Proc. Camb. Phil. Soc. (1974), **76**, 401
PCPS 76–38
Printed in Great Britain

On pro-reductive groups

By MARTIN MOSKOWITZ
The Graduate School of the City University of New York,
33 West 42 Street, New York, N.Y. 10036

(*Received* 21 *August* 1973)

In the proof of the Freudenthal–Weil theorem in, for example (5), essential use is made of the fact that if G and H are compact analytic groups and $\phi: G \to H$ is a continuous epimorphism then $\phi(Z(G)_0) = Z(H)_0$ where the subscript $_0$ denotes the identity component of a topological group G and $Z(G)$ its centre. Although this is sufficient for the proof of the Freudenthal–Weil theorem it raises the interesting question as to whether actually $\phi(Z(G)) = Z(H)$ (from which the above would follow) and, if so, in what generality this can be expected. The present paper deals with this question, in more general form, as well as certain of its structural consequences.

We recall that an analytic group G is called reductive, semisimple or of compact type if its Lie algebra $\mathfrak{g}$ is respectively reductive, semisimple, or the Lie algebra of some compact analytic group.

0.1 *Definition.* Let G be a locally compact, connected topological group. Then as is well-known (8), $G = \varprojlim G_\alpha$ (projective limit) of analytic groups G_α. If all the G_α are reductive or semisimple then we shall call G a pro-reductive or pro-semisimple group respectively.

As we shall see shortly (see (2·5)) these notions are intrinsic to G, i.e. do not depend on the way G is written as a projective limit. Thus if $G = \varprojlim H_\beta$ and is pro-reductive or pro-semisimple, then each H_β is reductive or semisimple respectively. In fact (2·10) characterizes these notions in invariant terms by means of the notion of radical due to Iwasawa (6).

Some of these results are extensions of known results in the case of Lie groups to pro-Lie groups while some are new even in the case of (reductive) Lie groups.

Our first section deals with Lie groups while the second with pro-Lie groups. In section 2 we generalize and put into perspective results proven for pro-semisimple groups in section 4 of (3) as well as sharpening certain of these. In particular, however, we merely refer to (4·2) of (3); for although it is formulated there only for pro-semisimple groups the proof in the reductive case is identical. On the other hand, in section 1 we give a complete proof in the reductive case of the very abbreviated (4·1) of (3).

1. Reductive groups

1.1. Theorem. *Let $\phi: G_1 \to G_2$ be a continuous epimorphism between analytic groups. If G_1 is reductive then $\phi(Z(G_1)) = Z(G_2)$. In particular, if G_1 is semisimple or of compact-type then this holds.*

https://doi.org/10.1017/S0305004100049070 Published online by Cambridge University Press

As we shall see in section 2 this result generalizes as it stands if G_1 is pro-reductive but G_2 is analytic and in a modified form if neither G_1 nor G_2 are analytic (see (2·3) and (2·4)).

Proof. Since ϕ is surjective it follows that

$$\phi([G_1, G_1]) = [G_2, G_2]. \tag{1}$$

This implies that the differential $\phi^*: \mathfrak{g}_1 \to \mathfrak{g}_2$ as well as its restriction.

$$(\phi^*)_{[\mathfrak{g}_1, \mathfrak{g}_1]}: [\mathfrak{g}_1, \mathfrak{g}_1] \to [\mathfrak{g}_2, \mathfrak{g}_2]$$

is surjective (see (5)). Since $\mathfrak{g}_1$ is reductive, $[\mathfrak{g}_1, \mathfrak{g}_1]$ is semisimple. It follows that since $[\mathfrak{g}_2, \mathfrak{g}_2]$ is a quotient algebra of a semisimple algebra it is itself semi-simple (5). This implies that $\mathfrak{g}_2$ is reductive (see Prop. 5*b*, p. 79 of (1)) and hence we have:

$$\mathfrak{g}_i = [\mathfrak{g}_i, \mathfrak{g}_i] \oplus \mathfrak{z}(\mathfrak{g}_i) \quad (i = 1, 2). \tag{2}$$

and therefore (see (5)) also

$$G_i = [G_i, G_i]\, Z(G_i)_0 \quad (i = 1, 2). \tag{3}$$

Since $\phi^*([\mathfrak{g}_1, \mathfrak{g}_1]) = [\mathfrak{g}_2, \mathfrak{g}_2]$ and $\phi^*(\mathfrak{z}(\mathfrak{g}_1)) \subseteq \mathfrak{z}(\mathfrak{g}_2)$ it follows easily from (2) by linear algebra that $\phi^*(\mathfrak{z}(\mathfrak{g}_1)) = \mathfrak{z}(\mathfrak{g}_2)$ and hence (see (5)) that:

$$\phi(Z(G_1)_0) = Z(G_2)_0. \tag{4}$$

Evidently (3) implies that $Z(G_i) = Z[G_i, G_i]\, Z(G_i)_0$ $(i = 1, 2)$. By (1), (3), (4) and this last remark we see that to show $\phi(Z(G_1)) = Z(G_2)$ it suffices to show that

$$\phi(Z[G_1, G_1]) = Z[G_2, G_2].$$

That is we may assume G_1 and G_2 are semisimple.

