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Birkhoff normalization and superintegrability of Hamiltonian systems

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Abstract. We study Birkhoff normalization in connection with superintegrability of an n-degree-of-freedom Hamiltonian system X_H with holomorphic Hamiltonian H. Without assuming any Poisson commuting relation among integrals, we prove that, if the system X_H has $n+q$ holomorphic integrals near an equilibrium point of resonance degree $q \geq 0$, there exists a holomorphic Birkhoff transformation φ such that $H \circ \varphi$ becomes a holomorphic function of $n-q$ variables and that $X_{H\circ\varphi}$ can be solved explicitly. Furthermore, the Birkhoff normal form $H \circ \varphi$ is determined uniquely, independently of the choice of φ, as convergent power series. We also show that the system X_H is superintegrable in the sense of Mischenko–Fomenko as well as Liouville-integrable near the equilibrium point.

1. *Introduction and statement of results*

Birkhoff normalization plays a basic role in the study of Hamiltonian systems. For instance, it can be used to study the behaviour of solutions near an equilibrium point. In particular, under the non-resonance condition up to some finite order, the resulting Birkhoff normal form gives a local approximation of a given Hamiltonian by an integrable one, and Kolmogorov–Arnold–Moser (KAM) theory and Nekhoroshev estimates can be applied (cf. [**2, 11**]). Also, it plays a key role in understanding the structure of solutions for a Liouville-integrable system near its singularities [**8, 13**]. Furthermore, it is also applied to other related areas such as Hamiltonian partial differential equations (see [**3, 11**] and references therein) and the inverse spectral problem [**22**], where the Birkhoff normal form is generalized to an infinite-dimensional or quantum form.

In this paper, we consider the Birkhoff normalization for finite-dimensional Hamiltonian systems and study its relationship with superintegrability (non-commutative integrability). Superintegrability is characterized by the existence of integrals whose numbers are more than one-half of the dimension of the phase space (see §6 for a precise

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definition). Such situations occur in many of systems in classical mechanics, such as the Kepler problem, the free rigid body motion (Euler–Poinsot system), etc. Nevertheless, it seems that the Birkhoff normal form for superintegrable systems has not been studied in detail until now. Throughout this paper, we consider the Birkhoff normal form in an analytic category and are mainly concerned with the complex normal form, which is more convenient to deal with than real one.

Let H be a holomorphic function of $z \in \mathbf{C}^{2n}$ near the origin, and assume that it has the power series expansion

$$H = H_2 + H_3 + H_4 + \cdots, \tag{1.1}$$

where H_j $(j = 2, 3, \ldots)$ are homogeneous polynomials in $z = (z_1, \ldots, z_{2n})$ with $z_i = x_i$, $z_{n+i} = y_i$ $(i = 1, \ldots, n)$. We denote by X_H the Hamiltonian vector field

$$\frac{dz}{dt} = J H_z, \quad J = \begin{pmatrix} O & I \\ -I & O \end{pmatrix},$$

where H_z is the gradient vector of the Hamiltonian H and I is the identity matrix of degree n. For any functions f and g, we define the Poisson bracket $\{f, g\}$ by

$$\{f, g\} = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial x_k} \frac{\partial g}{\partial y_k} - \frac{\partial f}{\partial y_k} \frac{\partial g}{\partial x_k} \right).$$

Let $\mathcal{P}_2$ be the Lie algebra consisting of all quadratic polynomials of z under the Poisson bracket defined above. Then it is isomorphic (as a Lie algebra) to the semi-simple Lie algebra $\mathfrak{sp}(2n, \mathbf{C})$ of the Lie group $\mathrm{Sp}(2n, \mathbf{C})$. Therefore, $\mathcal{P}_2$ is semi-simple and one can find a system of symplectic coordinates $z \in \mathbf{C}^{2n}$ so that

$$H_2 = S + N, \quad S = \sum_{k=1}^{n} \lambda_k x_k y_k, \quad \{S, N\} = 0, \tag{1.2}$$

where N is the quadratic form with the property that the coefficient matrix of X_N is nilpotent. We allow here the degenerate situation where some of λ_k are equal to zero. The Hamiltonian H is said to be in *Birkhoff normal form* (or we call H itself the Birkhoff normal form) if the identity $\{H, S\} = 0$ holds. In the present paper, we also say generally that a function f is in *S-normal form* if the identity $\{f, S\} = 0$ holds. One can find a formal symplectic transformation $\varphi\colon z \mapsto z + O(|z|^2)$ such that $H \circ \varphi$ is a formal power series in Birkhoff normal form. The near-to-identity transformation φ is called a *Birkhoff transformation*. Here we say that a transformation is symplectic if it preserves the standard symplectic structure $\sum_{k=1}^{n} dy_k \wedge dx_k$.

According to Siegel [**18**] (see also Pérez-Marco [**16**]), the Birkhoff transformation is generally divergent. Then, when does a convergent Birkhoff transformation exist? Here and throughout this paper, 'convergent' always means 'absolutely convergent' and hence is equivalent to 'holomorphic (analytic) in a neighbourhood of the origin'. The existence of a convergent Birkhoff transformation is closely related to the integrability of Hamiltonian vector field X_H. In fact, in the non-resonance case (i.e. when $\lambda_1, \ldots, \lambda_n$ are rationally independent), it is easy to see that the existence of a convergent Birkhoff transformation implies analytic Liouville-integrability of X_H, that is, there exist n holomorphic and

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functionally independent integrals $G_1, \ldots, G_n$ of X_H such that $\{G_i, G_j\} = 0$ for all $i, j = 1, \ldots, n$. Here the functions $G_1, \ldots, G_n$ are said to be *functionally independent* if their gradient vectors are linearly independent on a dense open subset of the domain considered. Conversely, there exists a convergent Birkhoff transformation if X_H is analytically Liouville-integrable near a non-resonant equilibrium point. This was proved by the author [**7**], generalizing the previous results by Rüssmann [**17**] and Vey [**21**] under some additional non-degeneracy conditions. The proof is constructive and uses the structure of simultaneous normalization of n integrals so that one can prove the convergence of iteration of transformations without any additional condition on small divisors. The same method was successfully used to generalize this result to simple resonance cases [**9, 10**]. More recently, Zung [**24**] proved it generally by developing a new geometric method based on the toric characterization of Birkhoff normalization. It includes all results referred to above as special cases. However, it does not give any detailed information about the Birkhoff normal form $H \circ \varphi$ in resonance cases, such as whether $X_{H\circ\varphi}$ can be solved explicitly or not. Incidentally, we refer to [**5, 13**] for the generalization of Rüssmann–Vey's result to the smooth case, and to [**19, 20, 25**] for related results on the normalization of commuting holomorphic vector fields.

We now recall some different features with Birkhoff normal forms between the non-resonance and resonance cases. In the non-resonance case, the Birkhoff normal form $H \circ \varphi$ is a power series in n variables $\omega_k = x_k y_k$ $(k = 1, \ldots, n)$ and is determined independently of the choice of the transformation φ, while φ is not determined uniquely. The vector field $X_{H\circ\varphi}$ admits n Poisson commuting integrals $\omega_1, \ldots, \omega_n$ and can be solved explicitly in the new coordinates. In resonance cases, the Birkhoff normal form $H \circ \varphi$ generally contains other 'resonant' terms in addition to those terms consisting of $\omega_1, \ldots, \omega_n$, and depends on the choice of the transformation φ. Furthermore, the existence of a convergent Birkhoff transformation does not necessarily imply integrability of X_H if the resonance degree q defined below is greater than one. The Birkhoff normal form becomes more complicated in general as the resonance degree q increases. On the other hand, it is possible in resonance cases that the number of functionally independent integrals exceeds n, while it is not in the non-resonance case. The aim of this paper is to show that the superintegrability gives rise to the non-resonance feature in resonance cases. More precisely, we show that if the resonance degree and the number of integrals are balanced, the Hamiltonian H is taken into a unique Birkhoff normal form by convergent Birkhoff transformations so that the transformed Hamiltonian vector field can be solved explicitly in the new coordinates.

To state the results, we first define the resonance degree following [**10, 24**]. Let $\mathcal{R}$ be the discrete subgroup of $\mathbf{Z}^n$ defined by

$$\mathcal{R} := \{k = (k_1, \ldots, k_n) \in \mathbf{Z}^n \mid \langle k, \lambda \rangle = 0\},$$

where $\langle k, \lambda \rangle = \sum_{j=1}^n k_j \lambda_j$. We call this group $\mathcal{R}$ the *resonance lattice* for the quadratic form $S = \sum_{k=1}^n \lambda_k x_k y_k$. If the dimension of $\mathcal{R}$ over $\mathbf{Z}$ is q, we say that the quadratic form S (or the equilibrium point $z = 0$) is of *resonance degree* q and call the discrete group $\mathcal{R}$ the *resonance lattice of degree* q. Here $0 \le q \le n - 1$ and the case $q = 0$ corresponds to the non-resonance case. The case $q = 1$ is the simple resonance case mentioned above.

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Let $\rho^{(1)}, \ldots, \rho^{(q)} \in \mathbf{Z}^n$ be the generators of the resonance lattice $\mathcal{R}$. Then, there exist $n-q$ linearly independent rational vectors that are orthogonal to those generators. Namely, multiplying by integers, one can find linearly independent integer vectors $\rho^{(q+1)}, \ldots, \rho^{(n)} \in \mathbf{Z}^n$ such that

$$\langle \rho^{(i)}, \rho^{(j)} \rangle = 0 \quad (i = 1, \ldots, q, \ \ j = q+1, \ldots, n). \tag{1.3}$$

We set

$$\omega_k = x_k y_k, \qquad \tau_k = \sum_{j=1}^{n} \rho_j^{(k)} \omega_j \quad (k = 1, \ldots, n), \tag{1.4}$$

where $\rho_1^{(k)}, \ldots, \rho_n^{(k)}$ are the components of the vector $\rho^{(k)}$. Furthermore, writing $\rho^{(k)}$ as $\rho^{(k)} = \rho_+^{(k)} - \rho_-^{(k)}$ with $\rho_+^{(k)}$ and $\rho_-^{(k)}$ being vectors whose components are non-negative integers, we define the monomial ω_{n+k} by

$$\omega_{n+k} = x^{\rho_+^{(k)}} y^{\rho_-^{(k)}} \quad (k = 1, \ldots, q), \tag{1.5}$$

where we have used multi-index notation. For example, if $\lambda_k = 0$, one may take $\rho^{(k)} = \boldsymbol{e}_k$ (the unit vector) and then $\omega_{n+k} = x_k$. If $\lambda_k \neq 0$ for all $k = 1, \ldots, n$, all monomials $\omega_{n+1}, \ldots, \omega_{n+q}$ are of degree at least two. One can easily see that the monomials $\omega_1, \ldots, \omega_{n+q}$ are in S-normal form (see Lemma 1 in §2).

We now state our main result.

THEOREM 1. *Let H be a holomorphic function of the form (1.1) with (1.2) such that the quadratic form S is of resonance degree q. Suppose that there exist $n+q$ integrals of X_H which are holomorphic and functionally independent near the origin. Then there exists a convergent Birkhoff transformation φ and $H \circ \varphi$ becomes a holomorphic function of $n-q$ variables $\tau_{q+1}, \ldots, \tau_n$. Furthermore, the following hold as direct consequences of this conclusion.*

(i) *The function $H \circ \varphi$ does not depend on the choice of φ and is determined uniquely as power series in $\omega_1, \ldots, \omega_n$. In particular, the nilpotent part of H_2 vanishes, i.e. $N = 0$.*

(ii) *The functions $\omega_1, \ldots, \omega_{n+q}$ are $n+q$ functionally independent integrals of $X_{H\circ\varphi}$.*

(iii) *For any integral G of X_H, $G \circ \varphi$ is in S-normal form. It is a function of $n+q$ variables $\omega_1, \ldots, \omega_{n+q}$ and can be written as Laurent series in those variables.*

Remarks. (1) In the case $q = 0$, this theorem reduces to the previous result [**7**] mentioned above.

