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Discrete coactions on Hilbert C*-modules

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1. *Introduction and terminology*

In this paper, we will investigate discrete coactions on Hilbert C*-modules. In particular, we obtain a one-to-one correspondence between Hilbert C*-modules with discrete coactions and Hilbert C*-modules over the crossed products of the original C*-algebras which satisfies some nice properties (see 3·6 and 3·7). Then we will give some applications of this correspondence in the last three sections.

In Section 2, we give some results about discrete coactions on Hilbert C*-modules which mainly correspond to those about discrete coactions on C*-algebras (see [7]).

The main result is in Section 3 which gives a one-to-one correspondence between Hilbert $C_r^*(G)$-A-modules and Hilbert $A \times_\delta G$-modules (3·6). Moreover, the correspondence is functorial (3·7).

By studying the corresponding equivalence relation in defining the KK-theory and the co-equivariant KK-theory (4.2), we obtain the Green–Julg theorem for discrete coactions (4.3) in Section 4.

In Section 5, we examine the condition on a Hilbert $C_r^*(G)$-A-module for which the corresponding Hilbert $A \times_\delta G$-module is full (5·1 and 5·4) and obtain some results concerning the Morita equivalence between $A \times_\delta G$ and A^δ (5·2). Furthermore, it is shown that $A \times_\delta G$ is simple if and only if A^δ is simple and $A_t \neq (0)$ for all $t \in G$ (5·4).

Finally, in Section 6, we show that $A \times_\delta G$ corresponds to $l^2(G) \otimes A$ under the correspondence in Section 3. This gives a co-equivariant version of the stabilization theorem which states that any countably generated Hilbert $C_r^*(G)$-A-module can be absorbed by $H \otimes l^2(G) \otimes A$ with a suitably defined coaction.

Throughout this paper, unless specified, G is a discrete group, A is a C*-algebra (usually with a G-coaction), δ is a reduced coaction of G on A and $\otimes$ is the spatial tensor product or tensor product of Hilbert modules. Moreover, we use the term co-equivariant for those operators that preserve coactions.

Before proceeding into Section 2, we recall some notations and results from [7]. For any $t \in G$, let λ_t and M_t be elements in $\mathscr{L}(l^2(G))$ defined by $\lambda_t(\epsilon_s) = \epsilon_{ts}$ and $M_t(\epsilon_s) = \delta_{t,s}\epsilon_t$, where $\{\epsilon_s\}$ is the canonical basis for $\mathscr{L}(l^2(G))$. Let φ_t be the continuous linear map on $C_r^*(G)$ such that $\varphi_t(\lambda_s) = \delta_{s,t}$ and denote by S_t the slice map corresponding to φ_t. Moreover, we denote by A_t the set $\{x \in A : \delta(x) = x \otimes \lambda_t\}$ and call it the t-component with respect to δ. It is easy to see that $\lambda_s M_t = M_{st}\lambda_s$ and $(1 \otimes M_t)x(1 \otimes M_s) = (S_{ts^{-1}}(x) \otimes \lambda_{ts^{-1}})(1 \otimes M_s)$ for all $x \in A \otimes C_r^*(G)$. Moreover, for any $a \in A$, define $a_t = S_t(\delta(a))$. Then $(a_t)^* = (a^*)_{t^{-1}}$ and $S_t(\delta(a)(b \otimes \lambda_s)) = a_{ts^{-1}}b$. If either a or b belongs to $\bigoplus_{t \in G} A_t$, then $(ab)_t = \sum_{r \in G} a_r b_{r^{-1}t}$.

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PROPOSITION. *Let A be a* C**-algebra.*

(a) Let δ be a coaction of G on A and let A_t be the t-component with respect to δ. Then $A = \overline{\bigoplus_{t\in G} A_t}$.

(b) Let $\{B_t\}$ be a family of linear subspaces of A such that $A = \overline{\bigoplus_{t\in G} B_t}$. Then

$$B_r.B_s \subseteq B_{rs},\ B_r^* = B_{r^{-1}} \quad \text{and} \quad \left\|\sum_{t\in G} b_t\right\| = \left\|\sum_{t\in G} b_t \otimes \lambda_t\right\|$$

(for a finite number of non-zero $b_t \in B_t$) if and only if there exists a reduced coaction δ of G on A such that $A_t = B_t$ for all $t \in G$.

2. *Discrete coactions on Hilbert* C**-modules*

We begin this section with a notion of 'slice maps' for Hilbert modules.

LEMMA 2·1. *Let A and B be* C**-algebras and let E be a Hilbert A-module. For any $f \in B'_+$, there exists a slice map S_f from $E \otimes B$ to E given by $S_f(x \otimes b) = x.f(b)$ with norm $\|S_f\| = \|f\|$.*

Proof. It is required to show that

$$\left\|\left\langle \sum_{i=1}^n x_i f(b_i), \sum_{i=1}^n x_i f(b_i)\right\rangle\right\| \leqslant \|f\|^2 \left\|\left\langle \sum_{i=1}^n x_i \otimes b_i, \sum_{i=1}^n x_i \otimes b_i\right\rangle\right\|$$

for any $x_i \in E$ and $b_i \in B$. But this is clear by expressing

$$\langle x_i, x_j\rangle = \sum_{k=1}^n (a_i^k)^*(a_j^k)$$

for a matrix (a_j^k) in A (see [3, 1·2·4]) and using the fact that the slice map L_f on $A \otimes B$ is completely positive.