1·2. LEMMA. *Let G and H be analytic groups and $\phi: G \to H$ be a covering epimorphism then $\phi(Z(G)) = Z(H)$.*

Proof. As above $\phi(Z(G))$ is clearly contained in $Z(H)$. Let $\phi(g_0) \in Z(H)$ then $\phi(g)\,\phi(g_0) = \phi(g_0)\,\phi(g)$ for all $g \in G$. This means that $[g, g_0] \in \operatorname{Ker}\phi$ for each $g \in G$. Since $[G, g_0]$ is connected and $\operatorname{Ker}\phi$ is discrete, $[G, g_0]$ is a single point which must be the identity since $[g_0, g_0] = 1$. Thus $g_0 \in Z(G)$.

Continuing the proof of (1·1) let $(G_1^{\sim}, \pi)$ be the universal covering group of G_1 then $\phi\pi: G_1^{\sim} \to G_2$ is a continuous epimorphism. If $\phi\pi(Z(G_1^{\sim})) = Z(G_2)$ then since

$$\pi(Z(G_1^{\sim})) = Z(G_1)$$

by the lemma we would have $\phi(Z(G_1)) = Z(G_2)$. Thus we may assume without loss of generality that G_1 is simply connected. Hence $G_1 = \prod_{i=1}^{r} H_i$, the direct product of simple groups. Now $\operatorname{Ker}\phi \cap H_i$ is a closed normal subgroup of H_i so that either $\operatorname{Ker}\phi \cap H_i$ coincides with H_i or is a discrete subgroup of $Z(H_i)$. That is either $\phi(H_i) = (1)$ or $(\phi)_{H_i}$ is a covering map of H_i onto its image in G_2. Clearly the image is some (closed) simple component of G_2. Therefore by rearranging the order of the H_i's and taking into account the absolute uniqueness of the simple components in the semisimple case we have $G_1 = H \times L$ (direct product) where $\phi(H) = 1$ and $(\phi)_L: L \to G_2$ is a covering map. Since $Z(G_1) = Z(H) \times Z(L)$ it follows that $\phi(Z(G_1)) = \phi(Z(H))\,\phi(Z(L)) = \phi(Z(L))$. The result now follows from (1·2).

We remark that an analytic group G is a $[Z]$-group in the sense of (2) if and only if it is of compact type. This follows from Theorem (4·3) of (2). This remark enables us to prove

https://doi.org/10.1017/S0305004100049070 Published online by Cambridge University Press

1·3. COROLLARY. *Let G be an analytic group of compact type then each quotient group of G is also of compact type.*

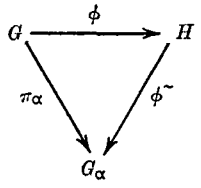
Proof. By the above G is a $[Z]$-group. It is also reductive. Hence the quotient map $\phi: G \to H$ takes $Z(G)$ onto $Z(H)$ and therefore induces a continuous epimorphism $\phi^{\sim}: G/Z(G) \to H/Z(H)$. Since $G/Z(G)$ is compact so is $H/Z(H)$. This implies H is of compact type.

2. PRO-REDUCTIVE GROUPS

2·1. LEMMA. *Let $G = \varprojlim G_\alpha$ be a pro-reductive group with $\pi_\alpha: G \to G_\alpha$ the corresponding projections. Then $\pi_\alpha(Z(G)) = Z(G_\alpha)$ and in fact via the restrictions $(\pi_\alpha)_{Z(G)}$ we have $Z(G) = \varprojlim Z(G_\alpha)$.*

Proof. The proof of the first statement proceeds exactly as in (4·2) of (3) except in this generality one employs (1·1) rather than the corresponding result for pro-semisimple groups. Concerning the second statement, we see that by considering the restrictions $(\pi_\alpha)_{Z(G)}$ and also the restrictions $(\pi_{\alpha\beta})_{Z(G_\beta)}$ for $\alpha \leqslant \beta$ of the coherence maps we get a projective system. It follows that since the $(\pi_\alpha)_{Z(G)}$ are surjective so are the $(\pi_{\alpha\beta})_{Z(G_\beta)}$. Since G is a locally compact connected group it is compactly generated and therefore σ-compact. The latter is therefore true of $Z(G)$ and $Z(G_\beta)$. It follows (see (5)) that $(\pi_\alpha)_{Z(G)}$ and $(\pi_{\alpha\beta})_{Z(G_\beta)}$ are open maps. Thus L.P. 1, 2 and 3 of p. 24 of (9) are satisfied. This means that $Z(G) = \varprojlim Z(G_\alpha)$.