(2) We do not assume any Poisson commuting relations among integrals. Therefore, Zung's theorem [**24**] and its generalizations [**25, 26**] do not apply and the existence of a convergent Birkhoff transformation is not trivial at all. As a matter of fact, $\omega_1, \ldots, \omega_n$ are n Poisson commuting integrals of $X_{H\circ\varphi}$ and, therefore, X_H is Liouville-integrable near the origin. However, it is not the assumption but a consequence of the theorem. It is to be noted that integrals ω_k are defined only locally since the transformation φ is holomorphic only near the origin in general.

(3) The vector field $X_{H\circ\varphi}$ can be solved explicitly for the new symplectic coordinates, and those solutions are confined on the level set of the map $F(z) = (\omega_1, \ldots, \omega_{n+q})$.

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It turns out that X_H is superintegrable near the equilibrium point in the sense of Mischenko–Fomenko formulated in [**6, 12**]. We show this in §6.

We can derive from Theorem 1 the following consequences about relation between the number of integrals and the resonance degree of equilibrium points. Here and below, the number of integrals means the number of integrals of X_H which are holomorphic and functionally independent near the origin.

COROLLARY 1. *Let H be a holomorphic function of the form (1.1) with (1.2) and q a non-negative integer. Then the following hold:*

(i) *if the quadratic form S is of resonance degree q, then the number of integrals is less than or equal to $n+q$;*

(ii) *if the number of integrals is $n+q$, then S is of resonance degree greater than or equal to q;*

(iii) *if the number of integrals is $2n-1$, then S is of resonance degree $n-1$ and the Birkhoff normal form $H \circ \varphi$ in Theorem 1 is a convergent power series in one variable $S = \sum_{k=1}^{n} \lambda_k x_k y_k$.*

We note that this corollary holds for any vector field X_H which is generally non-integrable. The proof is straightforward. In fact, items (i) and (ii) follow from Theorem 1(iii). Item (iii) follows from (ii) since the resonance degree q is at most $n-1$ and τ_n is an integer multiple of $H_2 = S$ in the case $q = n-1$.

It seems very special that an n-degree-of-freedom system possesses $2n-1$ integrals, but there are some well-known examples such as the Kepler problem and the Calogero model. The latter is a model describing the motions of interacting particles on the line (see [**1**, §4]) and its Hamiltonian is given by

$$H(x, y) = \frac{1}{2}\sum_{k=1}^{n}(y_k^2 + \alpha^2 x_k^2) + \sum_{1 \le k < l \le n} \frac{1}{(x_k - x_l)^2} \quad (\alpha \in \mathbf{R} \setminus \{0\}),$$

to which item (iii) above can be applied near equilibria.

If the Hamiltonian is real analytic, Theorem 1 can be stated in a real-analytic category with S replaced by a real-quadratic normal form. It can be seen by the arguments of [**7**, §7] or [**24**, Proposition 1.3]. We state it here only in the case of elliptic equilibrium point, where the real-quadratic normal form is given by

$$\widehat{S} = \sum_{k=1}^{n} \frac{\alpha_k}{2}(x_k^2 + y_k^2) \quad (\alpha_k \in \mathbf{R}). \tag{1.6}$$

We say that $\widehat{S}$ is of resonance degree q if the resonance lattice defined by

$$\mathcal{R} := \{k \in \mathbf{Z}^n \mid \langle k, \alpha \rangle = 0\}, \quad \alpha = (\alpha_1, \ldots, \alpha_n)$$

is of dimension q over $\mathbf{Z}$. Correspondingly, we replace ω_k and τ_k by

$$\begin{cases} \widehat{\omega}_k = \dfrac{1}{2}(x_k^2 + y_k^2), \quad \widehat{\tau}_k = \displaystyle\sum_{j=1}^{n} \rho_j^{(k)} \widehat{\omega}_j \quad (k = 1, \ldots, n), \\ \widehat{\omega}_{n+k} = \operatorname{Im} f_k(x, y) \quad (k = 1, \ldots, q), \end{cases}$$

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with

$$f_k(x, y) = \omega_{n+k}\left(\frac{x+iy}{\sqrt{2}}, \frac{y+ix}{\sqrt{2}}\right) = \prod_{j=1}^{n}\left(\frac{x_j+iy_j}{\sqrt{2}}\right)^{\rho_{+j}^{(k)}}\left(\frac{y_j+ix_j}{\sqrt{2}}\right)^{\rho_{-j}^{(k)}},$$

where $i = \sqrt{-1}$ and $\operatorname{Im} f_k(x, y)$ denotes the imaginary part of the complex-valued function $f_k(x, y)$ in the real variable $(x, y) \in \mathbf{R}^{2n}$. Then we have the following.

THEOREM 2. *In the assumption of Theorem 1, assume that H is real analytic near the origin and that S is replaced by the real polynomial $\widehat{S}$. Suppose that there exist $n+q$ integrals of X_H which are holomorphic and functionally independent near the origin. Then there exists a real-analytic symplectic transformation $\varphi\colon z \mapsto z + O(|z|^2)$ so that $H \circ \varphi$ becomes a real-analytic function of $n-q$ variables $\widehat{\tau}_{q+1}, \ldots, \widehat{\tau}_n$. As a direct consequence of this conclusion, items (i), (ii) and (iii) of Theorem 1, except the second assertion of (iii), hold with ω_k replaced by $\widehat{\omega}_k$. Moreover, each connected component of the regular level set of the real map $F(z) = (\widehat{\omega}_1, \ldots, \widehat{\omega}_{n+q})$ is an $(n-q)$-dimensional torus.*

Remark. This theorem also holds when $\widehat{\omega}_{n+k}$ are defined as the real parts of $f_k(x, y)$.

In the rest of this paper, we prove Theorem 1 by using the constructive method developed in [**7**] and Theorem 2 as its corollary. To this end, it is crucial to observe that the Lie algebra of all power series in S-normal form is generated by a finite number of monomials and, further, that in the case of resonance degree q, the S-normal form can be written as Laurent series in $n+q$ variables $\omega_1, \ldots, \omega_{n+q}$. We combine this fact with the techniques of [**7, 9**] to derive the desired estimates at each iteration step.

The choice of those variables ω_{n+k} $(k = 1, \ldots, q)$ is natural and already appears in [**10**, Appendix A], whose study is, however, restricted to a special class of resonances called decoupled resonance. In the present paper, we deal with general resonance cases and therefore do not know explicitly about generators of the algebra of all power series in S-normal form. However, it turns out that our method is valid by using the idea mentioned above. Also, we can deal with all cases including degenerate ones admitting zero eigenvalues in a unified manner. It is different from the previous work [**9**].

The paper is organized as follows. In §2, we prove the observation above and some preliminary facts including that the $n+q$ integrals as well as the Hamiltonian H can be taken into S-normal form simultaneously. They will lead to the proofs of Theorem 1(ii) and (iii). In §3, we prove the existence of a convergent Birkhoff transformation φ without taking care of the special form of $H \circ \varphi$. Here φ is obtained as the composition of the infinite number of symplectic transformations which take $n+q$ integrals into S-normal form step by step. We use the rapidly convergent iteration scheme to prove the convergence of φ. In §4, we show that the Birkhoff normal form $H \circ \varphi$ becomes a holomorphic function of $n-q$ variables $\tau_{q+1}, \ldots, \tau_n$ and is uniquely determined, i.e. the claim of (i). In §5, we prove Theorem 2 and finally show in §6 that X_H is superintegrable with singularities in the sense of Mischenko–Fomenko near the origin.

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2. *Preliminaries*

In this section, we first investigate the power series expansion of the S-normal form in detail. In particular, we show that any S-normal form can be written as a power series in a finite number of monomials. Furthermore, we present some preliminary facts for the proof of Theorem 1, including the existence of a formal Birkhoff transformation.

2.1. *Finiteness of the number of generators for S-normal forms.*

First of all, we prove the following fact.

LEMMA 1. *Let* $S = \sum_{j=1}^{n} \lambda_j x_j y_j$ *and* $\mathcal{R}$ *the resonance lattice of degree* q *for* S. *Let* f *be a function which is written as power series* $f = \sum_{\alpha,\beta \in \mathbf{Z}_+^n} c_{\alpha\beta} x^\alpha y^\beta$. *Then the following hold.*

(i) *The function* f *is in S-normal form if and only if*

$$c_{\alpha\beta} = 0 \quad \text{if } \alpha - \beta \notin \mathcal{R}.$$

(ii) *Let* $\tau_1, \ldots, \tau_n$ *be the functions given by (1.4). Then* f *is in S-normal form if and only if the following* $n - q$ *identities hold:*

$$\{f, \tau_k\} = 0 \quad (k = q+1, \ldots n).$$

(iii) *The monomials* $\omega_1, \ldots, \omega_{n+q}$ *given by (1.4) and (1.5) are in S-normal form and functionally independent.*

(iv) *If* f *is a power series in* $n - q$ *variables* $\tau_{q+1}, \ldots, \tau_n$, *then* $\omega_1, \ldots \omega_{n+q}$ *are integrals of* X_f.

Remarks. (1) Part (ii) implies that if there exists a convergent Birkhoff transformation in the case of resonance degree q, the vector field X_H generally possesses $n - q$ Poisson commuting integrals.

(2) Part (iv) together with (iii) proves Theorem 1(ii).

Proof. By direct computation, we have

$$\{f, S\} = \sum_{\alpha,\beta \in \mathbf{Z}_+^n} c_{\alpha\beta} \langle \alpha - \beta, \lambda \rangle x^\alpha y^\beta.$$

This implies part (i). In the same way, we have

$$\{f, \tau_k\} = \sum_{\alpha,\beta \in \mathbf{Z}_+^n} c_{\alpha\beta} \langle \alpha - \beta, \rho^{(k)} \rangle x^\alpha y^\beta.$$

One can see from (1.3) and (i) above that this implies (ii). Furthermore, it is clear from (i) that $\omega_1, \ldots, \omega_{n+q}$ are in S-normal form. By carrying out elementary (linear) transformations, one can easily prove that

$$\operatorname{rank}\left(\frac{\partial(\omega_1, \ldots, \omega_{n+q})}{\partial(z_1, \ldots, z_{2n})} \right) = n + q \quad \text{if } \prod_{j=1}^{2n} z_j \neq 0. \tag{2.1}$$

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This implies the functional independence of $\omega_1, \ldots, \omega_{n+q}$, leading to part (iii). Finally, it follows from (ii) and (iii) that, for f depending only on $\tau_{q+1}, \ldots, \tau_n$,

$$\{f, \omega_k\} = \sum_{j=q+1}^{n} \frac{\partial f}{\partial \tau_j}\{\tau_j, \omega_k\} = 0.$$

Therefore, $\omega_1, \ldots, \omega_{n+q}$ are integrals of X_f. This proves (iv). □

Next we investigate the S-normal form in the case of resonance degree q.

PROPOSITION 1. *Let $S = \sum_{j=1}^{n} \lambda_j x_j y_j$ be a quadratic form of resonance degree q and let $\mathcal{B}$ be the Lie algebra consisting of all power series of $z \in \mathbf{C}^{2n}$ in S-normal form. Then there exist a finite number of monomials $v_1, \ldots, v_N$ $(N \geq n+q)$ with identity coefficients such that the following hold:*

(i) *any series $f \in \mathcal{B}$ is a power series in $v_1, \ldots, v_N$;*

(ii) *$v_i = \omega_i$ for $i = 1, \ldots, n+q$ and the remaining monomials $v_{n+q+1}, \ldots, v_N$ can be expressed as the quotients of two monomials in $\omega_1, \ldots, \omega_{n+q}$.*

Proof. Let us consider the polynomial ring $\mathbf{C}[z]$ $(= \mathbf{C}[z_1, \ldots, z_{2n}])$ instead of the power series ring. As Zung observed [**24**], $X_{i\tau_{q+1}}, \ldots, X_{i\tau_n}$ $(i = \sqrt{-1})$ are commuting vector fields whose orbits are all periodic with period 2π (or its division). Then, their flows give rise to $\mathbf{T}^{n-q}$ action on $\mathbf{C}[z]$ and it follows from Lemma 1(ii) that those polynomials in S-normal form are invariants under this action. Therefore, we may conclude (i) by invariant theory; however, we now give a direct elementary proof for the sake of completeness.