LEMMA 2·2. *Let E be a Hilbert A-module and let (ψ, H) be a faithful representation of A. Then for any $x, y \in E$, we have $\|\langle x, y\rangle\| = \sup\{\|\langle x \otimes_\psi \xi, y \otimes_\psi \eta\rangle\| : \xi, \eta \in H;\ \|\xi\|, \|\eta\| \leqslant 1\}$. Moreover, for any $x \in E$, $\|x\| = \sup\{\|x \otimes_\psi \xi\| : \xi \in H;\ \|\xi\| \leqslant 1\}$.*

LEMMA 2·3. *Let E be a Hilbert $C_r^*(G)$-A-module and let φ_t be as in the last paragraph of Section 1. Then the slice map S_t corresponding to φ_t is well defined in $E \otimes C_r^*(G)$. Moreover, if $E_t = S_t(\delta_E(E))$, we have:*

(a) $x \in E_t$ if and only if $\delta_E(x) = x \otimes \lambda_t$ and $\langle E_s, E_t\rangle \subseteq A_{s^{-1}t}$ and $E_r.A_s \subseteq E_{rs}$;

(b) for any $x, y \in E$ and $a \in A$, $S_t(\delta(x)(a \otimes \lambda_s)) = x_{ts^{-1}}a$, $(xa)_t = \sum_{r\in G} x_r a_{r^{-1}t}$ and if either x or y belongs to $\bigoplus_{t\in G} E_t$ then $\langle x, y\rangle_t = \sum_{r\in G} \langle x_r, y_{rt}\rangle$ (where $x_t = S_t(\delta_E(x))$).

Proof. We only need to show that S_t is well defined and the rest is obvious. In fact, S_e is well defined by 2·1 and $S_t = S_e \circ \Psi_t$ (where Ψ_t is given by $\Psi_t(\xi \otimes x) = \xi \otimes \lambda_{t^{-1}}x$).

From now on, unless specified, S_t means the map as defined in 2·3 and x_t means $S_t(\delta_E(x))$ and $E_t = S_t(\delta_E(E))$.

PROPOSITION 2·4. *Let E be a Hilbert A-module.*

(a) If E has a G-coaction and E_t is as defined in 2·3, then $E = \overline{\bigoplus_{t\in G} E_t}$.

(b) Suppose that there is a family $\{D_t : t \in G\}$ of linear subspaces of E such that $E = \overline{\bigoplus_{t\in G} D_t}$. Then

$$\langle D_s, D_t\rangle \subseteq A_{s^{-1}t},\quad D_r.A_s \subseteq D_{rs} \quad \text{and} \quad \left\|\sum_{t\in G} x_t\right\| = \left\|\sum_{t\in G} x_t \otimes \lambda_t\right\|$$

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(where only a finite number of $x_t \in D_t$ are non-zero) if and only if there is a coaction δ_E of G on E such that $D_t = E_t$.

Proof. The proof is essentially the same as that of [7, 2·6].

COROLLARY 2·5. *(a) For any $x \in E$, $(x_t) \in c_0(G, E)$ and hence only a countable number of x_t's are non-zero.*

(b) Let E and F be Hilbert $C_r^*(G)$-A-modules. Then $T \in \mathscr{L}(E, F)$ is co-equivariant if and only if $T(E_t) \subseteq F_t$ for all $t \in G$.

Proof. (a) Let $T(x) = (x_t)$ for all $x \in E$. Then $T(\bigoplus_{t \in G} E_t) \subseteq c_0(G, E)$. The corollary now follows from the fact that T is continuous (since $\|Tx\| = \sup_{t \in G} \|x_t\| \leqslant \|x\|$).

(b) This is clear.

LEMMA 2·6. *Let A and B be C*-algebras with G-coactions and let E and F be Hilbert $C_r^*(G)$-A-module and Hilbert $C_r^*(G)$-B-module respectively. Let $\phi: A \to \mathscr{L}(F)$ be a co-equivariant *-homomorphism. Let δ be the natural coaction on $E \otimes_\phi F$ as defined in* [2, 2·10] *and $(E \otimes_\phi F)_t$ be the t-component with respect to δ. Then $(E \otimes_\phi F)_t = \overline{\sum_{rs=t} E_r \otimes_\phi F_s}$ (where $E_r \otimes_\phi F_s$ is the linear span of the set $\{x \otimes_\phi y \in E \otimes_\phi F : x \in E_r, y \in F_s\}$).*

Proof. Note that $\delta(x \otimes_\phi y) = \delta_E(x) \otimes_{\phi \otimes \mathrm{id}} \delta_F(y)$. For any $z \in (E \otimes_\phi F)_t$, there exist $x_r^{n,k} \in E_r$ and $y_s^{n,k} \in F_s$ such that the sum $\sum_{k \in \mathbb{N}} \sum_{r,s \in G} x_r^{n,k} \otimes_\phi y_s^{n,k}$ (with finite non-zero terms) converges to z (by 2·4). Therefore, we have

$$z = z_t = \lim_n S_t\left(\sum_{k \in \mathbb{N}} \sum_{r,s \in G} (x_r^{n,k} \otimes_\phi y_s^{n,k}) \otimes \lambda_{rs}\right) = \lim_n \sum_{k \in \mathbb{N}} \sum_{rs=t} x_r^{n,k} \otimes_\phi y_s^{n,k}.$$

PROPOSITION 2·7. *Let E be a Hilbert $C_r^*(G)$-A-module. Let $\delta_{\mathscr{K}}$ be the induced coaction on $\mathscr{K}(E)$ as defined in* [2, 2·8(a)] *and $T \in \mathscr{K}(E)$. Then $T \in \mathscr{K}(E)_t$ if and only if $T(E_s) \subseteq E_{ts}$ for all $s \in G$. Thus T is co-equivariant if and only if $T \in \mathscr{K}(E)_e$.*

Proof. Note that $\delta_E(T(x)) = \delta_{\mathscr{K}}(T)(\delta_E(x))$ for all $T \in \mathscr{K}(E)$ and $x \in E$. Hence if $T \in \mathscr{K}(E)_t$, $T(E_s) \subseteq E_{ts}$ for all $s \in G$. Conversely, assume that $T(E_s) \subseteq E_{ts}$ for all $s \in G$. Then by 2·4, $\delta_{\mathscr{K}}(T)(\delta_E(x)) = (T \otimes \lambda_t)(\delta_E(x))$ for all $x \in E$. Since $\delta_E(E).(A \otimes C_r^*(G))$ is dense in $E \otimes C_r^*(G)$ and both $\delta_{\mathscr{K}}(T)$ and $T \otimes \lambda_t$ are module maps, $\delta_{\mathscr{K}}(T) = T \otimes \lambda_t$.

