
$$

Hence, the proof is finished. ■

5. DISCUSSION

In [4], Pandelis and Teneketzis optimized the α-discounted holding cost problem with general service times and m servers. Translated into our situation (i.e., exponentially distributed service time, $m = 2$ servers and $\alpha = 0$), their result reads as follows.

THEOREM 3 [4]: *For $\mu_1 > \mu_2$ and $c_1/c_2 \geq 1 + p$, in the nonpreemptive case it is always optimal to allocate both servers to queue 1 (if possible).*

Note that if $\mu_2 > \mu_1(1-p)$, we have

$$p + \frac{\mu_2}{\mu_1} < 1 + p,$$

and if $\mu_2 > \mu_1(1-p)$, we have

$$\frac{1}{2}\left(1 + \frac{\mu_2}{\mu_1} + \frac{p^2\mu_1 + 2p\mu_2}{\mu_1 + \mu_2}\right) < 1 < 1 + p.$$

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Hence, in both cases the region for a pure "11" strategy that we found is larger than that in [4], where our assumptions are more restrictive than those in [4].

Now we separate the parameter region for c_1/c_2 into four regions, denoted by A, C, D, and B, as shown in Figs. 2a and 2b. For regions A and B, we have given the optimal control in Theorem 1 and Theorem 2. For regions C and D, we *conjecture* the following:

Conjecture 1:

(a) Let $\mu_2 > \mu_1(1-p)$ (see Fig. 2a).
  (i) For $c_1/c_2 \in C$, there exists an index j^* for the first column such that $\mathcal{C}(1,j) = 12$ if $j \le j^*$ and $\mathcal{C}(1,j) = 22$ if $j > j^*$ (see Fig. 3a, where $j^* = 4$). Further, $\mathcal{C}(i,1) = 12$, $i \ge 2$, and $\mathcal{C}(i,j) = 22$, $i \ge 2, j \ge 2$.
  (ii) For $c_1/c_2 \in D$, the control matrix $\mathcal{C}(i,j)$ consists of a triangular region of "11" and the rest of the control matrix is "22." This triangular region increases, if c_1/c_2 increases (see Figs. 3b and 1b).

(b) Let $\mu_2 < \mu_1(1-p)$ (see Fig. 2b).
  (i) For $c_1/c_2 \in D$, there exists an index i^* such that $\mathcal{C}(i,1) = 12$ if $i \le i^*$ and $\mathcal{C}(i,1) = 11$ if $i > i^*$ (see Fig. 4b, where $i^* = 2$). Further, $\mathcal{C}(1,j) = 12, j \ge 2$ and $\mathcal{C}(i,j) = 11$, $i \ge 2, j \ge 2$.
  (ii) For $c_1/c_2 \in C$, there is a triangular region of "22" and the rest of the control matrix is "11," where this triangular region increases if c_1/c_2 increases (see Figs. 4a and 1a).

The case $p = 1$ has been completely solved by the authors in [6] for the preemptive case. Note that the case $\mu_2 < \mu_1(1-p)$ is not possible; hence, only part (a) of the above conjecture is decisive and was proved in [6]. Note that the region for a pure "22" (resp. "11") strategy changes from $c_1/c_2 \le (\mu_1 + 2\mu_2)/(\mu_1 + \mu_2)$ (resp.

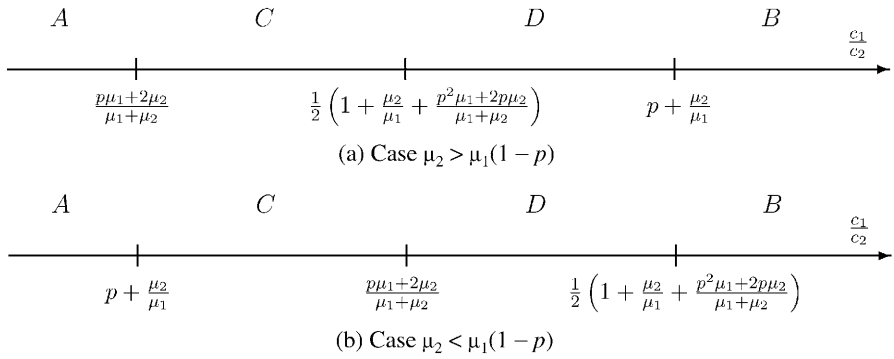
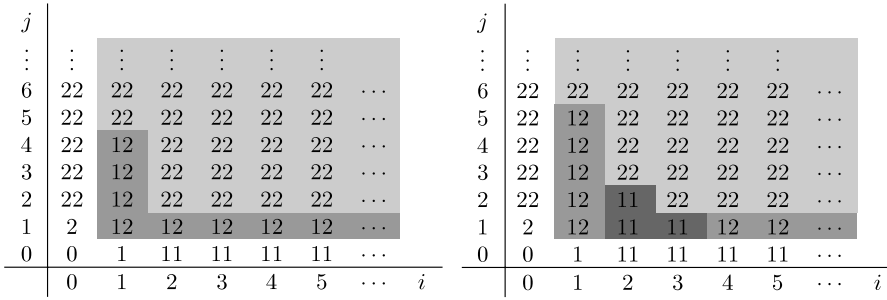


FIGURE 2. Parameter regions for c_1/c_2 in two cases.