Let $\mathcal{B}_P$ be the subring of $\mathbf{C}[z]$ consisting of all polynomials in S-normal form, and let $I \subset \mathbf{C}[z]$ be the ideal generated by all monomials in S-normal form of positive total degree. Then, by Hilbert's basis theorem, there exist finitely many monomials $v_1, \ldots, v_m \in \mathcal{B}_P$ of positive total degree which generate the ideal I. We now prove that $\mathcal{B}_P = \mathbf{C}[v_1, \ldots, v_m]$. The inclusion $\mathbf{C}[v_1, \ldots, v_m] \subset \mathcal{B}_P$ is trivial. Conversely, suppose that $\mathcal{B}_P \not\subset \mathbf{C}[v_1, \ldots, v_m]$. Then there exists a homogeneous polynomial $f \in \mathcal{B}_P$ of positive total degree such that $f \notin \mathbf{C}[v_1, \ldots, v_m]$. Let f_* be such a homogeneous polynomial of the minimal degree, which we put to be $d > 0$. Then, since $f_* \in I$, it can be written as

$$f_* = \sum_{i=1}^{m} f_i v_i$$

for some homogeneous polynomials $f_1, \ldots, f_m \in \mathbf{C}[z]$. Here, since $\deg f_* = d$, we may assume that $\deg f_i v_i = d$ or $f_i v_i \equiv 0$. Let us compare those terms in S-normal form contained in both sides of this identity. Then, in view of Lemma 1(i), we have

$$f_* = \sum_{i=1}^{m} (P_N f_i) v_i,$$

where $P_N f_i \in \mathcal{B}_P$ denotes the sum of all of those terms in S-normal form contained in f_i (see (2.4) below). We note that $(P_N f_i)v_i$ is either a homogeneous polynomial of degree d or identically zero. Since $\deg v_i > 0$, we have $\deg P_N f_i < d$. Therefore, by the minimality of d, we have

$$P_N f_i \in \mathbf{C}[v_1, \ldots, v_m] \quad (i = 1, \ldots, m)$$

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and consequently $f_* = \sum_{i=1}^{m}(P_N f_i)v_i \in \mathbf{C}[v_1, \ldots, v_m]$. This is a contradiction and hence we conclude $\mathcal{B}_P = \mathbf{C}[v_1, \ldots, v_m]$. Since $\omega_1, \ldots, \omega_{n+q}$ are monomials in S-normal form, we may assume that $\omega_1, \ldots, \omega_{n+q} \in \{v_1, \ldots, v_m\}$. Setting $v_i = \omega_i$ for $i = 1, \ldots, n+q$ and $m = N$, we have the claim of (i) as well as the first claim of (ii).

To prove (ii), we consider any monomial $x^\alpha y^\beta$ in S-normal form. Then we have $\alpha - \beta \in \mathcal{R}$ which can be written as

$$\alpha - \beta = \sum_{j=1}^{q} c_j \rho^{(j)} \quad (c_j \in \mathbf{Z}).$$

Therefore, we have the expression

$$x^\alpha y^\beta = x^{\alpha-\beta}(xy)^\beta, \quad x^{\alpha-\beta} = \prod_{j=1}^{q}(x^{\rho^{(j)}})^{c_j}, \tag{2.2}$$

where $(xy)^\beta = \prod_{j=1}^{n}(x_j y_j)^{\beta_j}$. Using (1.5), we can write $x^{\rho^{(j)}}$ in the form

$$x^{\rho^{(j)}} = x^{\rho_+^{(j)} - \rho_-^{(j)}} = \frac{x^{\rho_+^{(j)}} y^{\rho_-^{(j)}}}{(xy)^{\rho_-^{(j)}}} = \frac{\omega_{n+j}}{(xy)^{\rho_-^{(j)}}}. \tag{2.3}$$

Since the monomials $v_{n+q+1}, \ldots, v_N$ have the form $x^\alpha y^\beta$ with (2.2) and (2.3), we have the claim of (ii). □

By the arguments above, the monomials $v_1, \ldots, v_N$ have the expression of the form

$$v_k = \prod_{j=1}^{n+q} \omega_j^{\alpha_{kj}} \quad (\alpha_{kj} \in \mathbf{Z}), \quad k = 1, \ldots, N.$$

Here the integers α_{kj} are determined uniquely since $\omega_1, \ldots, \omega_{n+q}$ are functionally independent. The following formula will be useful in the proof of Theorem 1.

LEMMA 2. *We have*

$$\frac{\partial v_k}{\partial \omega_j} = \alpha_{kj}\frac{v_k}{\omega_j} \quad (k = 1, \ldots, N;\ j = 1, \ldots, n+q).$$

2.2. *Structure of Birkhoff normalization.* In this section, we clarify the structure of Birkhoff normalization which makes it possible to consider simultaneous normalization of $n+q$ integrals of X_H. Also, we show the existence of a formal Birkhoff transformation φ as the limit of the iteration procedure.

Let $\mathcal{P}$ (respectively $\mathcal{P}_m$) denote the Lie algebra of all formal power series at the origin (respectively all homogeneous polynomials of degree m) under the Poisson bracket $\{\cdot, \cdot\}$. For a quadratic form $f = S$ $(= \sum_{k=1}^{n} \lambda_k x_k y_k)$ or $f = H_2$, we consider the linear map (the adjoint representation of $\mathcal{P}$)

$$\operatorname{ad} f : \mathcal{P} \ni g \mapsto \{g, f\} \in \mathcal{P}.$$

Then the condition $g \in \operatorname{Ker} \operatorname{ad} S$ is equivalent to that g is in S-normal form. Let $\operatorname{ad}_m f$ be the restriction of $\operatorname{ad} f$ to $\mathcal{P}_m$, that is,

$$\operatorname{ad}_m f := \operatorname{ad} f | \mathcal{P}_m.$$

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The following lemma plays an important role in the construction of a normalizing transformation.

LEMMA 3. *We have:*
(i) $\mathcal{P}_m = \operatorname{Ker} \operatorname{ad}_m S \oplus \operatorname{Im} \operatorname{ad}_m S$, $\mathcal{P} = \operatorname{Ker} \operatorname{ad} S \oplus \operatorname{Im} \operatorname{ad} S$;
(ii) *the restriction of* $\operatorname{ad}_m H_2$ *to* $\operatorname{Im} \operatorname{ad}_m S$ *is an invertible map from* $\operatorname{Im} \operatorname{ad}_m S$ *onto itself;*
(iii) *if* $f, g \in \operatorname{Ker} \operatorname{ad} S$, *then* $\{f, g\} \in \operatorname{Ker} \operatorname{ad} S$;
(iv) *if* $f \in \operatorname{Ker} \operatorname{ad} S$ *and* $g \in \operatorname{Im} \operatorname{ad} S$, *then* $\{f, g\} \in \operatorname{Im} \operatorname{ad} S$.

Proof. This lemma is the same as [**9**, Lemma 3.1], to which we refer its proof. □

In view of (i) above, we introduce the projection operators

$$P_N : \mathcal{P} \to \operatorname{Ker} \operatorname{ad} S, \quad P_R : \mathcal{P} \to \operatorname{Im} \operatorname{ad} S \tag{2.4}$$

so that any function $f \in \mathcal{P}$ can be represented as $f = P_N f + P_R f$.

Now we consider the Birkhoff normalization. For any power series $f = f(z)$ with $f(0) = 0$, we use the notation

$$f = f^0 + f^1 + \cdots ; \quad f^0 \not\equiv 0 \quad (\deg f^0 = s \geq 1), \tag{2.5}$$

where f^d $(d = 0, 1, \ldots)$ are homogeneous polynomials in z of degree $s + d$. We call f^0 the *lowest-order part* of f. We say that f is in *S-normal form up to order* $s + d$ if $f^0 + \cdots + f^d$ is in S-normal form. The following lemma guarantees the simultaneous normalization of integrals.

LEMMA 4. *Let H be a function whose lowest-order part $H^0 = H_2$ is of the form (1.2). Let G be an integral of X_H and assume that H is in S-normal form up to order $s_0 + d$ ($s_0 = \deg H^0 = 2$). Then G is in S-normal form up to order $s + d$ ($s = \deg G^0$).*

Proof. By assumption, we have the identity $\{G, H\} = 0$. Its lowest-order part is $\{G^0, H^0\} = 0$. By Lemma 3, this identity holds only if $G^0 \in \operatorname{Ker} \operatorname{ad} S$. Comparing homogeneous parts in the identity $\{G, H\} = 0$, we can prove the assertion inductively. See [**9**, Proposition 3.2] for more details. □

By Lemma 4 and Proposition 1, we obtain Theorem 1(iii). To construct a Birkhoff transformation, we note the following fact.

LEMMA 5. (i) *Let φ be a symplectic transformation of the form $\varphi : z \mapsto z + O(|z|^{d+1})$. Then, it is written in the form*

$$\varphi(z) = z + J W_z(z) + O(|z|^{2d+1}) \tag{2.6}$$

with W being a polynomial of the form

$$W(z) = W^{d+2} + \cdots + W^{2d+1}, \tag{2.7}$$

where W^l is a homogeneous polynomial in z of degree l. The map φ is also expressed as

$$\varphi(z) = \exp X_W \circ \psi(z), \quad \psi(z) = z + O(|z|^{2d+1}),$$

where $\exp X_W$ is the time-one map of the flow of the Hamiltonian vector field X_W.

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(ii) *Let φ be a symplectic transformation of the form $\varphi\colon z \mapsto z + O(|z|^2)$. Then, it can be expressed as*

$$\begin{cases} \varphi = \varphi^{(\nu)} \circ \psi; \quad \varphi^{(\nu)} = \varphi_1 \circ \cdots \circ \varphi_\nu, \quad \varphi_\mu = \exp X_{W_\mu} (1 \le \mu \le \nu), \\ \psi(z) = z + O(|z|^{2^\nu + 1}), \end{cases} \tag{2.8}$$

where ν is an arbitrary positive integer and W_μ is a polynomial of the form (2.7) with $d = 2^{\mu-1}$ ($\nu = 1, 2, \ldots$).

Proof. The former assertion of (i) is proved easily by expressing φ in terms of the generating function (see [**7**, Lemma 2.7]). We have its latter assertion because $\exp X_W$ has the same form as (2.6). The claim of (ii) is a direct consequence of (i). □

In view of this lemma, we construct a formal Birkhoff transformation as composition of the infinite number of transformations of the form $\varphi = \exp X_W$ as follows.

PROPOSITION 2. *Let H be a holomorphic function of the form (1.1) with (1.2) and assume that it is in Birkhoff normal form up to order $s_0 + d - 1 (s_0 = 2, d \ge 1)$. Then there exists a unique polynomial W of the form*

$$W = W^{d+2} + \cdots + W^{2d+1} \quad \text{with } P_N W = 0, \tag{2.9}$$

with W^l being homogeneous polynomials of degree l, such that the transformation $\varphi = \exp X_W$ takes H into Birkhoff normal form up to order $s_0 + 2d - 1$.

We prove this proposition in the next section. Using this proposition successively, we have the formal existence of the desired transformation φ of Theorem 1.

3. *Proof of the existence of a convergent Birkhoff transformation*

In this section, we prove the existence of a convergent Birkhoff transformation of Theorem 1 in several steps. The proof follows the same line as in [**7, 9**], to which we refer for some details.

3.1. *Basic idea of using simultaneous normalization structure.*

Let $G_0 = H$ and let $G_1, \ldots, G_{n+q}$ be integrals of the vector field X_H which are holomorphic and functionally independent near the origin. Here one of those integrals $G_1, \ldots, G_{n+q}$ may be equal to H. By Ziglin's lemma [**23**] (see also [**7**, Appendix]), considering polynomials of these integrals with complex coefficients, we may assume that the lowest-order parts $G_1^0, \ldots, G_{n+q}^0$ are functionally independent.