Remark 2·8. $\mathscr{K}(E)_t$ is generated by the set $W = \{\theta_{x,y} : x \in E_r, y \in E_s; r, s \in G$ such that $rs^{-1} = t\}$. For any $T \in \mathscr{K}(E)_t$, T can be approximated by elements of the form: $\sum_{k=1}^n \theta_{x_k, y_k}$ (where $x_k, y_k \in \bigoplus_{t \in G} E_t$). Moreover, if $x, y \in \bigoplus_{t \in G} E_t$, then $\theta_{x,y} = \sum_{r,s} \theta_{x_r, y_s}$ and thus $S_t(\delta_{\mathscr{K}}(\theta_{x,y})) = \sum_{s \in G} \theta_{x_{ts}, y_s}$. Therefore, $T = S_t(\delta_{\mathscr{K}}(T))$ can be approximated by elements in the linear span of W.

We would also like to study the Hilbert $A \times_\delta G$-module in the remainder of this section.

LEMMA 2·9. *Let $(F, \langle\langle \cdot, \cdot \rangle\rangle)$ be a Hilbert $A \times_\delta G$-module and let $F_t = F.\delta(A)(1 \otimes M_{t^{-1}})$. Then $\langle\langle F_t, F_s \rangle\rangle \subseteq \delta(A_{t^{-1}s})(1 \otimes M_{s^{-1}})$ and $F = \overline{\bigoplus_{t \in G} F_t}$. Moreover, for any $y \in F$, $(y_t) \in c_0(G, F)$ (where $y_t = y.(1 \otimes M_{t^{-1}})$).*

Proof. Since $\langle\langle x\delta(a)(1 \otimes M_u), y\delta(b)(1 \otimes M_v) \rangle\rangle = (1 \otimes M_u)\delta(a^*)\langle\langle x, y \rangle\rangle \delta(b)(1 \otimes M_v)$, it is easy to obtain the first statement. Now consider F to be a Hilbert $M(A \times_\delta G)$-module. Let $m_\Lambda = \sum_{s \in \Lambda} (1 \otimes M_s)$ for any finite subset Λ of G. Then the projections m_Λ converge to 1 strictly in $M(A \times_\delta G)$. For any $x \in F$, we have $x.m_\Lambda \in \bigoplus_{t \in \Lambda} F_{t^{-1}}$.

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Moreover, there exist $y \in F$ and $a \in A \times_\delta G$ such that $x = ya$. Therefore, $x.m_\Lambda = y.(a.m_\Lambda)$ will converge to x.

LEMMA 2·10. *Let A and B be C*-algebras with coactions δ and ϵ respectively and let F be a Hilbert $B \times_\epsilon G$-module. If $\phi: B \to M(A)$ is a non-degenerate co-equivariant *-homomorphism (in the sense that $\bar{\delta} \circ \phi = \overline{(\phi \otimes \mathrm{id})} \circ \epsilon$), ϕ induces a *-homomorphism ψ from $B \times_\epsilon G$ to $M(A \times_\delta G)$ such that $\psi(\epsilon(b)(1 \otimes M_f)) = \bar{\delta}(\phi(b))(1 \otimes M_f)$. Moreover $(F \otimes_\psi A \times_\delta G)_t = \overline{\sum_{rs=t} F_r \otimes_\psi \delta(a_s)(1 \otimes M_{t^{-1}})}$.*

Proof. The existence of ψ follows from the definition of crossed product. Notice first that $((\bigoplus_{r \in G} F_r) \otimes_\psi A \times_\delta G)(1 \otimes M_{t^{-1}})$ is dense in $(F \otimes_\psi A \times_\delta G)_t$ by 2·9. Let $x = (\sum_{r \in G} x_r) \otimes_\psi \delta(a)(1 \otimes M_{t^{-1}})$ for some $x_r \in F_r$ and $a \in A$. Then we have

$$x = \sum_{r \in G} x_r(1 \otimes M_{r^{-1}}) \otimes_\psi \delta(a)(1 \otimes M_{t^{-1}}) = \sum_{r \in G} x_r \otimes_\psi \delta(a_{r^{-1}t})(1 \otimes M_{t^{-1}}).$$

3. *Hilbert $C_r^*(G)$-A-modules and Hilbert $A \times_\delta G$-modules*

We begin with one direction of the correspondence:

PROPOSITION 3·1. *Let E be a Hilbert $C_r^*(G)$-A-module. Consider the $C_c(G, A)$-scalar multiplication $x.\delta(a)(1 \otimes M_{s^{-1}}) = (xa)_s$ and the inner product $\langle\langle x, y\rangle\rangle = \sum_{t \in G} \delta \langle x, y_t\rangle (1 \otimes M_{t^{-1}})$ (where $s \in G$, $a \in A$ and $x, y \in \bigoplus_{t \in G} E_t$) on $\bigoplus_{t \in G} E_t$. Then $\bigoplus_{t \in G} E_t$ is a pre-Hilbert $C_c(G, A)$-module; denote by $E^\sim$ its completion. Moreover, if (ψ, H) is a faithful representation of A. Then*

$$\| \langle\langle x, x\rangle\rangle \| = \sup\left\{ \left\| \sum_{r \in G} x_r \otimes_\psi \xi_r \right\|^2 : \xi_r \in H; \sum_{r \in G} \|\xi_r\|^2 \leqslant 1 \right\},$$

for any $x \in \bigoplus_{t \in G} E_t$.

