Proc. Camb. Phil. Soc. (1973), **74**, 39
With 4 text-figures
PCPS 74–5
Printed in Great Britain

Collapsing $K \times I$

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(*Received 5 June* 1972)

Zeeman(5) once bravely made the following conjecture:

Conjecture. Suppose K is a contractible 2-complex and I is the unit interval $[0,1]$. *Then $K \times I$ collapses.*

This conjecture is interesting since it trivially implies the Poincaré conjecture. For if C is a homotopy 3-cell, then it has a contractible 2-dimensional spine, so that $C \times I$ would be a collapsible 4-manifold. But then $C \times I$ would be a 4-cell, with $C = C \times 0$ in its 3-sphere boundary, and the PL Schoenfliess theorem shows that C is a 3-cell.

Moreover, Zeeman also pointed out that this conjecture is actually stronger than the Poincaré conjecture, since there are contractible 2-complexes which cannot be embedded (or even immersed) in any 3-manifold. Consequently, everyone seems convinced that the above conjecture is false. But even though probable counter-examples are known, no one seems to know how to show that they are indeed counter-examples.

In this paper, we shall show that the conjecture is true for certain kinds of 2-complexes, or, in view of the general negativism indicated above, that no counter-examples to Zeeman's conjecture can be found among these kinds of 2-complexes.

Let p, q, r, and s be positive integers and $K(p,q,r,s)$ denote the wedge $a \vee b$ of two circles a and b with two 2-cells D_1 and D_2 attached along their boundaries by the words $a^p b^q$ and $a^r b^s$, respectively.

It is easily verified that $K(p,q,r,s)$ is contractible if and only if $ps - qr = \pm 1$. In (5) Zeeman remarked, without proof, that his conjecture held for $K(1,1,n,n+1)$ and $K(1,n,1,n+1)$. We generalize his remarks to $K(1,n,m,nm \pm 1)$. Our proof is independent of Zeeman's special cases and thus proves them. In fact our proof can be viewed as a sort of 'linear combination' of proofs for $K(1,1,n,n+1)$ and $K(1,n,1,n+1)$ We express an indebtedness to a paper by Wright(4) whose techniques suggested a portion of our approach. We also note that Robertson(2) has obtained similar results, though the methods used here appear to be simpler.

THEOREM. *$K(1,n,m,nm \pm 1) \times I$ collapses for all positive integers n and m.*

Proof. We shall dispense with many of the combinatorial details and presume that the diagrams will make the procedure quite clear. Let $q = m + nm$ and

$$K_{\pm} = K(1,n,m,nm \pm 1).$$

We envisage D_1 as having $n+1$ sides, one labelled 'a' followed by n labelled 'b' and

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https://doi.org/10.1017/S0305004100047757 Published online by Cambridge University Press