Let $\deg G_i^0 = s_i \ge 1$, where we note $s_0 = 2$. Suppose that the Hamiltonian H is in Birkhoff normal form up to order $s_0 + d - 1$. Then it follows from Lemma 4 that G_i is in S-normal form up to order $s_i + d - 1$ and can be written in the form

$$G_i(z) = g_i(z) + \widehat{G}_i(z); \quad g_i = P_N g_i, \quad \widehat{G}_i = O(|z|^{s_i + d}), \tag{3.1}$$

where g_i is a polynomial in z of degree $\le s_i + d - 1$. Let $\varphi = \exp X_W$ be a symplectic transformation with W being a polynomial of the form (2.7). Then it can be written in the form (2.6) and we have

$$G_i \circ \varphi(z) = g_i(z) + \{g_i(z), W(z)\} + \widehat{G}_i(z) + O(|z|^{s_i + 2d}). \tag{3.2}$$

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As g_i is in S-normal form, it follows from Lemma 3(iv) that $P_N\{g_i, W\} = 0$ under the assumption $P_N W = 0$. Therefore, $G_i \circ \varphi$ is in S-normal form up to order $s_i + 2d - 1$ if and only if W satisfies

$$\{g_i(z), W(z)\} = -P_R \widehat{G}_i(z) + O(|z|^{s_i+2d}) \quad (i = 0, 1, \ldots, n+q). \tag{3.3}$$

By comparing homogeneous parts of degree $s_i + l$, we obtain the recursive relations

$$\{g_i^0, W^{l+2}\} = -P_R \widehat{G}_i^l - \sum_{\nu=1}^{l-d} \{g_i^\nu, W^{l+2-\nu}\} \quad (i = 0, 1, \ldots, n+q) \tag{3.4}$$

for $l = d, d+1, \ldots, 2d-1$.

Proof of Proposition 2. Let us consider (3.4) for $i = 0$. Suppose that there exist polynomials $W^{l+2}(l = d, \ldots, d+j-1)$ satisfying (3.4) and the condition $P_N W^{l+2} = 0$. Then, by Lemma 3, the right-hand side of (3.4) for $l = d + j$ belongs to $\operatorname{Im} \operatorname{ad}_{s_i+d+j} S$, and hence there exists a unique polynomial W^{d+2+j} such that (3.4) holds for $l = d + j$ and $P_N W^{d+2+j} = 0$. By induction, this implies the existence of a unique polynomial W of the form (2.9) satisfying (3.3) with $i = 0$. □

Next we consider (3.4) for $i = 1, \ldots, n+q$ to estimate the polynomial W determined above. Since g_i^ν are functions of $n+q$ variables $\omega_1, \ldots, \omega_{n+q}$, we have

$$\{g_i^\nu, W^{l+2-\nu}\} = \sum_{j=1}^{n+q} \frac{\partial g_i^\nu}{\partial \omega_j} \{\omega_j, W^{l+2-\nu}\} \quad (\nu = 0, 1, \ldots, l-d).$$

Here $\partial g_i^\nu / \partial \omega_j$ is not generally holomorphic at the origin. Let us consider the polynomial $g_i^\nu(z)$ in S-normal form as a polynomial of N variables $v_1, \ldots, v_N$ and write

$$g_i^\nu(z) = \boldsymbol{g}_i^\nu(v_1, \ldots, v_N).$$

Here the polynomial $\boldsymbol{g}_i^\nu$ may not be determined uniquely since our proof of Proposition 1 does not assert the minimality of the number N of the basis $v_1, \ldots, v_N$. However, it will turn out that our proof of Theorem 1 works because our estimates are independent of how the polynomials $\boldsymbol{g}_i^\nu$ are determined (see (3.29) and (3.30)).

Now we note

$$\frac{\partial g_i^\nu}{\partial \omega_j} = \sum_{k=1}^{N} \frac{\partial \boldsymbol{g}_i^\nu}{\partial v_k} \frac{\partial v_k}{\partial \omega_j} \quad (j = 1, \ldots, n+q). \tag{3.5}$$

Then, in view of Lemma 2, $(\partial g_i^\nu / \partial \omega_j)\omega_j$ are polynomials in z. We set

$$a_{ij}^\nu(z) := M(z) \frac{\partial g_i^\nu}{\partial \omega_j}, \quad M(z) = \prod_{j=1}^{n+q} \omega_j. \tag{3.6}$$

They are polynomials in z and hence holomorphic at the origin. Multiplying the both sides of (3.4) by $M(z)$, we have the system of $n+q$ equations

$$\sum_{j=1}^{n+q} a_{ij}^0(z) \{\omega_j, W^{l+2}\} = F_i^l(z) \quad (i = 1, \ldots, n+q), \tag{3.7}$$

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where

$$F_i^l(z) = -M(z)P_R\widehat{G}_i^l - \sum_{\nu=1}^{l-d}\sum_{j=1}^{n+q} a_{ij}^{\nu}(z)\{\omega_j,\, W^{l+2-\nu}\}. \tag{3.8}$$

Since $g_1^0, \ldots, g_{n+q}^0$ are functionally independent functions of z and can be considered as functions of $n+q$ variables $\omega_1, \ldots, \omega_{n+q}$, we have

$$\det(a_{ij}^0(z))_{i,j=1,\ldots,n+q} \neq 0 \quad \text{on a dense open subset } \Omega' \text{ of } \mathbf{C}^{2n}. \tag{3.9}$$

Summarizing, we have proved the following.

PROPOSITION 3. *In addition to the assumption of Proposition 2, let $G_0 = H$ and let G_i $(i = 1, \ldots, n+q)$ be holomorphic integrals of X_H. Then the following hold.*

(i) *The polynomial W in Proposition 2 satisfies the system of equations (3.3).*

(ii) *If the quadratic form S is of resonance degree q, then each homogeneous part $W^{l+2}(l = d, \ldots, 2d-1)$ of W satisfies (3.7) with (3.8).*

(iii) *If, in addition to the assumption of (ii), the lowest-order parts of $G_1, \ldots, G_{n+q}$ are functionally independent, then condition (3.9) holds.*

Furthermore, we can prove the following lemma.

LEMMA 6. *Let $G_i(i = 1, \ldots, n+q)$ be holomorphic functions of the form (3.1), and assume that their lowest-order parts are functionally independent. Let W be a polynomial of the form (2.9) whose homogeneous parts W^{l+2} satisfy (3.7) with (3.8) under condition (3.9). Then $D_k W^{l+2} := \{\omega_k, W^{l+2}\}$ can be expressed as follows:*

$$D_k W^{l+2} = \frac{q_k^l(z)}{p(z)} \quad (k = 1, \ldots, n+q;\ l = d, \ldots, 2d-1) \tag{3.10}$$

with

$$\begin{cases} p(z) = \det(a_{ij}^0(z))_{i,j=1,\ldots,Q}, \\ \\ q_k^l(z) = \det \begin{pmatrix} a_{11}^0 & \ldots & \overset{(k)}{F_1^l} & \ldots & a_{1Q}^0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{Q1}^0 & \ldots & F_Q^l & \ldots & a_{QQ}^0 \end{pmatrix}, \quad (Q = n+q). \end{cases} \tag{3.11}$$

Here the polynomials $q_k^l(z)$ are divisible by $p(z)$.

Proof. Let us consider the system of $n+q$ equations (3.7) as linear equations for $D_k W^{l+2}(k = 1, \ldots, n+q)$ under condition (3.9). Then, at any point $z \in \Omega'$ fixed, one can solve (3.7) uniquely as (3.10) with (3.11) and, hence, the polynomial $W(z)$ in Proposition 2 satisfies (3.10) at any point $z \in \Omega'$. Since Ω' is dense and open in $\mathbf{C}^{2n}$, this implies that the equalities

$$D_k W^{l+2}(z)\, p(z) = q_k^l(z) \quad (k = 1, \ldots, n+q;\ l = d, \ldots, 2d-1)$$

hold for any $z \in \mathbf{C}^{2n}$. This means that the polynomials $q_k^l(z)$ are divisible by $p(z)$ and expression (3.10) holds. □

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3.2. *Estimates of W.* We make use of expression (3.10) to obtain the estimates of W and its derivatives. We first determine domains where we consider those estimates. We note the following lemma.

LEMMA 7. *Let s be the degree of the polynomial $p(z)$ given in (3.11). Then there exist constants δ_i $(i = 1, \ldots, 2n)$ such that $0 < \delta_i < 1$ and*

$$|p(z)| \geq c_1 r^s \quad \text{on } \Delta_r := \{z \in \mathbf{C}^{2n} \mid |z_i| = \delta_i r \ (i = 1, \ldots, 2n)\}, \tag{3.12}$$

where $z = (z_1, \ldots, z_{2n})$ and $c_1 > 0$ is a constant which is independent of r.

Proof. This is the same as [**9**, Lemma 5.1], to which we refer its proof (see also the proof of [**7**, Lemma 4.1]). □

Using the constants $\delta_1, \ldots, \delta_{2n}$ given above, we introduce the polydisks

$$\Omega_r := \{z \in \mathbf{C}^{2n} \mid |z_i| < \delta_i r \ (i = 1, \ldots, 2n)\},$$

where $r > 0$ is considered as a small parameter. Let $A(\Omega_r)$ be the space of power series in z which are absolutely convergent on $\overline{\Omega}_r$, where $\overline{\Omega}_r$ is the closure of Ω_r. Writing a function $f \in A(\Omega_r)$ in the form (2.5), we define

$$|f|_r := \max_{z \in \overline{\Omega}_r} |f(z)|, \quad \|f\|_r := \sum_{d=0}^{\infty} |f^d|_r, \quad \|f\|_{r,m} := \frac{\|f\|_r}{r^m},$$

where m is an arbitrary integer.

We note that the maximum $\|D_k W^{l+2}\|_r (= |D_k W^{l+2}|_r)$ is attained on Δ_r. Then it follows from expressions (3.10) and (3.12) that

$$\|D_k W^{l+2}\|_r \leq \frac{\|q_k^l(z)\|_r}{\min_{z \in \Delta_r} |p(z)|} \leq \frac{1}{c_1} \|q_k^l(z)\|_{r,s}. \tag{3.13}$$

Using this formula, we derive the estimate of $\|D_k W^{l+2}\|_r$. To this end, we introduce the following constants:

$$d_k = \deg \omega_k, \quad m = \deg M(z) \left(= \sum_{k=1}^{n+q} d_k\right), \quad m_{ik} = m + s_i - d_k,$$
$$(i, k = 1, \ldots, n+q). \tag{3.14}$$

Then we have the following lemma.

LEMMA 8. *The homogeneous polynomial W^{l+2} in Lemma 6 satisfies*

$$\|D_k W^{l+2}\|_r \leq c_2 \sum_{i=1}^{n+q} \|F_i^l\|_{r,m_{ik}} \quad (k = 1, \ldots, n+q;\ l = d, \ldots, 2d-1),$$

where c_2 is a positive constant which is independent of r.