2·2. LEMMA. *Let $\phi: G \to H$ be a continuous homomorphism where $G = \varprojlim G/K_\alpha$ is a projective limit and H is a Lie group. Then $\phi(K_\alpha) = (1)$ for some α,* i.e. *ϕ induces a continuous homomorphism $\phi^{\sim}$ such that the diagram below is commutative.*



Proof. Let V_0 be a neighbourhood of 1 in H without non-trivial subgroups and choose a neighbourhood U of 1 in G such that $\phi(U) \subseteq V_0$. Since G is the projective limit of the G_α's we can choose a compact normal subgroup K_α of G such that $K_\alpha \subseteq U$. Then $\phi(K_\alpha) \subseteq \phi(U) \subseteq V_0$ and hence $\phi(K_\alpha)$ is trivial.

2·3. THEOREM. *Let $\phi: G \to H$ be a continuous epimorphism between locally compact groups. If G is pro-reductive then $\phi(Z(G))$ is dense in $Z(H)$. In particular, if G is a compact connected group then $\phi(Z(G)) = Z(H)$.*

Proof. We first observe that H is pro-reductive. H is connected so $H = \varprojlim H_\beta$. Let H_β be one of the approximating analytic groups and $\pi_\beta: H \to H_\beta$. Then $f = \pi_\beta\phi$ satisfies the hypothesis of Lemma (2·2). Hence there exist an α and a continuous homomorphism $f^{\sim}$ such that $f^{\sim}: G_\alpha \to H_\beta$ and $f^{\sim}\pi_\alpha = f$. Since ϕ and π_β are surjective so is f and since π_α is also surjective this implies $f^{\sim}$ is surjective and H_β is actually a quotient group of G_α. As was noted during the course of proving (1·1) since G_α is a reductive analytic group so is H_β.

https://doi.org/10.1017/S0305004100049070 Published online by Cambridge University Press

To prove the theorem we first assume that H is itself reductive. Then by (2·2) there exists an α and a continuous epimorphism $\phi^{\sim}$ such that $\phi = \phi^{\sim}\pi_{\alpha}$. Since G_{α} and H are reductive analytic groups, $\phi^{\sim}(Z(G_{\alpha})) = Z(H)$. By (2·1), $\pi_{\alpha}(Z(G)) = Z(G_{\alpha})$. Hence

$$\phi(Z(G)) = \phi^{\sim}\pi_{\alpha}(Z(G)) = Z(H).$$

In general, let $H = \varprojlim H_{\beta}$. For each β, $\pi_{\beta}\phi$ is a continuous epimorphism $G \to H_{\beta}$. It follows that $\pi_{\beta}\phi(Z(G)) = Z(H_{\beta})$. On the other hand by (2·1), $\pi_{\beta}(Z(H)) = Z(H_{\beta})$. Therefore $Z(H)K_{\beta} = \phi(Z(G))K_{\beta}$ for all β, so that

$$Z(H) \subseteq \bigcap_{\beta} Z(H)K_{\beta} = \bigcap_{\beta} \phi(Z(G))K_{\beta} \subseteq \bigcap_{V} \phi(Z(G))V,$$

where V is an arbitrary neighbourhood of 1 in H. As is well-known the latter coincides with $\phi(Z(G))^{-}$. On the other hand, $\phi(Z(G)) \subseteq Z(H)$ and since $Z(H)$ is closed we have $\phi(Z(G))^{-} = Z(H)$. If G is compact then so is $Z(G)$ and hence $\phi(Z(G))$ is closed.

2·4 COROLLARY. *Let $\phi\colon G \to H$ be a continuous epimorphism where G is pro-reductive and H is analytic then $\phi(Z(G)) = Z(H)$.*

Proof. By (2·2) ϕ actually lives on some G_{α}. The result now follows from the proof of (2·3).

2·5. COROLLARY. *The notion of being pro-reductive or pro-semisimple is an intrinsic one.*

Proof. Let $G = \varprojlim G_{\alpha}$ be pro-reductive or pro-semisimple and suppose $G = \varprojlim H_{\beta}$. Then as in the proof of the first paragraph of (2·3) with ϕ replaced by the identity map we see that each H_{β} is a quotient group of some G_{α}. This together with the fact that the quotient group of a reductive or semisimple analytic group is again reductive or semisimple respectively, completes the proof.