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(a) Parameters $c_1 = 2$, $c_2 = 1$, $\mu_1 = 1$, $\mu_2 = 2$, and $p = 0.75$

(b) Parameters $c_1 = 2.45$, $c_2 = 1$, $\mu_1 = 1$, $\mu_2 = 2$, and $p = 0.75$

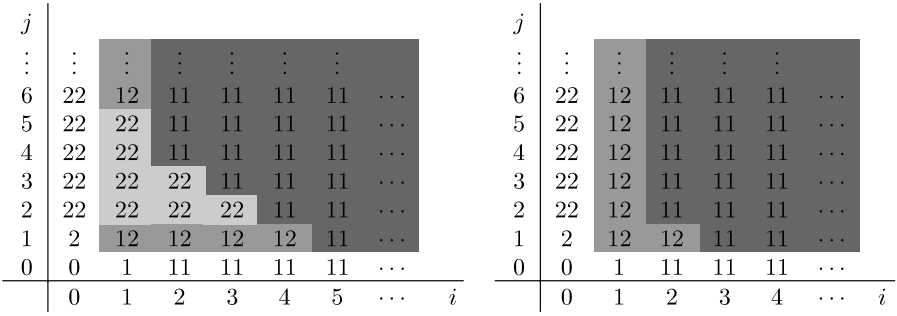
FIGURE 3. Control matrix in the case $\mu_2 > \mu_1(1 - p)$.

$c_1/c_2 \geq 1 + (\mu_2/\mu_1)$) in the case $p = 1$ to $c_1/c_2 \leq (p\mu_1 + 2\mu_2)/(\mu_1 + \mu_2)$ (resp. $c_1/c_2 \geq p + (\mu_2/\mu_1)$) in the case of general $p \in [0,1]$.

For the general model with m servers, it is very hard to prove the optimal strategy. The case $p = 0$ has partly been solved by the authors in [7]. Furthermore, numerical examples show that the conditions (for general p) for the pure strategies are analogous to the conditions in the two-server model. Therefore, we state the following conjecture:

Conjecture 2:

(a) Assume that $\mu_2 > \mu_1(1 - p)$; then in the m-server model, it is optimal to allocate all servers to queue 2 or queue 1 if and only if



(a) Parameters $c_1 = 0.61$, $c_2 = 1$, $\mu_1 = 3$, $\mu_2 = 1$, and $p = 0.25$

(b) Parameters $c_1 = 0.7$, $c_2 = 1$, $\mu_1 = 3$, $\mu_2 = 1$, and $p = 0.25$

FIGURE 4. Control matrix in the case $\mu_2 < \mu_1(1 - p)$.

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$$\frac{c_1}{c_2} < \frac{p\mu_1 + m\mu_2}{\mu_1 + (m-1)\mu_2} \quad \text{or} \quad \frac{c_1}{c_2} > p + \frac{\mu_2}{\mu_1},$$

respectively.

(b) Assume that $\mu_2 < \mu_1(1-p)$; then in the m-server model, it is optimal to allocate all servers to queue 2 if and only if

$$\frac{c_1}{c_2} < p + \frac{\mu_2}{\mu_1}.$$

(c) Assume that $\mu_2 = \mu_1(1-p)$; then in the m-server model, it is optimal to allocate all servers to queue 2 or queue 1 if and only if

$$\frac{c_1}{c_2} < 1 \quad \text{or} \quad \frac{c_1}{c_2} > 1,$$

respectively.

For the sake of completeness, we state the optimal control for the one-server problem. The proof is straightforward and therefore omitted.

THEOREM 4: *In the case of one server, it is always optimal to serve a job of queue 1 or queue 2 (if possible) if and only if*

$$\frac{c_1}{c_2} > p + \frac{\mu_2}{\mu_1} \quad \textit{or} \quad \frac{c_1}{c_2} < p + \frac{\mu_2}{\mu_1},$$

respectively.

Note that, if $c_1/c_2 = p + \mu_2/\mu_1$, then in every point (i,j) of the control matrix $\mathcal{C}(i,j)$ $(i,j \geq 1)$, we have two possible strategies, namely to take a job of queue 1 or to take a job of queue 2.

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