Proof. Expanding the determinant $q_k^l(z)$ in (3.11) according to the kth column, we obtain

$$q_k^l(z) = \sum_{i=1}^{n+q} (-1)^{i+k} F_i^l(z) \det Q_{ik}(z),$$

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where $Q_{ik}(z)$ denotes the matrix obtained from $(a^0_{ij}(z))_{i,j=1,\dots,n+q}$ by deleting the ith row and kth column. From (3.6), we see that $a^0_{ik}(z)$ is a homogeneous polynomial of degree $m_{ik} = m + s_i - d_k$ or $a^0_{ik}(z) \equiv 0$ and, hence, $\det Q_{ik}(z)$ is a homogeneous polynomial of degree $s - m_{ik}$ or $\det Q_{ik}(z) \equiv 0$. Therefore, there exists a positive constant c_3 such that

$$\|\det Q_{ik}(z)\|_{r,s-m_{ik}} \le c_3 \quad (i,k = 1,\dots,n+q).$$

Hence, we have

$$\|q^l_k(z)\|_r \le r^s \sum_{i=1}^{n+q} \|F^l_i(z)\|_{r,m_{ik}} \|\det Q_{ik}(z)\|_{r,s-m_{ik}} \le c_3 r^s \sum_{i=1}^{n+q} \|F^l_i(z)\|_{r,m_{ik}}.$$

Here we note that $F^l_i(z)$ is a homogeneous polynomial of degree $m + s_i + l > m + s_i - d_k$. Setting $c_2 = c_3/c_1$, the assertion follows from this and (3.13). □

To proceed further, let us introduce some notation. Given a power series $f = \sum_{\alpha,\beta} c_{\alpha\beta} x^\alpha y^\beta$, we denote by $\widetilde{f}$ the power series obtained from f with its coefficients replaced by their absolute values, namely,

$$\widetilde{f} := \sum_{\alpha,\beta} |c_{\alpha\beta}| x^\alpha y^\beta.$$

We introduce further notation

$$|||\partial \widehat{g}|||_r := \sum_{i,j=1}^{n+q} \left\| M(z) \frac{\partial(\widetilde{g_i} - \widetilde{g^0_i})}{\partial \omega_j} \right\|_{r,m_{ij}} \left(= \sum_{i,j=1}^{n+q} \sum_{l=1}^{d-1} \|\widetilde{a}^l_{ij}(z)\|_{r,m_{ij}} \right) \tag{3.15}$$

and

$$|||\widehat{G}|||_r := \sum_{i=1}^{n+q} \|\widetilde{G_i} - \widetilde{g_i}\|_{r,s_i-2}, \tag{3.16}$$

where we write $G_i - g_i$ instead of $\widehat{G}_i$ for notational reason. The following is the key lemma in the proof of Theorem 1.

LEMMA 9. *In addition to the assumption of Lemma 6, assume that* $G_i \in A(\Omega_r)$ $(i = 1,\dots,n+q)$ *and that*

$$c_2 |||\partial \widehat{g}|||_r < \tfrac{1}{2}. \tag{3.17}$$

Then the polynomial $W(z)$ *in Lemma 6 satisfies*

$$\|D_k W\|_{r,d_k-2} \le 2c_2 |||\widehat{G}|||_r \quad (k = 1,\dots,n+q). \tag{3.18}$$

Proof. We introduce the notation

$$\|\partial \widehat{g}\|_r := \sum_{i,j=1}^{n+q} \left\| M(z) \frac{\partial(g_i - g^0_i)}{\partial \omega_j} \right\|_{r,m_{ij}}$$

and set

$$U = c_2 \sum_{l=0}^{d-1} (c_2 \|\partial \widehat{g}\|_r)^l, \qquad V = \sum_{i=1}^{n+q} \sum_{l=d}^{2d-1} \|P_R G^l_i\|_{r,s_i-2},$$

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where we note that $G_i^l = \widehat{G}_i^l$. By (3.6) and (3.14), we have $\|a_{ij}^l\|_{r,m_{ij}} = O(r^l)$ and $\|P_R G_i^l\|_{r,s_i-2} = O(r^{l+2})$ as $r \to 0$. Motivated by this fact, we write

$$u_{ij}^l := \|a_{ij}^l\|_{r,m_{ij}}, \quad v_i^{l+2} := \|P_R G_i^l\|_{r,s_i-2}$$

and consider U and V as polynomials in u_{ij}^l and v_i^{l+2}, respectively. In view of (3.15), any monomial in U takes the form

$$(\text{constant}) \prod_{i,j=1}^{n+q} u_{ij}^{\alpha_{ij}}$$

with non-negative integers α_{ij}. Let us define the degree of this monomial by $\mu = \sum_{i,j=1}^{n+q} \alpha_{ij}$ and denote by U^μ the sum of monomials (homogeneous polynomial) of degree μ in U. We define the homogeneous polynomial V^μ in the same way.

We now prove

$$\|D_k W^{l+2}\|_{r,d_k-2} \le \sum_{\mu+\lambda=l+2} U^\mu V^\lambda \quad (k = 1, \dots, n+q) \tag{3.19}$$

inductively for $l = d, \dots, 2d-1$. Let $l = d$. Then, it follows from Lemma 8 that

$$\|D_k W^{d+2}\|_r \le c_2 \sum_{i=1}^{n+q} \|F_i^d\|_{r,m_{ik}}.$$

Here $F_i^d = -M(z) P_R G_i^d$ by (3.8) and $\|M(z)\|_r \le r^m$. Therefore, we have

$$\|F_i^d\|_{r,m_{ik}} \le \frac{\|P_R G_i^d\|_r}{r^{s_i-d_k}},$$

which leads to

$$\|D_k W^{d+2}\|_{r,d_k-2} \le c_2 \sum_{i=1}^{n+q} \|P_R G_i^d\|_{r,s_i-2} = U^0 V^{d+2}.$$

Suppose that (3.19) holds for $l = d, d+1, \dots, d+\kappa-1$. Then, using Lemma 8 and (3.8) again, for $l = d+\kappa$ we have

$$\|D_k W^{l+2}\|_{r,d_k-2} \le c_2 \sum_{i=1}^{n+q} \left(\|P_R G_i^l\|_{r,s_i-2} + \sum_{\nu=1}^{\kappa} \sum_{j=1}^{n+q} \|a_{ij}^\nu\|_{r,m_{ij}} \|D_j W^{l+2-\nu}\|_{r,d_j-2} \right).$$

This leads to the estimate (3.19) for $l = d+\kappa$. It can be proved in the same way as in the proof of [**7**, Proposition 4.3], and so we omit it. Clearly estimate (3.19) implies

$$\|D_k W\|_{r,d_k-2} \le UV$$

and by the relations $\|\partial \widehat{g}\|_r \le \|\partial \widehat{g}\|\!\|_r$ and $\|P_R G_i^l\|_{r,s_i-2} \le \|\widetilde{G}_i^l\|_{r,s_i-2}$, estimate (3.18) follows from this inequality and (3.17). □

Now we derive the estimates of W and its derivatives by using Lemma 9. For a power series $f \in \mathcal{P}$, let $\widehat{P}_N f$ denote the sum of all terms in f consisting of $\omega_1, \dots, \omega_n$ only. It is given by

$$\widehat{P}_N f(x, y) = \int_0^1 \cdots \int_0^1 f(e^{2\pi i\theta} x, e^{-2\pi i\theta} y)\, d\theta_1 \cdots d\theta_n, \tag{3.20}$$

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where $i = \sqrt{-1}$, $\theta_j \in \mathbf{R}$ and $e^{2\pi i\theta}x$ and $e^{-2\pi i\theta}y$ are n-dimensional vectors defined by

$$e^{2\pi i\theta}x := (e^{2\pi i\theta_1}x_1, \ldots, e^{2\pi i\theta_n}x_n), \quad e^{-2\pi i\theta}y := (e^{-2\pi i\theta_1}y_1, \ldots, e^{-2\pi i\theta_n}y_n).$$

We introduce the constants

$$c_4 := 4\pi c_2 n, \quad \delta := \min_{1\le k\le 2n} \delta_k.$$

Then we have the following lemma.

LEMMA 10. *Let $0 < \rho < r$. Under the assumption of Lemma 9, the polynomial W in Lemma 6 satisfies the following estimates:*

(i) $\|W\|_r \le c_4 \|\widehat{G}\|_r$;

(ii) $\|W_{z_i}\|_\rho \le (c_4/\delta(r-\rho))\|\widehat{G}\|_r$ $(i = 1, \ldots, 2n)$;

(iii) $\|W_{z_i z_j}\|_\rho \le (4c_4/\delta^2(r-\rho)^2)\|\widehat{G}\|_r$ $(i, j = 1, \ldots, 2n)$.

Proof. Let $z = (x, y) \in \Omega_r$ be taken arbitrarily and fixed. We consider the function

$$\widehat{W}^{l+2}(\theta) := W^{l+2}(e^{2\pi i\theta}x, e^{-2\pi i\theta}y), \quad \theta = (\theta_1, \ldots, \theta_n) \in [0, 1]^n.$$

Here we note that $(e^{2\pi i\theta}x, e^{-2\pi i\theta}y) \in \Omega_r$ and $\widehat{W}^{l+2}(0) = W^{l+2}(z)$. Then by the mean value theorem we have

$$|\widehat{W}^{l+2}(\theta) - \widehat{W}^{l+2}(0)| \le \sum_{k=1}^{n} \max_{\theta\in[0,1]^n} \left|\frac{\partial \widehat{W}^{l+2}}{\partial\theta_k}\right| \le 2\pi \sum_{k=1}^{n} |\{W^{l+2}, \omega_k\}|_r.$$

In view of (3.20) and by the condition $P_N W^{l+2} = 0$, the integration of the above with respect to $\theta_1, \ldots, \theta_n$ on $[0, 1]$ gives the estimate

$$|W^{l+2}(z)| = |\widehat{W}^{l+2}(0)| \le 2\pi \sum_{k=1}^{n} |D_k W^{l+2}|_r.$$

Since $d_k = 2$ for $k = 1, \ldots, n$ in (3.18), this implies estimate (i). Estimates (ii) and (iii) follow from Cauchy's estimate. □

Remark. In the proof of (i) above, we do not need the estimates of $D_k W^{l+2}$ with $k = n+1, \ldots, n+q$. However this does not imply the uselessness of those estimates. In fact, the proof of Lemma 9 shows that we need the estimates of $D_j W^{l+2-\nu}$ $(\nu = 1, \ldots, l-d)$ with $j = n+1, \ldots, n+q$ to get the estimate of $D_k W^{l+2}$ for $k = 1, \ldots, n$.

3.3. *Estimates at one iteration step.* Next we derive the estimates for the transformation φ and for new remainder and normal form parts at one iteration step. They will be described in Lemmas 12 and 13.

First we introduce a constant

$$c_5 = 2nc_4\delta^{-2}.$$

Then we have the following lemma.

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LEMMA 11. *In addition to the assumption of Lemma 6, assume that* $G_i \in A(\Omega_r)$ $(i = 1, \ldots, n+q)$ *satisfy* $c_2 |||\partial \widehat{g}|||_r < \frac{1}{2}$ *and* $c_5 (r-\rho)^{-2} |||\widehat{G}|||_r < 1$. *Let* $0 < \sigma < \rho < r$ *with* $\rho - \sigma = r - \rho$. *Then, for the polynomial* W *in Lemma 6, the time-one map* $\varphi = \exp X_W$ *defines a holomorphic transformation from* Ω_σ *into* Ω_ρ.

Proof. By assumption and Lemma 10(ii), we have $\|W_{z_k}\|_\rho < (1/2n)\delta(r-\rho)$. Therefore, the assertion follows from the existence theorem for solutions of ordinary differential equations. See the proof of [**7**, Lemma 5.1]. □

We turn to the estimate of new remainder terms. Let us recall formulae (3.2) and (3.3), which can be rewritten as

$$G_i \circ \varphi(z) = g_i(z) + \{g_i(z), W(z)\} + \widehat{G}_i(z) + R_i^1(z); \quad R_i^1(z) = O(|z|^{s_i+2d}) \tag{3.21}$$

with

$$\{g_i(z), W(z)\} + \widehat{G}_i(z) = P_N^{2d-1}\widehat{G}_i(z) + R_i^2(z); \quad R_i^2(z) = O(|z|^{s_i+2d}), \tag{3.22}$$

where

$$P_N^{2d-1}\widehat{G}_i := P_N(G_i^d + G_i^{d+1} + \cdots + G_i^{2d-1}).$$

Then $G_i \circ \varphi$ is written as

$$G_i \circ \varphi(z) = g_i'(z) + \widehat{G}_i'(z)$$

with

$$g_i'(z) = g_i(z) + P_N^{2d-1}\widehat{G}_i(z), \quad \widehat{G}_i'(z) = R_i^1(z) + R_i^2(z). \tag{3.23}$$

Here we assume that $z \in \Omega_\sigma$ in view of Lemma 11.