Proof. For the first statement, the only thing that is not straightforward is to show that $\langle\langle x, x\rangle\rangle \geqslant 0$ for any $x \in \bigoplus_{t \in G} E_t$. Let $\{t_1, \ldots, t_n\}$ be the set of all $t \in G$ such that $x_t \neq 0$. Since $(\langle x_{t_i^{-1}}, x_{t_j^{-1}}\rangle)$ is a positive matrix, there exists a matrix (a_j^i) in $M_n(A)$ such that $\langle x_{t_i^{-1}}, x_{t_j^{-1}}\rangle = \sum_k (a_i^k)^*(a_j^k)$ for all i, j. Now define a family of function $f^r \in C_c(G, A)$ by $f^r(s) = a_j^i$ when $r = t_i$ and $s = t_j$ and $f^r(s) = 0$ otherwise. Then we have

$$\left(\sum_{r \in G} (f^r)^* . f^r \right)(t) = \sum_{u \in G} \langle x_{(ut)^{-1}}, x_{t^{-1}}\rangle_u = \langle x, x_{t^{-1}}\rangle.$$

Hence $\langle\langle x, x\rangle\rangle = \sum_{r \in G} (f^r)^* . f^r \geqslant 0$. For the second statement, consider the reduced representation $(\psi \times \lambda, l^2(G, H))$ of $A \times_\delta G$. Then by 2·2, we have

$$\begin{aligned} \|\langle\langle x, x\rangle\rangle\| &= \sup\left\{ \|\langle x \otimes_{\psi \times \lambda}(\xi_r), x \otimes_{\psi \times \lambda}(\xi_r)\rangle\| : (\xi_r) \in l^2(G, H); \sum_{r \in G} \|\xi_r\|^2 \leqslant 1 \right\} \\ &= \sup\left\{ \left\| \left\langle \sum_{r \in G} x_r \otimes_\psi \xi_r, \sum_{r \in G} x_r \otimes_\psi \xi_r \right\rangle \right\| : (\xi_r) \in l^2(G, H); \sum_{r \in G} \|\xi_r\|^2 \leqslant 1 \right\}. \end{aligned}$$

Remark 3·2. The map Ψ from $E \times_\delta G = E \otimes_{i_A}(A \times_\delta G)$ to $E^\sim$ defined by

$$\Psi(x \otimes \delta(a)(1 \otimes M_{s^{-1}})) = (xa)_s$$

is a well defined surjective module map. However, it may fail to be injective.

Thus we get a map from the collection of Hilbert $C_r^*(G)$-A-modules to the collection of Hilbert $A \times_\delta G$-modules. We are now going to give a map in the opposite direction.

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We will show in 3·6 that they are inverses of each other. Let F be a Hilbert $A \times_\delta G$-module and F_t be as defined in 2·9. Define a $\bigoplus_{t\in G} A_t$-module structure and a $\bigoplus_{t\in G} A_t$-valued inner product on $\bigoplus_{t\in G} F_t$ by:

(i) $x_s . a_t = x_s \delta(a_t)(1 \otimes M_{(st)^{-1}})$ for $x_s \in F_s$ and $a_t \in A_t$;

(ii) $\langle x_r, y_s \rangle = \text{«}x_r, y_s\text{»}(s^{-1})$ for $x_r \in F_r$ and $y_s \in F_s$.

Consider also the map $\delta: \bigoplus_{t\in G} F_t \to (\bigoplus_{t\in G} F_t) \otimes_{alg} C_r^*(G)$ defined by $\delta(x_t) = x_t \otimes \lambda_t$. Then

PROPOSITION 3·3. *$\bigoplus_{t\in G} F_t$ becomes a pre-Hilbert $\bigoplus_{t\in G} A_t$-module and δ extends to a coaction $\delta_{F^\#}$ on the completion $F^\#$ of $\bigoplus_{t\in G} F_t$. In this case, $F^\#$ is a Hilbert $C_r^*(G)$-A module.*

Proof. We first show that $\langle x, x\rangle \geqslant 0$ for any $x \in \bigoplus_{t\in G} F_t$ and $\langle x, x\rangle = 0$ if and only if $x = 0$. Note that $(\text{«}x_r, x_s\text{»})_{r,s\in G}$ is a positive matrix with entries in $A \times_\delta G \subseteq \mathscr{L}(l^2(G, H))$ (if A is faithfully represented in $\mathscr{L}(H)$). Hence $\sum_{r,s\in G} b_r^* \text{«}x_r, x_s\text{»} b_s \geqslant 0$ for all $b_r \in \mathscr{L}(l^2(G, H))$. By putting $b_r^t = 1 \otimes \lambda_{r^{-1}t}$, we have $(b_r^t)^* \text{«}x_r, x_s\text{»} (b_s^t) = \langle x_r, x_s\rangle \otimes M_{t^{-1}}$ and so $\langle x, x\rangle \otimes 1 \geqslant 0$. Now if $\langle x, x\rangle = \sum_{r,s\in G} \langle x_r, x_s\rangle = 0$, then $\sum_{r\in G} \langle x_r, x_{rt}\rangle = 0$ for all $t \in G$. Hence $\sum_{r\in G} \langle x_r, x_r\rangle = 0$ and so $\langle x_r, x_r\rangle = 0$ for all $r \in G$. Therefore $\text{«}x_r, x_r\text{»} = \delta_A(\langle x_r, x_r\rangle)(1 \otimes M_{r^{-1}}) = 0$, which implies that $x_r = 0$. For the fact that δ can be extended to $F^\#$, it suffices to show that δ is continuous. However, δ is obviously norm preserving since δ_A is. Finally, it is easily seen that δ is a coaction on $F^\#$.

Let $F^\#$ and $\langle , \rangle$ be as in 3·3 and let (ψ, H) be a faithful representation of $A \times_\delta G$. Then $\varphi = \bar{\psi} \circ \delta$ is a faithful representation of A (here we regard δ as a homomorphism from A to $M(A \times_\delta G)$). We can then define the reduced representation $\phi = \varphi \times \lambda$ of $A \times_\delta G$. Let $H_t = \bar{\psi}(1 \otimes M_t)H$ and let $\xi_t = \bar{\psi}(1 \otimes M_t)(\xi)$ for any $\xi \in H$. Then $H = \bigoplus_{t\in G} H_t$ as a Hilbert space direct sum. Moreover, we have $\bar{\psi}(1 \otimes \lambda_r) H_s \subseteq H_{rs}$. Now, we can express the norm $\|\langle , \rangle\|$ in terms of ϕ as follows:

LEMMA 3·4. *For any $x \in \bigoplus_{t\in G} F_t$, $\|\langle x, x\rangle\| = \sup\{\sum_{r\in G} \|x \otimes_\phi \xi^r\|^2 : \xi \in H; \|\xi\| \leqslant 1\}$ (where $\xi^r \in l^2(G, H)$ is defined by $\xi^r(s) = \xi_{sr}$).*