2·6. COROLLARY. *Let G be a pro-reductive group then $Z(G)_0[G, G]$ is dense in G. In particular, if G is a compact connected group then $G = Z(G)_0([G, G]^{-})$.*

Proof. We write $G = \varprojlim G_{\alpha}$ then for each α,

$$G_{\alpha} = Z(G_{\alpha})_0[G_{\alpha}, G_{\alpha}] \quad \text{and} \quad \pi_{\alpha}[G, G] = [G_{\alpha}, G_{\alpha}].$$

Moreover, as was noted in (2·1), $(\pi_{\alpha})_{Z(G)}\colon Z(G) \to Z(G_{\alpha})$ is a continuous open epimorphism. Since $\operatorname{Ker}(\pi_{\alpha})_{Z(G)}$ is a compact subgroup the map is closed. It follows from this that $\pi_{\alpha}(Z(G)_0) = Z(G_{\alpha})_0$ (see (4), p. 63). Thus $\pi_{\alpha}(Z(G)_0[G, G]) = G_{\alpha}$ for all α. This means that for each α,

$$Z(G_0)[G, G]K_{\alpha} = G.$$

Arguing as in (2·3) we have $G = (Z(G)_0[G, G])^{-}$. Since if G is compact

$$(Z(G)_0[G, G])^{-} \subseteq Z(G)_0[G, G]^{-}$$

and the latter is closed. The proof is complete.

Our next result has some overlap with corollary (4·4) of the recent article of Minbashian (8).

https://doi.org/10.1017/S0305004100049070 Published online by Cambridge University Press

2·7. COROLLARY. *Let G be a pro-reductive group then the following conditions are equivalent*:

(i) *G is pro-semisimple,*

(ii) *$Z(G)$ is totally disconnected,*

(iii) *$[G, G]$ is dense in G.*

Proof. Since $G = (Z(G)_0 [G, G])^-$ it follows that (ii) implies (iii). If $[G, G]$ is dense in G and $G = \varprojlim G_\alpha$ then the same is true of the G_α. As is well-known for reductive analytic groups G_α, $[G_\alpha, G_\alpha]$ is closed and semisimple. Thus each G_α is semisimple and (i) holds. Finally, if each G_α is semisimple then $Z(G_\alpha)$ is discrete and by (2·1)

$$Z(G) = \varprojlim Z(G_\alpha)$$

and therefore as a projective limit of discrete groups $Z(G)$ is totally disconnected.

2·8. COROLLARY. *Let G be a pro-reductive group then $G/Z(G)$ is centreless and pro-semisimple.*

Proof. Taking ϕ to be the canonical epimorphism $G \to G/Z(G)$ and applying (2·3) we see that $Z(G/Z)$ contains (1) as a dense (and closed) subgroup. Hence G/Z is centreless. If $\pi_\alpha \colon G \to G_\alpha$ are the projections then by (2·1) $\pi_\alpha(Z(G)) = Z(G_\alpha)$ and hence π_α induces a continuous epimorphism, $\tilde{\pi_\alpha} \colon G/Z(G) \to G_\alpha/Z(G_\alpha)$. Evidently one also has coherence maps $\tilde{\pi_{\alpha\beta}}$ which satisfy the coherence relations and are continuous epimorphisms. Arguing as in (2·1) it follows that $G/Z(G) = \varprojlim G_\alpha/Z(G_\alpha)$. Since G_α is reductive, $G_\alpha/Z(G_\alpha)_0$ is semisimple. The same is then true of the quotient group

$$\frac{G_\alpha/Z(G_\alpha)_0}{Z(G_\alpha)/Z(G_\alpha)_0} = G_\alpha/Z(G_\alpha).$$

In strong contrast to the nilpotent case we have

2·9. COROLLARY. *Let G be a pro-reductive group, ι_x the inner automorphism determined by $x \in G$ and $\tilde{\iota_x}$ the associated automorphism of $G/Z(G)$. If $\tilde{\iota_x}$ is trivial then so is ι_x.*

Proof. Clearly $\tilde{\iota_x}$ being trivial is evidently equivalent to the statement $[x, G] \subseteq Z(G)$. If ϕ denotes the canonical epimorphism $G \to G/Z(G)$, this means that $[\phi(x), \phi(g)] = (1)$ for all $g \in G$, that is $\phi(x) \in Z(G/Z)$. By (2·8) the latter is trivial so that $x \in Z(G)$.