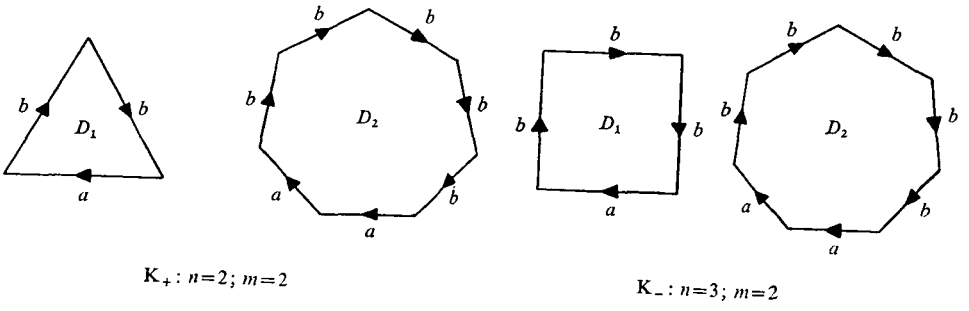


FIG. 1

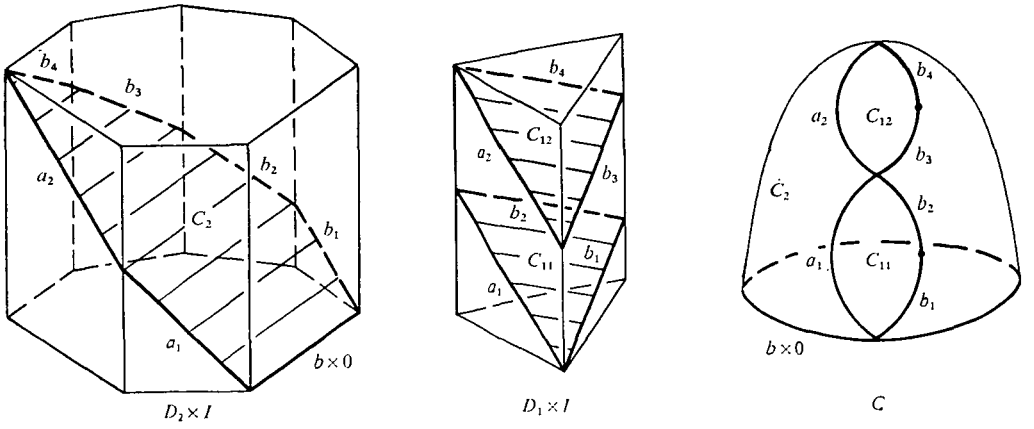


FIG. 2

D_2 as having $q \pm 1$ sides, the first m labelled 'a' followed by $nm \pm 1$ labelled 'b'. (We order the sides clockwise as in Fig. 1.)

Now consider K_+ separately. Let C_2 be a 2-cell which is properly embedded in $D_2 \times I$, meets $D_2 \times 0$ in the last edge $b \times 0$, and meets $D_2 \times 1$ in the vertex shared by the last $a \times 1$ and first $b \times 1$, the remainder of Bd C_2 being the sum of the q segments $a_1, \ldots, a_m, b_1, \ldots, b_{mn}$, as shown in Fig. 2. (It is convenient to index the a_i's in a clockwise fashion and the b_j's in a counter-clockwise fashion. The a_i's and b_j's must be chosen so that the vertex common to a_i and a_{i+1} will be identified in $K_+ \times I$ with the vertex common to b_{in} and b_{in+1}. Of course, the initial points of a_1 and b_1 will also be identified. But there must be no other identifications among any of the a_i's and b_j's.)

Clearly, these segments determine m simple closed curves $\Gamma_1, \ldots, \Gamma_m$ where Γ_i is identified with the sum of a_i, $b_{(i-1)n+1}, \ldots, b_{in}$ in $K_+ \times I$. Let C_{1i} be a nice properly embedded 2-cell in $D_1 \times I$ bounded by Γ_i, as illustrated in Fig. 2. In the identification space $K_+ \times I$, $\left(\bigcup_{i=1}^{m} C_{1i}\right) \cup C_2$ determines a 2-cell C attached to $(a \vee b) \times 0$ via the word $b = b \times 0$ (see Fig. 2).

We now proceed to collapse $K_+ \times I$. It is convenient (as well as illustrative) to describe the collapse in $D_1 \times I$ and $D_2 \times I$, prior to identification. The collapse can be

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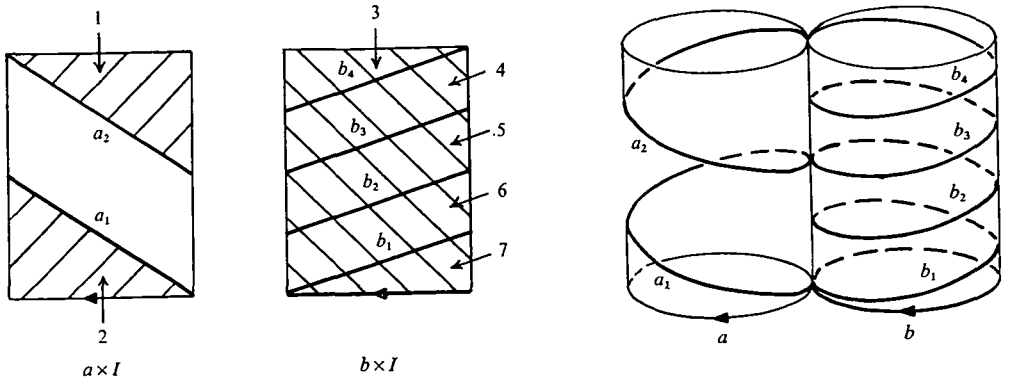


FIG. 3

realized in $K_+ \times I$, since the free faces used remain free after identification, although minor subdivisions may be required to undo any identifications on the boundaries of these faces.

First, collapse the 3-dimensional stuff as follows. Collapse $D_2 \times I$ vertically down from the top $D_2 \times 1$ and up from the bottom $D_2 \times 0$ to C_2. Also collapse $D_1 \times I$ down to C_{1m} and up to C_{11}. Note that the collapses in $D_2 \times I$ have liberated the faces in $(\mathrm{Bd}\, D_1) \times I$ between each a_i and a_{i+1}. So we can collapse the 3-cells between each C_{1i} and C_{1i+1}.

All that remains of the identification space $K_+ \times I$ is the cylinder $b \times I$, the cylinder $a \times I$ minus the interiors of the free faces between the a_i's, and C, which is attached to $b \times 0$ by the word b. Fig. 3 depicts the situation, both before and after identification, although C is not shown.