Let us introduce the notation

$$|||g|||_r := \sum_{i=1}^{n+q} \|\widetilde{g}_i\|_{r,s_i-2}.$$

Here we note that $\widetilde{g}_i$ begins with terms of order $s_i \geq 1$. Our purpose is to derive from (3.21)–(3.23) the estimates of $|||g'|||_{r'}$ and $|||\widehat{G}'|||_{r'}$ for some positive value r' smaller than r. To this end, we introduce a constant c_6 satisfying

$$\left(\frac{r}{r'}\right)^{s_i-2} \leq c_6 \quad (i = 1, \ldots, n+q) \tag{3.24}$$

and a constant $A > 0$ satisfying

$$|||g|||_r \leq Ar^2.$$

Furthermore, we set

$$c_7 = \frac{5^{2n+5} \cdot 10}{3} c_5^2.$$

Then, the desired estimates can be obtained in the same way as in [**9**, pp. 433–435] and are summarized as follows.

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LEMMA 12. *In addition to the assumption of Lemma 6, assume that* $G_i \in A(\Omega_r)$ $(i = 1, \dots, n+q)$ *satisfy*

$$\|\!|g|\!\|_r \le Ar^2, \quad c_2\|\!|\partial\widehat{g}|\!\|_r < \tfrac{1}{2} \quad \text{and} \quad c_5(r-\rho)^{-2}\|\!|\widehat{G}|\!\|_r < 1. \tag{3.25}$$

Let W *be the polynomial in Lemma 6, and let* $0 < r' < r$ *and* $\sigma = r - \frac{2}{5}(r-r')$, $\rho = r - \frac{1}{5}(r-r')$. *Then* $\varphi = \exp X_W$ *is a holomorphic symplectic transformation from* Ω_σ *into* Ω_ρ, *and the normal form part* g_i' *and the remainder part* $\widehat{G}_i'$ *of* $G_i \circ \varphi$ *satisfy the following estimates:*

$$\|\!|g'|\!\|_{r'} \le \left(\frac{r'}{r}\right)^2 \left\{ \|\!|g|\!\|_r + \|\!|\widehat{G}|\!\|_r \left(\frac{r'}{r}\right)^d \right\}, \tag{3.26}$$

$$\|\!|\widehat{G}'|\!\|_{r'} \le c_6c_7(A+1)\|\!|\widehat{G}|\!\|_r \left\{ \frac{\|\!|\widehat{G}|\!\|_r}{r^2(1-(r'/r))^{2n+5}} + \frac{(5r'/(4r'+r))^{2d+2}}{(1-(r'/r))^{2n+2}} \right\}. \tag{3.27}$$

Finally, we derive the estimate of $\|\!|\partial\widehat{g}'|\!\|_{r'}$. By (3.15) and (3.23), we have

$$\|\!|\partial\widehat{g}'|\!\|_{r'} = \|\!|\partial\widehat{g}|\!\|_r + \sum_{i,j=1}^{n+q} \sum_{l=d}^{2d-1} \left\| M(z) \frac{\partial \widetilde{g}_i^l}{\partial \omega_j} \right\|_{r',m_{ij}} \quad (\widetilde{g}_i^l = P_N \widetilde{G}_i^l).$$

We now consider $\widetilde{g}_i(z)$ as a function $\widetilde{\boldsymbol{g}}_i$ of $v_1, \dots, v_N$. Then we have identity (3.5). Let us define functions $f_j(z)$ by $v_j = f_j(z)$ and introduce positive constants β_j $(0 < \beta_j < 1)$ by

$$f_j(\delta_1 r, \delta_2 r, \dots, \delta_{2n} r) = \beta_j r^{d_j} \quad (j = 1, \dots, N), \tag{3.28}$$

where d_j is the degree of the monomial v_j in $z_1, \dots, z_{2n}$. Then, introducing the domain

$$\Pi_r = \{v = (v_1, \dots, v_N) \in \mathbf{C}^N \mid |v_j| < \beta_j r^{d_j} (j = 1, \dots, N)\},$$

we have the following estimate

$$|\widetilde{\boldsymbol{g}}_i^l|_r := \max_{v \in \overline{\Pi}_r} |\widetilde{\boldsymbol{g}}_i^l(v)| \le |\widetilde{g}_i^l(z)|_r. \tag{3.29}$$

We set $\beta = \prod_{\nu=1}^{n+q} \beta_\nu$ and note the definitions of the constants (3.14). Let $0 < \rho < r$ again. Then, by Lemma 2 and the definition (3.6) of $M(z)$ and using Cauchy's estimate, we have

$$\begin{aligned} \left\| M(z) \frac{\partial \widetilde{\boldsymbol{g}}_i^l}{\partial v_k} \frac{\partial v_k}{\partial \omega_j} \right\|_\rho &\le \beta\rho^m \cdot \frac{|\widetilde{\boldsymbol{g}}_i^l|_r}{\beta_k(r^{d_k} - \rho^{d_k})} \cdot \frac{|\alpha_{kj}|\beta_k\rho^{d_k}}{\beta_j\rho^{d_j}} \\ &\le \frac{\beta|\alpha_{kj}|}{\beta_j} \cdot \frac{\|\widetilde{g}_i^l\|_r}{1-(\rho/r)} \cdot \rho^{m-d_j} \\ &\le \frac{\beta|\alpha_{kj}|}{\beta_j} \cdot \frac{\|\widetilde{g}_i^l\|_{r,s_i-2}}{1-(\rho/r)} \cdot \rho^{m_{ij}-2} \left(\frac{r}{\rho}\right)^{s_i-2}, \end{aligned} \tag{3.30}$$

where we used the relations $(\rho/r)^{d_k} \le \rho/r < 1$ to obtain the second inequality. Then, by replacing ρ with r' and using (3.24), it follows from (3.5) that

$$\left\| M(z) \frac{\partial \widetilde{g}_i^l}{\partial \omega_j} \right\|_{r',m_{ij}} \le \sum_{k=1}^{N} \frac{c_6\beta|\alpha_{kj}|}{\beta_j} \cdot \frac{\|\widetilde{g}_i^l\|_{r,s_i-2}}{(r')^2(1-(r'/r))}.$$

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Therefore, introducing a new constant

$$c_8 = c_6\beta \sum_{j=1}^{n+q} \sum_{k=1}^{N} \frac{1}{\beta_j} |\alpha_{kj}|,$$

we have

$$\sum_{i,j=1}^{n+q} \sum_{l=d}^{2d-1} \left\| M(z) \frac{\partial \widetilde{g}_i^l}{\partial \omega_j} \right\|_{r',m_{ij}} \leq \frac{c_8 |||\widehat{G}|||_r}{(r')^2(1-(r'/r))}.$$

Hence, we have proved the following lemma.

LEMMA 13. *Under the assumption of Lemma 12, we have*

$$|||\partial \widehat{g}'|||_{r'} \leq |||\partial \widehat{g}|||_r + \frac{c_8 |||\widehat{G}|||_r}{(r')^2(1-(r'/r))}. \tag{3.31}$$

3.4. *Convergence proof.* Finally we consider iteration of the transformations described above to obtain the desired convergent Birkhoff transformation. To this end, let $r_0 > 0$ and $G_k \in \mathcal{A}(\Omega_{r_0})$ for $k = 0, 1, \ldots, n+q$. Let us define a sequence $\{r_\nu\}$ by

$$r_\nu = \frac{r_0}{2}\left(1 + \frac{1}{\nu+1}\right) \quad \nu = 0, 1, 2, \ldots,$$

and consider the νth step by setting

$$r = r_\nu, \quad r' = r_{\nu+1}, \quad A = A_\nu, \quad d = 2^\nu,$$

and

$$g_i = g_i^{(\nu)}, \quad g_i' = g_i^{(\nu+1)}, \quad \widehat{G}_i = \widehat{G}_i^{(\nu)}, \quad \widehat{G}_i' = \widehat{G}_i^{(\nu+1)}, \quad \varphi = \varphi_\nu.$$

Then it follows from Lemma 12 that

$$\varphi_\nu : \Omega_{\sigma_\nu} \to \Omega_{\rho_\nu}; \quad \sigma_\nu = r_\nu - \tfrac{2}{5}(r_\nu - r_{\nu+1}), \quad \rho_\nu = r_\nu - \tfrac{1}{5}(r_\nu - r_{\nu+1})$$

and that one can give the estimate of $|||g^{(\nu+1)}|||_{r_{\nu+1}}$ and $|||\widehat{G}^{(\nu+1)}|||_{r_{\nu+1}}$. For this sequence $\{r_\nu\}$, we have

$$1 - \frac{r_{\nu+1}}{r_\nu} = \frac{1}{(\nu+2)^2}$$

and, therefore,

$$1 < \frac{r_\nu}{r_{\nu+1}} \leq \frac{4}{3}.$$

This implies that the constant c_6 satisfying (3.24) can be chosen independently of r_ν. For later use, we choose the constants c_2 and c_6 such that

$$c_2 > 1, \quad c_6 \geq (\tfrac{4}{3})^2.$$

In order to justify the iteration procedure with this sequence, we have to prove that the three conditions of (3.25) hold at each step. For this purpose, let us take a constant $A_0 > 0$ such that

$$|||g^{(0)}|||_{r_0} \leq A_0 r_0^2.$$

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We now set

$$c_9 = c_6 c_7 (c_8 + 1)(A_0 + \tfrac{3}{2})$$

and consider the condition

$$c_9 \epsilon_\nu < 1 \quad (\nu = 0, 1, \ldots); \qquad \epsilon_\nu = \frac{|||\widehat{G}^{(\nu)}|||_{r_\nu}}{r_\nu^2 (1 - (r_{\nu+1}/r_\nu))^{2n+5}}. \tag{3.32}$$

Since $\widehat{G}_i^{(0)}$ begins with terms of order $\geq s_i + 1$, it follows from the definition (3.16) of $|||\widehat{G}^{(0)}|||_{r_0}$ that $\epsilon_0 > 0$ can be arbitrarily small if $r_0 > 0$ is chosen appropriately. Suppose that condition (3.32) holds for $\nu = 0, 1, \ldots, m$. Then, by estimates (3.26)–(3.27) and (3.31) together with the definitions of the constants, it turns out that the three conditions

$$|||g^{(\nu)}|||_{r_\nu} < (A_0 + \tfrac{1}{2}) r_\nu^2, \quad c_2 |||\partial \widehat{g}^{(\nu)}|||_{r_\nu} < \tfrac{1}{2}, \quad c_5 (r_\nu - \rho_\nu)^{-2} |||\widehat{G}^{(\nu)}|||_{r_\nu} < 1$$

hold for $\nu = 0, 1, \ldots, m+1$ (see [**9**, Lemma 7.1]) and, by estimate (3.27), we have

$$\epsilon_{\nu+1} \leq c \epsilon_\nu (\epsilon_\nu + \lambda_\nu) \quad (\nu = 0, 1, \ldots, m),$$

where

$$c = c_9 \left(\frac{4}{3}\right)^2 \left(\frac{3}{2}\right)^{2(2n+5)}, \qquad \lambda_\nu = (\nu + 2)^{2(2n+2)} \left(1 - \frac{1}{5(\nu+2)^2 - 4}\right)^{2^{\nu+1}+2}.$$

Here $\lambda_\nu \to 0$ as $\nu \to \infty$, and therefore, one can prove in the same way as in [**9**, §7] that, if $r_0 > 0$ is chosen sufficiently small, there exists a positive integer N such that

$$\begin{cases} c\epsilon_\nu < (4c)^{-1} & \text{for } \nu = 0, 1, \ldots, N, \\ c\epsilon_\nu < (4c)^{-1} 2^{-\nu+N} & \text{for } \nu \geq N, \end{cases} \tag{3.33}$$

proving condition (3.32) for all $\nu \geq 0$ and, hence, justifying the iteration procedure. Finally in view of (3.33), deriving and using the estimate of the derivatives of φ_ν, one can prove the uniform convergence of the sequence of mappings

$$\varphi^{(\nu)} = \varphi_0 \circ \varphi_1 \circ \cdots \circ \varphi_\nu \colon \Omega_{\sigma_\nu} \to \Omega_{\rho_0} \quad (\nu = 0, 1, \ldots)$$

to a symplectic mapping which is holomorphic in the domain $\Omega_{r_0/2}$. See [**7**, §6] for details. We have thus completed the proof of convergence of the Birkhoff transformation.