Proof. By 2·2, we have

$$\|\langle x, x\rangle\| = \sup\left\{\left| \sum_{s,t\in G} \langle \bar{\psi}(\langle x_s, x_t\rangle \otimes \lambda_{s^{-1}t})\xi, \xi\rangle \right| : \xi \in H; \|\xi\| \leqslant 1\right\}.$$

Hence

$$\|\langle x, x\rangle\| = \sup\left\{\left| \sum_{r,s,t\in G} \langle \varphi(\langle x_s, x_t\rangle)\xi_{t^{-1}r}, \xi_{s^{-1}r}\rangle \right| : \xi \in H; \|\xi\| \leqslant 1\right\}$$

$$= \sup\left\{ \sum_{r\in G} \langle x \otimes_\phi \xi^r, x \otimes_\phi \xi^r\rangle : \xi \in H; \|\xi\| \leqslant 1\right\}.$$

Let $\mathfrak{E} = \mathfrak{E}(A)$ (respectively $\mathfrak{E}_c(A)$) be the collection of all (resp. countably generated) Hilbert $C_r^*(G)$-A-modules and $\mathfrak{F} = \mathfrak{F}(A \times_\delta G)$ (resp. $\mathfrak{F}_c(A \times_\delta G)$) be the collection of all (resp. countably generated) Hilbert $A \times_\delta G$-modules.

LEMMA 3·5. *Let $E \in \mathfrak{E}$ and $F \in \mathfrak{F}$. Then $E_t = (E^\sim)_t$ and $F_t = (F^\#)_t$.*

Proof. By 3·1 and 2·2, $\|\text{«}, \text{»}\| = \|\langle , \rangle\|$ on $E_t \times E_t$. Hence E_t is closed in $E^\sim$ (since it is complete). Since $(E^\sim)_t = (E^\sim)(1 \otimes M_{t^{-1}})$ is the closure of E_t in $E^\sim$, we obtain the first statement. Similarly, $(F^\#)_t = S_t(\delta(F^\#))$ is the closure of F_t in $F^\#$ and 3·4 shows that $\|\text{«}, \text{»}\| = \|\langle , \rangle\|$ on $F_t \times F_t$.

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THEOREM 3·6. *The map Θ from $\mathfrak{E}$ to $\mathfrak{F}$ which sends E to $E^\sim$ is a bijection. It respects direct sums of modules and sends co-equivariant submodules to submodules. Moreover, $\Theta(\mathfrak{E}_c) = \mathfrak{F}_c$.*

Proof. We first show that $F = (F^\#)^\sim$. Let $\langle\langle ,\rangle\rangle'$ be the inner product on $F' = (F^\#)^\sim$. By 3·5, F' is the completion of $\bigoplus_{t\in G} F_t$ under the norm defined by $\langle\langle ,\rangle\rangle'$. However, it is clear that $\langle\langle ,\rangle\rangle'$ coincides with $\langle\langle ,\rangle\rangle$ on $\bigoplus_{t\in G} F_t$ and so $F' = F$ by 2·9. The proof of the fact that $E = (E^\sim)^\#$ is just the same by using 3·5 and 2·4. For the last statement, let $E\in\mathfrak{E}_c$ and $(\eta_k)_{k\in\mathbb{N}}$ be a generating set for E. For any $x\in E_t, x = \lim_n \sum_{k\in\mathbb{N}} \eta_k a_k^{(n)}$ (where $a_k^{(n)}\in\bigoplus_{t\in G} A_t$). Thus, $x = x_t = \lim_n \sum_{k\in\mathbb{N}}\sum_{r\in G}(\eta_k)_r(a_k^{(n)})_{r^{-1}t}$ and so $\{(\eta_k)_t : k\in\mathbb{N}; t\in G\}$ is also a generating set for E. Since $\{(\eta_k)_t : k\in\mathbb{N}; t\in G\}$ is countable (by 2·5), we can assume that all η_k are taken from $\bigoplus_{t\in G} E_t$. Let $E_0 = \{\sum_{k\in\mathbb{N}}\eta_k a_k : a_k\in\bigoplus_{t\in G} A_t;$ $a_k = 0$ except for finitely many $k\}$ and $F_0 = \{\sum_{k\in\mathbb{N}}\eta_k b_k : b_k\in\bigoplus_{t\in G}\delta(A)(1\otimes M_{t^{-1}});$ $b_k = 0$ except for finitely many $k\}$. Then $E_0 = F_0$ as vector subspaces of $\bigoplus_{t\in G} E_t$. It is required to show that F_0 is dense in $E^\sim$. Let F be the closure of F_0 in $E^\sim$. Then F is a Hilbert $A\times_\delta G$-submodule of $E^\sim$. Now $\bigoplus_{t\in G} F_t \supseteq \bigoplus_{t\in G}(F_0)_t = E_0$. Furthermore, the inner product on $\bigoplus_{t\in G} F_t (\subseteq \bigoplus_{t\in G} E_t^\sim)$ defining $F^\#$ is the same as the original inner product on $\bigoplus_{t\in G} E_t$ (as above). Therefore, $F^\# = E$ (E_0 is dense in E) and so $F = (F^\#)^\sim = E^\sim$. Similarly, if $\Theta(E)$ is countably generated, then so is E.

From now on, we denote by $\mathscr{L}^e$ (respectively $\mathscr{K}^e$) the set of all co-equivariant operators in $\mathscr{L}$ (respectively $\mathscr{K}$).