Finally, we compare our notion of pro-reductive and pro-semisimple with properties of the radical of a locally compact connected group due to Iwasawa. We do this by use of a recent result of Lee (7).

2·10. DEFINITION. For a locally compact connected group G, $\operatorname{rad} G$, the radical of G denotes the largest (closed) normal connected solvable sub-group. For a proof of the existence of $\operatorname{rad}(G)$ as well as its fundamental properties see (6).

2·11. THEOREM. *Let G be a locally compact group then G is pro-reductive if and only if* $\operatorname{rad} G = Z(G)_0$. *Hence by* (2·7) (ii) *we also have the result, G is pro-semisimple if and only if* $\operatorname{rad} G = (1)$.

https://doi.org/10.1017/S0305004100049070 Published online by Cambridge University Press

Proof. Let G be pro-reductive and $\pi_\alpha: G \to G_\alpha$ be as usual. Since $\operatorname{rad} G$ is a connected solvable normal subgroup of G, $\pi_\alpha(\operatorname{rad} G)$ has the same properties with respect to G_α. By maximality $\pi_\alpha(\operatorname{rad} G) \subseteq \operatorname{rad} G_\alpha$. Since G_α is one of the approximating reductive analytic groups $\operatorname{rad}(G_\alpha) = Z(G_\alpha)_0$ so that $\pi_\alpha(\operatorname{rad} G) \subseteq Z(G_\alpha)_0$. By Lemma (2·1) together with the fact that the K_α's are compact we have $\pi_\alpha(Z(G)_0) = Z(G_\alpha)_0$. Since $\operatorname{rad} G$ clearly contains $Z(G)_0$ this means that $\pi_\alpha(\operatorname{rad} G) = \pi_\alpha(Z(G)_0)$. This implies

$$\operatorname{rad} G \cdot K_\alpha = Z(G)_0 K_\alpha$$

for all α, so that $\operatorname{rad} G \subseteq \bigcap_\alpha Z(G)_0 K_\alpha$. Arguing as in (2·3) we see that $\operatorname{rad} G \subseteq Z(G)_0^-$. Since the latter is closed and contained in $\operatorname{rad} G$ we have $\operatorname{rad} G = Z(G)_0$.

Conversely suppose $\operatorname{rad} G = Z(G)_0$. Let $G = \varprojlim G_\alpha$ be an approximation of G by analytic groups. By (2·5) it suffices to show each G_α is reductive. By (2·1) and the fact that $\operatorname{Ker} \pi_\alpha$ is compact, $\pi_\alpha(Z(G)_0) = Z(G_\alpha)_0$ for each α. Arguing as in (2·8) it follows that $G/Z(G)_0 = \varprojlim G_\alpha/Z(G_\alpha)_0$. Since $\operatorname{rad} G = Z(G)_0$ it follows from Lemma B of (7) that $G/Z(G)_0$ is pro-semisimple and hence by (2·5) $G_\alpha/Z(G_\alpha)_0$ is semisimple. This means that the analytic group G_α is an extension of a connected central subgroup by a semisimple group. As such it is reductive.

REFERENCES

(1) BOURBAKI, N. *Éléments de Mathematique*, Livre 7, Chapaître 1, Groupes et Algèbres de Lie (Hermann; Paris, 1960.)

(2) GROSSER, S. and MOSKOWITZ, M. On central topological groups. *Trans. Amer. Math. Soc.* **127** (1967), 317–340.

(3) GROSSER, S., LOOS, O. and MOSKOWITZ, M. Über Automorphismengruppen lokal-kompakter Gruppen und Derivationen von Lie-Gruppen, *Math. Z.* **114** (1970), 321–339.

(4) HEWITT, E. and ROSS, K. *Abstract harmonic analysis*, vol. I (Springer; Berlin–Gottingen–Heidelberg, 1963).

(5) HOCHSCHILD, G. *The structure of Lie groups* (Holden Day; San Francisco, 1965).

(6) IWASAWA, K. On some types of topological groups. *Ann. of Math.* (2) **50** (1949), 507–558.

(7) LEE, D. H. On the homomorphisms of locally compact groups. *Proc. Amer. Math. Soc.* **37** (1973), 246–254.

(8) MINBASHIAN, F. Pro Affine Algebraic Groups. *Amer. J. Math.* **95** (1973), 174–192.

(9) MONTGOMERY, D. and ZIPPIN, L. *Topological transformation groups* (Interscience; New York, 1955).

(10) WEIL, A. *L'integration dans les groupes topologiques et ses applications* (Hermann; Paris, 1953).

https://doi.org/10.1017/S0305004100049070 Published online by Cambridge University Press