4. *The special Birkhoff normal form and its uniqueness*

In this section, we complete the proof of Theorem 1 by showing that the Birkhoff normal form $H \circ \varphi$ becomes a holomorphic function of $n - q$ variables $\tau_{q+1}, \ldots, \tau_n$ and satisfies Theorem 1(i).

Let $G_0 = H$ and let $G_1, \ldots, G_{n+q}$ be integrals of the vector field X_H. Let φ be the convergent Birkhoff transformation obtained in the previous section. Then the functions $G_k \circ \varphi$ $(k = 0, 1, \ldots, n+q)$ are in S-normal form and holomorphic in the domain $\Omega_{r_0/2}$. Let

$$R_k = \beta_k R^{d_k} \quad (k = 1, \ldots, N) \quad \text{with } R = r_0/2,$$

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where β_k are the constants defined by (3.28). Then $G_k \circ \varphi$ are holomorphic functions of $v_1, \ldots, v_N$ in the domain

$$\Pi_R = \{(v_1, \ldots, v_N) \in \mathbf{C}^N \mid |v_k| < R_k \ (k = 1, \ldots, N)\}.$$

Writing G_k for $G_k \circ \varphi$, let us consider G_k as function of $n+q$ variables $\omega_1, \ldots, \omega_{n+q}$. Then we have the following lemma.

LEMMA 14. *Let λ be an arbitrary constant such that $0 < \lambda < 1$ and let $R_k(\lambda) := \beta_k(\lambda R)^{d_k}$ $(k = 1, \ldots, N)$. Then there exists a constant $\delta > 0$ such that $R_k(\lambda) + \delta < R_k$ and that G_k $(k = 0, 1, \ldots, n+q)$ are holomorphic functions of $\omega = (\omega_1, \ldots, \omega_{n+q})$ in the domain*

$$\Omega_\delta(\lambda) := \{\omega \in \mathbf{C}^{n+q} \mid R_k(\lambda) - \delta < |\omega_k| < R_k(\lambda) + \delta \ (k = 1, \ldots, n+q)\}.$$

They can be written as absolutely convergent Laurent series in $\Omega_\delta(\lambda)$.

Proof. Recall that v_k are monomials in $z_1, \ldots, z_{2n}$ and are also expressed as the quotients of two monomials in $\omega_1, \ldots, \omega_{n+q}$ Then, if we write $v_k(\omega_1, \ldots, \omega_{n+q})$ for v_k as functions of $\omega_1, \ldots, \omega_{n+q}$, we have the relations

$$|v_k| = v_k(|\omega_1|, \ldots, |\omega_{n+q}|) \quad (k = 1, \ldots, N)$$

and $v_k(\beta_1 r^{d_1}, \ldots, \beta_{n+q} r^{d_{n+q}}) = \beta_k r^{d_k}$ for any constant r with $0 < r \le R$. Since $v_k(|\omega_1|, \ldots, |\omega_{n+q}|)$ are continuous functions of $|\omega_1|, \ldots, |\omega_{n+q}|$ in a real neighbourhood of $(R_1(\lambda), \ldots, R_{n+q}(\lambda))$, there exists a constant $\delta > 0$ such that

$$|v_k(\omega_1, \ldots, \omega_{n+q})| < R_k \quad (k = 1, \ldots, N) \quad \text{for } (\omega_1, \ldots, \omega_{n+q}) \in \Omega_\delta(\lambda).$$

Since G_k are holomorphic functions of $v_1, \ldots, v_N$ in Π_R, this implies that G_k are holomorphic functions of $\omega = (\omega_1, \ldots, \omega_{n+q})$ in $\Omega_\delta(\lambda)$. □

We now prove that H depends only on $n-q$ variables $\tau_{q+1}, \ldots, \tau_n$. Since $G_1, \ldots, G_{n+q}$ are integrals of X_H, we have the identities $\{H, G_k\} = 0$ $(k = 1, \ldots, n+q)$. They lead to

$$\{H, G_k\} = \sum_{i=1}^{n+q} \left(\sum_{j=1}^{n+q} \frac{\partial G_k}{\partial \omega_j} \{\omega_i, \omega_j\} \right) \frac{\partial H}{\partial \omega_i} = 0 \quad (k = 1, \ldots, n+q),$$

which can be written in vector form

$$DG(\omega) A H_\omega = 0.$$

Here

$$DG(\omega) = \left(\frac{\partial G_i}{\partial \omega_j} \right)_{i,j=1,\ldots,n+q}, \quad A = (\{\omega_i, \omega_j\})_{i,j=1,\ldots,n+q}, \quad H_\omega = \begin{pmatrix} \partial H/\partial \omega_1 \\ \vdots \\ \partial H/\partial \omega_{n+q} \end{pmatrix}.$$

The functional independence of $G_1, \ldots, G_{n+q}$ implies that the matrix $DG(\omega)$ is non-singular on a dense open subset of $\Omega_\delta(\lambda)$ for any λ fixed. Therefore, it follows from the identity above that

$$A H_\omega = 0. \tag{4.1}$$

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By a simple computation, we see that

$$\begin{cases} \{\omega_i, \omega_j\} = 0 & (i, j = 1, \dots, n), \\ \{\omega_i, \omega_{n+j}\} = -\rho_i^{(j)} \omega_{n+j} & (i = 1, \dots, n,\ j = 1, \dots, q). \end{cases} \tag{4.2}$$

Therefore, the comparison of the first n rows of (4.1) gives the identity

$$A_1 \begin{pmatrix} \partial H/\partial \omega_{n+1} \\ \vdots \\ \partial H/\partial \omega_{n+q} \end{pmatrix} = 0 \quad \text{with } A_1 = (-\rho_i^{(j)} \omega_{n+j})_{i=1,\dots,n;\, j=1,\dots,q},$$

which can be written as

$$\sum_{j=1}^{q} \left(\omega_{n+j} \frac{\partial H}{\partial \omega_{n+j}} \right) \rho^{(j)} = 0.$$

Since $\rho^{(1)}, \dots, \rho^{(q)}$ are linearly independent, this implies

$$\frac{\partial H}{\partial \omega_{n+j}} = 0 \quad (j = 1, \dots, q) \tag{4.3}$$

and, hence, H is a Laurent series in $\omega_1, \dots, \omega_n$. Here H is also a power series in $z_1, \dots, z_{2n}$ and there is no relation among $\omega_1, \dots, \omega_n$. Therefore, H has to be a power series in $\omega_1, \dots, \omega_n$. By Lemma 14, it is absolutely convergent for any ω contained in

$$\Omega := \{\omega = (\omega_1, \dots, \omega_n) \in \mathbf{C}^n \mid |\omega_k| < R_k \ (k = 1, \dots, n)\},$$

and, hence, it is holomorphic in Ω. Next, in view of (4.2) and (4.3), comparing $(n + j)$th row of (4.1) for $j = 1, \dots, q$, we have

$$\sum_{i=1}^{n} \rho_i^{(j)} \omega_{n+j} \frac{\partial H}{\partial \omega_i} = 0 \quad (j = 1, \dots, q).$$

In view of (1.4), let us consider H as a function of $\tau_1, \dots, \tau_n$ and note that

$$\frac{\partial H}{\partial \omega_i} = \sum_{k=1}^{n} \frac{\partial H}{\partial \tau_k} \frac{\partial \tau_k}{\partial \omega_i} = \sum_{k=1}^{n} \frac{\partial H}{\partial \tau_k} \rho_i^{(k)}.$$

Then, by the orthogonality relation (1.3), the identity above leads to

$$\sum_{k=1}^{q} \langle \rho^{(j)}, \rho^{(k)} \rangle \omega_{n+j} \frac{\partial H}{\partial \tau_k} = 0 \quad (j = 1, \dots, q).$$

Since $\rho^{(1)}, \dots, \rho^{(q)}$ are linearly independent, the matrix $(\langle \rho^{(j)}, \rho^{(k)} \rangle)_{j,k=1,\dots,q}$ is non-singular. Therefore, we have

$$\frac{\partial H}{\partial \tau_k} = 0 \quad (k = 1, \dots, q).$$

Hence we have proved that H is a holomorphic function, namely a convergent power series, in $n - q$ variables $\tau_{q+1}, \dots, \tau_n$ and that $N = 0$ in decomposition (1.2).

We now turn to the proof of the uniqueness of the Birkhoff normal form $H \circ \varphi$. Let φ_1 be a convergent Birkhoff transformation such that $H \circ \varphi_1$ can be written as convergent

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power series in $n-q$ variables $\tau_{q+1}, \ldots, \tau_n$. Suppose that H is taken into Birkhoff normal form by another Birkhoff transformation φ_2. Then the transformation $\varphi = \varphi_1^{-1} \circ \varphi_2$ takes Birkhoff normal form $K_1 = H \circ \varphi_1$ into another Birkhoff normal form $K_2 = H \circ \varphi_2$. We prove $K_1 = K_2$ without assuming that $H \circ \varphi_2$ is a function of $\tau_{q+1}, \ldots, \tau_n$. We first prove the following lemma.

LEMMA 15. *Let $h(z)$ be a power series in z which depends only on $\tau_{q+1}, \ldots, \tau_n$, and let $W(z)$ be a power series with $W(0) = 0$ in S-normal form. Then h is invariant under the map* $\exp X_W$.

Proof. First we note that $\{h, W\} = 0$. In fact, by Lemma 1(ii), we have

$$\{h, W\} = \sum_{k=q+1}^{n} \frac{\partial h}{\partial \tau_k} \{\tau_k, W\} = 0.$$

We recall the Baker–Cambell–Hausdorff formula

$$h \circ \exp X_W = \sum_{m=0}^{\infty} \frac{1}{m!} \operatorname{ad}_W^m h; \quad \operatorname{ad}_W^0 h = h,$$
$$\operatorname{ad}_W^m h = \{\operatorname{ad}_W^{m-1} h, W\} \quad (m = 1, 2, \ldots).$$

Then we see that the identity $\{h, W\} = 0$ implies the identity $h \circ \exp X_W = h$. □

Recall that the transformation φ can be written in the form (2.8). We now prove that each polynomial W_ν is in S-normal form. By induction, let us assume that $W_1, \ldots, W_\nu$ are in S-normal form. Then, since K_1 depends only on $\tau_{q+1}, \ldots, \tau_n$, we have $K_1 \circ \varphi^{(\nu)} = K_1$ by Lemma 15. Let $h = K_1 \circ \varphi^{(\nu)} (= K_1)$ and $\varphi_{\nu+1} = \exp X_{W_{\nu+1}}$ with $W_{\nu+1}$ of the form (2.7) with $d = 2^\nu$. Then we have

$$h \circ \varphi_{\nu+1}(z) = h(z) + \{h(z), W_{\nu+1}(z)\} + O(|z|^{s_0+2d}) \quad (d = 2^\nu).$$

In view of Lemma 5(ii), this function has to be in S-normal form at least up to order $s_0 + 2d - 1$. Therefore, $\{h, W_{\nu+1}\}$ is in S-normal form up to order $s_0 + 2d - 1$. Since $\{h^0, W^{d+2}\}$ is the lowest-order part of $\{h, W_{\nu+1}\}$, it is in S-normal form. In the same way as in the proof of Lemma 4, this implies that $\{h^0, W^{d+2}\} = 0$ ($h^0 = S$). Similarly, comparing homogeneous parts of degree $s_0 + l$ for $l = d, \ldots, 2d-1$, we see inductively that $W^{d+2}, \ldots, W^{2d+1}$ are in S-normal form and, therefore, $W_{\nu+1}$ is in S-normal form. Hence, we have $h \circ \varphi_{\nu+1} = h$ by Lemma 15. By induction, we see that $K_1 \circ \varphi$ is equal to K_1 up to arbitrary order. This implies $K_1 = K_2$ and completes the proof of Theorem 1(i). We have thus completed the proof of Theorem 1.