THEOREM 3·7. *For any $E_1, E_2\in\mathfrak{E}$, there exists an isometric isomorphism Θ from the Banach space $\mathscr{L}^e(E_1,E_2)$ to $\mathscr{L}(E_1^\sim, E_2^\sim)$ which maps $\mathscr{K}^e(E_1,E_2)$ onto $\mathscr{K}(E_1^\sim,E_2^\sim)$. Furthermore, let $E_1,E_2,E_3\in\mathfrak{E}$ and let $T\in\mathscr{L}^e(E_1,E_2)$ and $U\in\mathscr{L}^e(E_2,E_3)$. Then $\Theta(T^*) = \Theta(T)^*$ and $\Theta(U\circ T) = \Theta(U)\circ\Theta(T)$.*

Proof. For $T\in\mathscr{L}^e(E_1,E_2)$, define $T^\sim(x) = T(x)\in\bigoplus_{t\in G}(E_2)_t$ for all $x\in\bigoplus_{t\in G}(E_1)_t$. It is required to show that $T^\sim$ is continuous and hence extends to an operator from $E_1^\sim$ to $E_2^\sim$. Let (ψ, H) be a faithful representation of A. Then $T\otimes_\psi 1$ is well defined in $\mathscr{L}(E_1\otimes_\psi H, E_2\otimes_\psi H)$. Now

$$\|\langle\langle T^\sim x, T^\sim x\rangle\rangle\| = \sup\left\{\left\|\sum_{r\in G} Tx_r\otimes_\psi\xi_r\right\|^2 : \xi_r\in H; \sum_{r\in G}\|\xi_r\|^2\leqslant 1\right\}$$
$$\leqslant \|T\otimes_\psi 1\|^2\,\|\langle\langle x,x\rangle\rangle\|$$

for any $x\in\bigoplus_{t\in G}(E_1)_t$,

(by 3·1). Hence $\|T^\sim\|\leqslant\|T\otimes_\psi 1\|\leqslant\|T\|$. We can also define $(T^*)^\sim$ (T is co-equivariant if and only if T^* is) and show that $(T^*)^\sim$ is the adjoint of $T^\sim$. Similarly, for $S\in\mathscr{L}(E_1^\sim, E_2^\sim)$, define $S^\#(y) = S(y)$ for all $y\in\bigoplus_{t\in G}(E_1)_t$. Then $\|S^\#\|\leqslant\|S\|$ (by 3·4) and $S^\#$ is co-equivariant (by 2·5(*b*)). Since T and $(T^\sim)^\#$ agree on $\bigoplus_{t\in G}(E_1)_t$, the map $\Theta(T) = T^\sim$ gives the required isomorphism. For the second statement, it suffices to show that $\{\Theta(\theta_{y,x}) : x\in(E_1)_t; y\in(E_2)_{t^{-1}}; t\in G\}$ generates $\mathscr{K}(E_1^\sim, E_2^\sim)$ (by 2·7 and 2·8). However, it is easily checked that $\Theta(\theta_{y,x}) = \theta_{y,x}$. The final statement is obvious.

LEMMA 3·8. *Let (A,δ), (B,ϵ), ϕ and ψ be as in Lemma 2·10. Let E and F be Hilbert $C_r^*(G)$-B-module and Hilbert $B\times_\epsilon$-G-module respectively. Let $T\in\mathscr{L}^e(E)$ and $S\in\mathscr{L}(F)$.*

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Then

(*a*) $(E \otimes_\phi A)^\sim \cong (E^\sim) \otimes_\psi (A \times_\delta G)$ *and* $(T \otimes_\phi 1)^\sim = (T^\sim) \otimes_\psi 1$ *under this isomorphism*;

(*b*) $(F \otimes_\psi (A \times_\delta G))^\# \cong (F^\#) \otimes_\phi A$ *and* $(S \otimes_\psi 1)^\# = (S^\#) \otimes_\phi 1$ *under this isomorphism.*

Proof. (*a*) By 2·4 and 2·9, it suffices to find an inner product preserving map Φ from $\bigoplus_{t\in G} (\sum_{rs=t} E_r \otimes_\phi A_s)$ to $\bigoplus_{t\in G} (E^\sim \otimes_\psi A \times_\delta G)_t$ with dense image (by 2·6). Now, we define $\Phi(x_r \otimes_\phi a_s) = x_r \otimes_\psi (a_s \otimes \lambda_s)(1 \otimes M_{(rs)^{-1}})$. We first show that it preserves the inner products.

$$
\begin{aligned}
&\langle\!\langle x_r \otimes_\psi (a_s \otimes \lambda_s)(1 \otimes M_{(rs)^{-1}}), x_u \otimes_\psi (a_v \otimes \lambda_v)(1 \otimes M_{(uv)^{-1}})\rangle\!\rangle_{(E^\sim \otimes_\psi A \times_\delta G}\\
&\quad = (1 \otimes M_{(rs)^{-1}})(a_s^* \otimes \lambda_{s^{-1}})\,\psi(\delta(\langle x_r, x_u\rangle)(1 \otimes M_{u^{-1}}))(a_v \otimes \lambda_v)(1 \otimes M_{(uv)^{-1}})\\
&\quad = (1 \otimes M_{(rs)^{-1}})[a_s^*(\phi\langle x_r, x_u\rangle)a_v \otimes \lambda_{(rs)^{-1}uv}](1 \otimes M_{(uv)^{-1}})\\
&\quad = \langle\!\langle x_r \otimes_\phi a_s, x_u \otimes_\phi a_v\rangle\!\rangle_{(E \otimes_\phi A)^\sim}.
\end{aligned}
$$

It is clear by 2·10 that $\Phi(\sum_{rs=t} E_r \otimes_\phi A_s)$ is dense in $(E^\sim \otimes_\psi A \times_\delta G)_t$. Finally, it is obvious that $\Phi \circ (T \otimes_\phi 1)^\sim = ((T^\sim) \otimes_\psi 1) \circ \Phi$ as T is co-equivariant and part (*a*) is proved.

Part (*b*) follows from (*a*) and 3·6.

Remark 3·9. (*a*) $A^\sim \cong (1 \otimes M_e)(A \times_\delta G)$ *since* $\mathbb{C}^\sim \cong \mathbb{C}(1 \otimes M_e)$ *as an ideal of* $C_0(G)$.

(*b*) *Lemma* 3·8 *is also true for* $T \in \mathscr{L}^e(E_1, E_2)$ *and* $S \in \mathscr{L}(F_1, F_2)$.