Now collapse the top triangles of $a \times I$ and $b \times I$ and the bottom triangle of $a \times I$ via the free faces $a \times 1$, $b \times 1$, and $a \times 0$, respectively. (Note that $b \times 0$ is not free.) A face of the top parallelogram in $b \times I$ is thereby liberated, viz. the face in $v \times I$, where v is the wedge point of the two circles a and b. Hence this top parallelogram can be collapsed, whereupon a face is liberated at the next lower level. Continuing in this fashion, we can ultimately collapse the bottom triangle of $b \times I$ via the free face on $v \times I$. (The arrows and numbers in Fig. 3 indicate the collapses and their order.) Thus $K_+ \times I \searrow C \searrow 0$.

The differences in the case for K_- are slight but important. The 2-cell C_2 must be constructed so that it meets $D_2 \times 0$ in the vertex between the last edge $b \times 0$ and the first $a \times 0$ and meets $D_2 \times 1$ in the vertex between the last $a \times 1$ and first $b \times 1$. $\mathrm{Bd}\, C_2$ is the sum of the $q-1$ segments $a_1, \ldots, a_m, b_1, \ldots, b_{mn-1}$ as shown in Fig. 4. These segments, along with the last edge $b \times 0$ of $D_1 \times 0$, determine m simple closed curves $\Gamma_1, \ldots, \Gamma_m$ in $(\mathrm{Bd}\, D_1) \times 1$, where $\Gamma_1 - b \times 0$ is identified in $K_- \times I$ with the sum of a_1 and $b_1, \ldots, b_{n-1}$, while, for $i = 2, \ldots, m$, Γ_i is identified with the sum of a_i and

$$b_{(i-1)n}, \ldots, b_{in-1}.$$

The 2-cells C_{1i} in $D_1 \times I$ and C in $K_- \times I$ are defined just as before (see Fig. 4) and the collapsing can be described similarly. So the theorem has been established.

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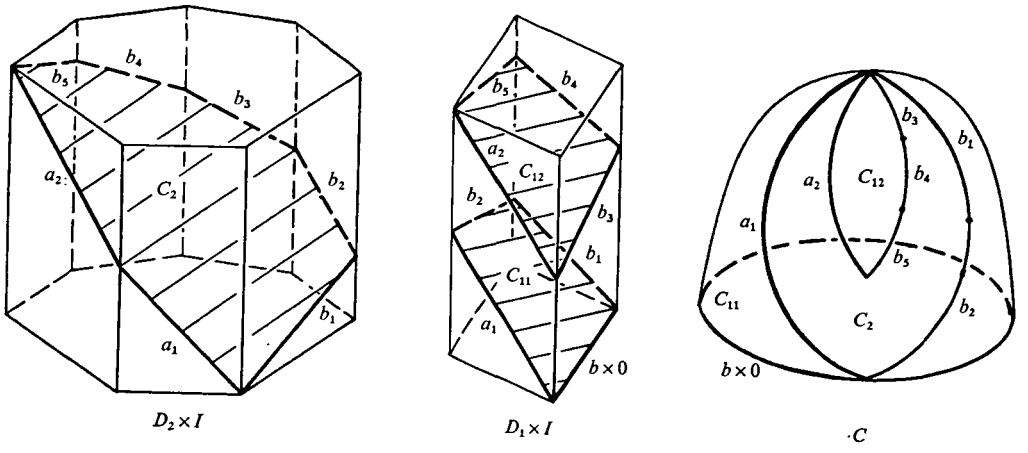


FIG. 4

In (1) Lickorish asked whether or not there was a contractible 2-complex K such that $K \times I$ collapses but $K \times I$ does not collapse to $p \times I$ for any $p \in K$. The complexes $K_{\pm} \times I$ all collapse to $p \times I$ for any p in $\operatorname{Int} D_2$, since the initial downward and upward collapses of $D_2 \times I$ to $[(\operatorname{Bd} D_2) \times I] \cup C_2$ can be replaced by collapses to

$$[(\operatorname{Bd} D_2) \times I] \cup C_2 \cup [p \times I]$$

and $p \times I$ can be ignored thereafter. We might also note that $K_{\pm} \times I$ does not collapse to $p \times I$ for $p \in \operatorname{Int} D_1$ as the homotopy is wrong (cf. the proposition in (1) or Theorem 1 of (3)). At any rate the complexes $K_{\pm}$ do not settle Lickorish's question. (Added in proof: Lickorish apparently has shown that $K(2, 3, m, n) \times I \searrow 0$ for $|2n - 3m| = 1$. I myself have a proof for the case $m = 3$ and $n = 4$. This not only answers the above question, but eliminates the reason for asking it. Moreover, since these complexes were considered prime candidates for counter-examples, the approach to Zeeman's conjecture has become somewhat more complicated.)

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