Finally, we conjecture that the Spieker centre S lies on the Lester circle for all triangles in U but not for those in V.

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103.12 A note on Problem 101.F

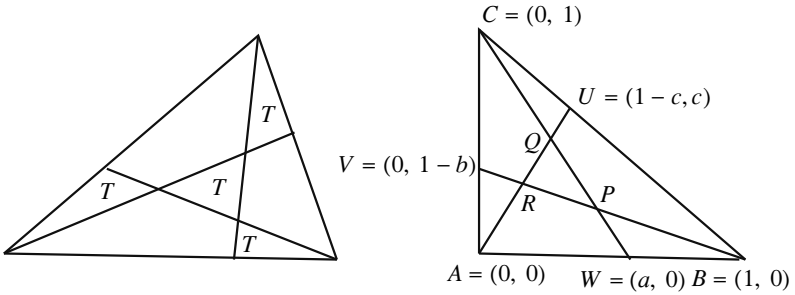


FIGURE 1

A small extension of one of the results of Stan Dolan's problem 101.F in the March 2018 *Gazette*, p. 177 is as follows. Suppose, in the left-hand diagram, only that the four triangles marked T have equal area. Then it follows without further assumption that the ratios of lengths along all three sides of the triangle are equal, and equal to the 'golden ratio' $\frac{1}{2}(\sqrt{5}+1)$. Of course, equality of the four areas gives us three (presumably independent) equations, so there is some expectation of, at any rate, a finite number of possible solutions for the three ratios.

The only proof I have of this is a rather complicated calculation. Since the problem is affinely invariant (involving ratios of areas and ratios of lengths along a straight line) we may assume the triangle is any size and shape we please, and I take it as in the right-hand diagram, with vertices at (0, 0), (1, 0) and (0, 1). With the points on the three sides being $(a, 0)$, $(0, 1-b)$ and $(1-c, c)$ as shown, where $0 < a, b, c < 1$, then the object is to prove, from the equality of areas of the four triangles AVR, BPW, CQU and PQR, that $a = b = c = \frac{1}{2}(\sqrt{5}-1)$, making the ratios all equal to $a/(1-a) = \frac{1}{2}(\sqrt{5}+1)$.

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The lines have equations

$$PQ : x + ay - a = 0; \qquad PR : (1-b)x + y + b - 1 = 0;$$

$$QR : cx - y(1-c) = 0.$$

From this we can calculate the coordinates of P, Q, R:

$$P = \left(\frac{ab}{ab-a+1}, \frac{(1-a)(1-b)}{ab-a+1}\right), Q = \left(\frac{a(1-c)}{ac-c+1}, \frac{ac}{ac-c+1}\right),$$

$$R = \left(\frac{(1-b)(1-c)}{bc-b+1}, \frac{c(1-b)}{bc-b+1}\right).$$

Armed with these formulas it is straightforward, but rather tedious, to write down the areas of the various triangles. For example the area of triangle ARV is half the determinant of the vectors $\mathbf{AR} = R - A$ and $\mathbf{AV} = V - A$, which is $\frac{(1-b)^2(1-c)}{2(bc-b+1)}$. Similarly the area of PQR is the rather more complicated expression

$$\frac{(2abc - bc - ac - ab + a + b + c - 1)^2}{2(bc-b+1)(ac-c+1)(ab-a+1)}.$$

Then the equality of areas of triangles ARV, PQR can be written as

$$ab^2c = (1-b)(1-c)(1-a+b). \tag{1}$$

I thank the referee for pointing out that this equation can be written in the following more suggestive form and thereby shortening the conclusion of the argument:

$$\frac{abc}{(1-a)(1-b)(1-c)} = \frac{1}{1-a} + \frac{1}{b}.$$

Area CQU = area PQR is obtained by a cyclic permutation of the letters a, b, c namely $a, b, c \to b, c, a$, and area BPW = area PQR is given by $a, b, c \to c, a, b$. Thus we obtain

$$\frac{1}{1-a} + \frac{1}{b} = \frac{1}{1-b} + \frac{1}{c} = \frac{1}{1-c} + \frac{1}{a}.$$

But we may assume without loss of generality that $1 > a \geqslant b \geqslant c > 0$ and if for example $a > b \geqslant c$ then $\frac{1}{a} < \frac{1}{b}$ and $\frac{1}{1-c} < \frac{1}{1-a}$ so that the above equalities cannot hold. Hence $a = b = c$ and (1) gives $(a^2 + a - 1)(a^2 - a + 1) = 0$ where the second factor has no real roots so a, b, c all have the required value $\frac{1}{2}(\sqrt{5} - 1)$.

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