4. *Green–Julg theorem for discrete coactions*

Let $M_G^c(A)$ be the collection of all pairs (E, T) where E is a countably generated $\mathbb{Z}_2$-graded Hilbert $C_r^*(G)$-A-module and $T \in \mathscr{L}(E)$ is a co-equivariant degree 1 operator. Let $M^c(A \times_\delta G) = M_{(e)}^c(A \times_\delta G)$ (where (e) means the trivial group). Results in Section 3 imply

PROPOSITION 4·1. $M_G^c(A) \cong M^c(A \times_\delta G)$ *as semi-groups.*

Before proving the main result (Theorem 4·3) of this section, we need to consider the equivalence relations on M^c and M_G^c that define the corresponding KK-theories.

LEMMA 4·2. (*a*) *If* (E_0, T_0) *and* (E_1, T_1) *in* $M_G^c(A)$ *are co-equivariantly homotopy equivalent, then* $(E_0^\sim, T_0^\sim)$ *and* $(E_1^\sim, T_1^\sim)$ *in* $M^c(A \times_\delta G)$ *are homotopy equivalent.*

(*b*) *If* (F_0, S_0) *and* (F_1, S_1) *are homotopy equivalent in* $M(A \times_\delta G)$ (*resp.* $M^c(A \times_\delta G)$), *then* $(F_0^\#, S_0^\#)$ *and* $(F_1^\#, S_1^\#)$ *are co-equivariantly homotopy equivalent in* $M_G(A)$ (*resp.* $M_G^c(A)$).

Proof. (*a*) Let $p_t : IA \to A$ and $q_t : IA \times_\delta G \to A \times_\delta G$ be the evaluation maps at $t \in [0, 1]$ (where IA is the C*-algebra $C([0,1]) \otimes A$; note that $I(A \times_\delta G) = (IA) \times_\epsilon G$ with ϵ being the induced coaction on IA). Let $(E, T) \in E_G(IA)$ be such that it implements the homotopy between (E_0, T_0) and (E_1, T_1). It is required to show that $(E^\sim, T^\sim)$ implements a homotopy between $(E_0^\sim, T_0^\sim)$ and $(E_1^\sim, T_1^\sim)$, i.e. $(E^\sim) \otimes_{q_t} A \times_\delta G \cong E_t^\sim$ and $(T^\sim) \otimes_{q_t} 1 = T_t^\sim$ under these isomorphisms. But we already have $E_t \cong E \otimes_{p_t} A$ and

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$T_i' = T \otimes_{p_i} 1$ under these isomorphisms. Now the result follows from 3·8(a) (more precisely, 3·9(b)).

For (b), the proof is the same, by using 3·8(b).

THEOREM 4·3. $KK_{C_r^*(G)}(\mathbb{C}, A) = KK(\mathbb{C}, A \times_\delta G)$.

Proof. First of all we note that for any $[E, \phi, T] \in KK_{C_r^*(G)}(\mathbb{C}, A)$, we can assume that ϕ is unital and hence trivial. Since the same is true for elements in $KK(\mathbb{C}, A \times_\delta G)$, we can ignore ϕ in our consideration. Let Θ be the isomorphism between $M_G^c(A)$ and $M^c(A \times_\delta G)$ as in 3·6 and 3·7. For any $(E, T) \in E_{C_r^*(G)}(\mathbb{C}, A)$ (see [2, 3·1], note that we can assume T to be co-equivariant by [2, 3·7]), let $(F, S) = \Theta(E, T)$. S is clearly of degree 1. By 4·1 and 4·2, it is required to show that $(F, S) \in E(\mathbb{C}, A \times_\delta G)$. By 3·6, it remains to show that S-S^* and S^2-1 are in $\mathscr{K}(F)$, which is clear by 3·7.

COROLLARY 4·4. *Let G be a countable discrete group and B be any* C*-*algebra with any G-action α. Then $K_0(B) = KK_G(C_0(G), B)$.*

Proof. By [2, 6·21], $KK_G(C_0(G), B) = KK_{C_r^*(G)}(\mathscr{K}, B \times_\alpha G) = KK_{C_r^*(G)}(\mathbb{C}, B \times_\alpha G)$. Hence $KK_G(C_0(G), B) = KK(\mathbb{C}, B \otimes \mathscr{K}) = KK(\mathbb{C}, B)$ by 4·3.

Remark 4·5. If we can prove Corollary 4·4 directly, then using [2, 6·21], we can give another proof for Theorem 4·3 in the case of countable groups (note that [2, 6·21] is only true for separable groups).

5. *Morita equivalence between $A \times_\delta G$ and A^δ*

In this section we will consider conditions on a Hilbert $C_r^*(G)$-A-module so that it corresponds to a full Hilbert $A \times_\delta G$-module. We will then use this to give sufficient conditions for which $A \times_\delta G$ is strongly Morita equivalent to A^δ (5·2 and 5·4). As a consequence, we obtain a result that relates the simplicity of $A \times_\delta G$ and A^δ.

PROPOSITION 5·1. *Let $(F, \langle\!\langle\, , \rangle\!\rangle)$ be a Hilbert $A \times_\delta G$-module and $E = F^\#$. Then F is full if and only if the linear span of $\langle E_r, E_s \rangle$ is dense in $A_{r^{-1}s}$ for all $r, s \in G$.*

Proof. We denote by $[S]$ the linear span of elements in S. Now F is full if and only if $[\langle\!\langle F, F_s \rangle\!\rangle]$ is dense in $\delta(A)(1 \otimes M_{s^{-1}})$. This is the case if and only if $[\langle\!\langle F_r, F_s \rangle\!\rangle]$ is dense in $(1 \otimes M_{r^{-1}})\delta(A)(1 \otimes M_{s^{-1}}) = A_{r^{-1}s} \otimes \lambda_{r^{-1}s} M_{s^{-1}}$. Note that $[\langle\!\langle F_r, F_s \rangle\!\rangle] = [\delta\langle E_r, E_s \rangle (1 \otimes M_{s^{-1}})] = [\langle E_r, E_s \rangle] \otimes \lambda_{r^{-1}s} M_{s^{-1}}$ and that the map $\phi: A_r \to \delta(A_r)(1 \otimes M_s)$ defined by $\phi(a) = \delta(a)(1 \otimes M_s)$ is isometric. (In fact, if $A \subseteq \mathscr{L}(H)$, we have

$$\begin{aligned} \|\phi(a)\| &= \sup\{\|(a \otimes \lambda_r M_s)(\xi_t)\| : \xi_t \in H \quad \text{and} \quad \textstyle\sum_{t \in G} \|\xi_t\|^2 \leqslant 1\} \\ &= \sup\{\|a\xi_s \otimes \delta_{rs}\| : \xi_t \in H \quad \text{and} \quad \textstyle\sum_{t \in G} \|\xi_t\|^2 \leqslant 1\} \\ &= \|a\|.) \end{aligned}$$

Thus, F is full if and only if $[\langle E_r, E_s \rangle]$ is dense in $A_{r^{-1}s}$.