5. *Proof of Theorem 2*

Let $H(z)$ be a real-analytic Hamiltonian of the form (1.1)–(1.2) with S replaced by the real-quadratic polynomial $\widehat{S}$ given by (1.6). We prove Theorem 2 by imposing reality conditions on Theorem 1.

Let $T : z = (x, y) \mapsto \zeta = (\xi, \eta)$ be a complex linear transformation defined by

$$\xi_k = \frac{x_k + i y_k}{\sqrt{2}}, \quad \eta_k = \frac{y_k + i x_k}{\sqrt{2}} \quad (k = 1, \ldots, n).$$

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This is symplectic and, hence, the system X_H is taken by its inverse into the system with Hamiltonian $H \circ T^{-1}(\zeta)$. Since $\frac{1}{2}(x_k^2 + y_k^2) = -i\xi_k\eta_k$, we have

$$\widehat{S} \circ T^{-1}(\zeta) = \sum_{k=1}^{n} \lambda_k \xi_k \eta_k \quad (\lambda_k = -i\alpha_k).$$

Then, under the assumption of Theorem 2, one can apply Theorem 1 to the Hamiltonian $H \circ T^{-1}(\zeta)$ and see that there exists a holomorphic symplectic transformation $\varphi\colon \zeta \mapsto \zeta + O(|\zeta|^2)$ such that $\{H \circ T^{-1} \circ \varphi, S\} = 0$ for $S = \sum_{k=1}^{n} \lambda_k \xi_k \eta_k$. By the arguments of [**7**, §7], one can prove that $\widehat{\varphi} := T^{-1} \circ \varphi \circ T$ is a real-analytic symplectic transformation such that $\{H \circ \widehat{\varphi}, \widehat{S}\} = 0$. Item (i) and the first assertion of (iii) follow from the corresponding claims of Theorem 1.

To prove (ii), we first note that

$$\widehat{\omega}_k = \operatorname{Im} \omega_k \circ T(z) \quad (k = 1, \ldots, n+q).$$

We introduce another variable

$$\widehat{\omega}_{n+k+q} = \operatorname{Re} \omega_{n+k} \circ T(z) \quad (k = 1, \ldots, q),$$

and consider the following Jacobian matrices:

$$A_1 = \frac{\partial(\widehat{\omega}_1, \ldots, \widehat{\omega}_n, \widehat{\omega}_{n+1}, \ldots, \widehat{\omega}_{n+q})}{\partial(z_1, \ldots, z_{2n})},$$

$$A_2 = \frac{\partial(\widehat{\omega}_1, \ldots, \widehat{\omega}_n, \widehat{\omega}_{n+1}^2, \ldots, \widehat{\omega}_{n+q}^2)}{\partial(z_1, \ldots, z_{2n})},$$

$$A_3 = \frac{\partial(\widehat{\omega}_1, \ldots, \widehat{\omega}_n, \widehat{\omega}_{n+1}^2, \ldots, \widehat{\omega}_{n+q}^2, \widehat{\omega}_{n+q+1}^2, \ldots, \widehat{\omega}_{n+2q}^2)}{\partial(z_1, \ldots, z_{2n})},$$

$$A_4 = \frac{\partial(\widehat{\omega}_1, \ldots, \widehat{\omega}_n, \widehat{\omega}_{n+1}, \ldots, \widehat{\omega}_{n+q}, \widehat{\omega}_{n+q+1}, \ldots, \widehat{\omega}_{n+2q})}{\partial(z_1, \ldots, z_{2n})}.$$

Since we have

$$\frac{\partial \widehat{\omega}_{n+k}^2}{\partial z_j} = 2\widehat{\omega}_{n+k} \frac{\partial \widehat{\omega}_{n+k}}{\partial z_j} \quad (j = 1, \ldots, 2n),$$

we see, by carrying out elementary transformations, that rank $A_1 =$ rank A_2 and rank $A_3 =$ rank A_4 provided that $\prod_{k=1}^{2q} \widehat{\omega}_{n+k} \neq 0$. Furthermore, in view of

$$\widehat{\omega}_{n+k}^2 + \widehat{\omega}_{n+k+q}^2 = |\omega_{n+k} \circ T|^2 = \omega_{n+k}\overline{\omega}_{n+k} = \prod_{j=1}^{n} \widehat{\omega}_j^{|\rho_j^{(k)}|} \quad (k = 1, \ldots, q), \tag{5.1}$$

we see similarly that rank $A_2 =$ rank A_3. Finally, using the relations

$$\begin{cases} i\widehat{\omega}_k = \omega_k \circ T \quad (k = 1, \ldots, n), \\ \widehat{\omega}_{n+k+q} + i\widehat{\omega}_{n+k} = \omega_{n+k} \circ T \quad (k = 1, \ldots, q), \end{cases} \tag{5.2}$$

we see that rank $A_4 =$ rank A_5, where

$$A_5 := \frac{\partial(\omega_1 \circ T, \ldots, \omega_n \circ T, \omega_{n+1} \circ T, \ldots, \omega_{n+q} \circ T, \widehat{\omega}_{n+q+1}, \ldots, \widehat{\omega}_{n+2q})}{\partial(z_1, \ldots, z_{2n})}.$$

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Hence, we have rank $A_1 = \text{rank } A_5$ if $\prod_{k=1}^{2q} \widehat{\omega}_{n+k} \neq 0$. Here, $\omega_1, \ldots, \omega_{n+q}$ are functionally independent in the sense of (2.1) and, hence, rank $A_5 \geq n+q$ on a dense open subset of $\mathbf{R}^{2n}$ as well as of $\mathbf{C}^{2n}$. On the other hand, we have rank $A_1 \leq n+q$ obviously. Therefore, we have proved that rank $A_1 = n+q$ on a dense open subset of $\mathbf{R}^{2n}$ as well as of $\mathbf{C}^{2n}$, i.e. the functional independence of $\widehat{\omega}_1, \ldots, \widehat{\omega}_{n+q}$. Moreover, since $\omega_k \circ T$ $(k = 1, \ldots, n)$ and $\omega_{n+k} \circ T$ $(k = 1, \ldots, q)$ are integrals of the real vector field $X_{H \circ \widehat{\varphi}}$, their real and imaginary parts are its integrals. Then it follows from (5.2) that $\widehat{\omega}_1, \ldots, \widehat{\omega}_{n+q}$ are real integrals of $X_{H \circ \widehat{\varphi}}$.

Finally, let Γ be a regular level set of $F(z) = (\widehat{\omega}_1, \ldots, \widehat{\omega}_{n+q})$. Then, we see that $\widehat{\omega}_j > 0$ on Γ for $j = 1, \ldots, n$, and therefore Γ is a closed subset of the n-dimensional torus. By Lemma 1(ii), $\widehat{\tau}_j$ $(j = q+1, \ldots, n)$ are Poisson commuting functions satisfying the identities

$$\{\widehat{\omega}_k, \widehat{\tau}_j\} = 0 \quad (k = 1, \ldots, n+q).$$

This implies that the flows of $X_{\widehat{\tau}_{q+1}}, \ldots, X_{\widehat{\tau}_n}$ commute on Γ and give rise to an $\mathbf{R}^{n-q}$-action on Γ. Since any connected components of Γ is a compact manifold of dimension $n-q$, it has to be an $(n-q)$-dimensional torus. This completes the proof of Theorem 2.

6. *Superintegrability with singularities*

Finally, we briefly discuss about superintegrability of the vector field X_H. Let us first give a definition of superintegrability of complex Hamiltonian systems. In the definition below, the number q $(0 \leq q \leq n-1)$ is independent of resonance degree we considered up to now.

Let X_H be an n-degree-of-freedom Hamiltonian vector field with Hamiltonian H being holomorphic in a domain D of $\mathbf{C}^{2n}$. We say that X_H is *analytically superintegrable* if X_H possesses $n+q(=2n-(n-q))$ holomorphic and functionally independent integrals $F_1, \ldots, F_{n+q}$ such that, for the map $F = (F_1, \ldots, F_{n+q}) \colon D \to \mathbf{C}^{n+q}$, the following two conditions hold.

(i) There exist functions $P_{ij} \colon F(D) \to \mathbf{C}$ such that

$$\{F_i, F_j\} = P_{ij} \circ F \quad (i, j = 1, \ldots, n+q).$$

(ii) The matrix P with entries P_{ij} has rank $2q(=2n-2(n-q))$ on a dense open subset of $F(D)$.

The notion of superintegrability (non-commutative integrability) was first studied by Nekhoroshev [**15**] and further developed by Mischenko and Fomenko [**14**] and others (see [**6, 12**]). The definition above can be considered as a complex analytic version of the definition in the sense of Mischenko–Fomenko formulated in [**6, 12**], where the map F is assumed to be a real smooth submersion with compact and connected fibers (i.e. level sets) and rank $P = 2q$ everywhere. In the definition above, we consider the holomorphic setting without any condition on the fibers of F and allow singularities of F and P. If the Hamiltonian H and its integrals $F_1, \ldots, F_{n+q}$ are real analytic in a real domain D, then we way that X_H is *real-analytically superintegrable* in D.

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We now consider the situation of Theorem 1. Let D be a neighbourhood of the origin $z = 0$ and let $F = (\omega_1, \ldots, \omega_{n+q})$. Then we have (4.2) and

$$\{\omega_{n+i}, \omega_{n+j}\} = \sum_{k=1}^{n} (\rho_{+\,k}^{(i)} \rho_{-\,k}^{(j)} - \rho_{-\,k}^{(i)} \rho_{+\,k}^{(j)}) \frac{\omega_{n+i}\omega_{n+j}}{\omega_k} \quad (i, j = 1, \ldots, q).$$

This implies that conditions (i) and (ii) above hold. More precisely, by (2.1), the gradient vectors of integrals $\omega_1, \ldots, \omega_{n+q}$ are linearly independent on the set $\Omega = \{z \in D \mid \prod_{j=1}^{2n} z_j \neq 0\}$, and rank $P = 2q$ on the set $F(\Omega)$. Therefore, the vector field X_H in Theorem 1 is superintegrable with singularities in a complex neighbourhood of the origin. In the case of Theorem 2, using (5.1) and (5.2), we have similar formulae for Poisson brackets between any two of $\widehat{\omega}_1, \ldots, \widehat{\omega}_{n+q}$ and see that X_H is real-analytically superintegrable in a real neighbourhood of the origin.

One may also choose the map $F = (F_1, \ldots, F_{n+q})$ as

$$\begin{cases} F_i = \tau_{q+i} & (i = 1, \ldots, n-q), \\ F_{n-q+i} = \tau_i & (i = 1, \ldots, q), \\ F_{n+i} = \omega_{n+i} & (i = 1, \ldots, q). \end{cases}$$

This satisfies the above conditions of superintegrability and, in particular, we have

$$\{F_i, F_j\} = 0 \quad \text{for } i = 1, \ldots, n-q, \quad j = 1, \ldots, n+q.$$

This is the situation which Nekhoroshev first considered as superintegrable systems. In this case, we have $n - q$ commuting vector fields $X_{F_1}, \ldots, X_{F_{n-q}}$ as well as their integrals $F_1, \ldots, F_{n+q}$. Since H is a function of $\tau_{q+1}, \ldots, \tau_n$, one may replace one of $\tau_{q+1}, \ldots, \tau_n$ by H. Then X_H is an integrable system in the extended sense of Bogoyavlenski [**4**], to which a theorem of Zung [**25**] can be applied to obtain the Poincaré–Dulac normal form by convergent normalization. It is a result of Theorem 1 (and Theorem 2) that these considerations are valid.

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