According to 5·1, we say that a coaction δ on A is full if the linear span of $A_r . A_s$ is dense in A_{rs} for all $r, s \in G$.

PROPOSITION 5·2. *If the coaction δ on A is full, then $A \times_\delta G$ is strongly Morita equivalent to A^δ.*

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Proof. Consider $E = A$ to be a Hilbert $C_r^*(G)$-A-module. Then $F = E^\sim$ is a full Hilbert $A \times_\delta G$-module by 5·1. Hence, $A \times_\delta G$ is strongly Morita equivalent to $\mathscr{K}(F)$ which is isomorphic to $\mathscr{K}^e(E) = A^\delta$ by 3·7.

Remark 5·3. (*a*) If $A \times_\delta G$ is simple, then every non-zero Hilbert $C_r^*(G)$-A-module will satisfy the hypothesis in 5·1 and so δ is full in the sense of 5·2.

(*b*) In general, A^δ is strongly Morita equivalent to $(A \times_\delta G)(1 \otimes M_e)(A \times_\delta G)$ by 3·9. Note that $(A \times_\delta G)(1 \otimes M_e)(A \times_\delta G)$ equals the closed linear span of $\{(1 \otimes M_s)\delta(a_s a_t)(1 \otimes M_{t^{-1}}) : s, t \in G;\ a_s \in A_s;\ a_t \in A_t\}$. We can also get 5·2 from this.

THEOREM 5·4. *δ is full if and only if the linear span of $A_r . A_{r^{-1}}$ is dense in A_e for all $r \in G$. Hence if $A_e (= A^\delta)$ is simple and $A_t \neq (0)$ for all $t \in G$, then δ is a full coaction. Moreover, $A \times_\delta G$ is simple if and only if A^δ is simple and $A_t \neq (0)$ for all $t \in G$.*

Proof. Suppose that the linear span of $A_r . A_{r^{-1}}$ is dense in A_e for all $r \in G$. Then $A_r . A_s \supseteq A_{rs} . A_{s^{-1}} . A_s$ for any $r, s \in G$. Hence the linear span of $A_r . A_s$ contains $A_{rs} . A_e$, which coincides with A_{rs} (since A_e contains an approximate identity for A). The second statement follows from the fact that the closed linear span of $A_r . A_{r^{-1}}$ is an ideal in A_e. The last statement follows from the previous one together with 5·3 (*a*), 5·2 and the fact that Morita equivalence preserves ideals structure.

6. *Stabilization theorem for co-equivariant Hilbert C*-modules*

The stabilization theorem (see, for example, [**1**, 13·6·2]) says that for any countably generated Hilbert A-module E, $E \oplus \mathscr{H}_A \cong \mathscr{H}_A$ (where $\mathscr{H}_A = H \otimes A$ with H being the separable Hilbert space). There is an equivariant version of this theorem in case of compact groups (see e.g. [**6**]). In this section, we will give a co-equivariant version for this theorem. First of all, we need a candidate to replace the standard Hilbert A-module $\mathscr{H}_A$. Thanks to Lemma 2·6, we can give a G-coaction on the Hilbert A-module $l^2(G) \otimes A = l^2(G) \otimes_\phi A$ (where $\phi : \mathbb{C} \to M(A)$ is the unique unital map). It is then natural to define a G-coaction on $\mathscr{H}_A^{\hat{G}} = H \otimes l^2(G) \otimes A$ where H is the separable Hilbert space with trivial coaction.

LEMMA 6·1. *Let E be a countably generated Hilbert $C_r^*(G)$-A-module. Then $E \oplus (\mathscr{H}_{A \times_\delta G})^\# \cong (\mathscr{H}_{A \times_\delta G})^\#$ where $(\mathscr{H}_{A \times_\delta G})^\#$ is the Hilbert $C_r^*(G)$-A-module as in 3·3.*

Proof. By 3·6 and 3·7, it suffices to see that $E^\sim \oplus \mathscr{H}_{A \times_\delta G} \cong \mathscr{H}_{A \times_\delta G}$; but this is just the ordinary stabilization theorem for Hilbert C*-modules.

PROPOSITION 6·2. *$(A \times_\delta G)^\# \cong l^2(G) \otimes A$ as Hilbert $C_r^*(G)$-A-modules.*

Proof. Let ϕ be the unital map from $\mathbb{C}$ to $M(A)$ and ψ the induced map from $C_0(G)$ to $M(A \times_\delta G)$ as in 2·10. Then $C_0(G) \otimes_\psi (A \times_\delta G) \cong A \times_\delta G$ as Hilbert $A \times_\delta G$-modules. Now the proposition follows from 3·8, since $(C_0(G))^\# \cong l^2(G)$ as Hilbert spaces with coactions.

THEOREM 6·3. *Let E be a countably generated Hilbert $C_r^*(G)$-A-module. Then $E \oplus \mathscr{H}_A^{\hat{G}} \cong \mathscr{H}_A^{\hat{G}}$ as Hilbert $C_r^*(G)$-A-modules.*

Proof. Since the coactions of G on H in both $\mathscr{H}_A^{\hat{G}} = H \otimes l^2(G) \otimes A$ and $(\mathscr{H}_{A \times_\delta G})^\# = H \otimes (A \times_\delta G)^\#$ are trivial, $\mathscr{H}_A^{\hat{G}}$ and $(\mathscr{H}_{A \times_\delta G})^\#$ are isomorphic as Hilbert $C_r^*(G)$-A-modules. The result now follows from 6·1.

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