It is worth noting that a similar argument works for roots $\tilde{r}_i = a + (i-1)d$ in an arithmetic sequence and yields

$$\sum_{i=1}^{n} \big(a + (i-1)d\big) = n\,\frac{(2a + (n-1)d)}{2}.$$

(As an outline of the proof, first consider

$$\tilde{p}_n(x) = \big(x - r_1\big)\big(x - r_2\big)\ldots\big(x - r_n\big)$$

to show that the polynomial

$$\tilde{q}_n(x) = \tilde{p}_n\big(a + \big(\tfrac{1}{2}(n-1)\big)d + x\big)$$

verifies $\tilde{q}_n(-x) = (-1)^n \tilde{q}_n(x)$; now, if $\tilde{s}_i$ stands for the ith root of $\tilde{q}_n$, then $a + \frac{1}{2}(n-1)d + \tilde{s}_i = \tilde{r}_i = a + (i-1)d$, and the proof ends as before.)

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103.06 Another approach to $\int_0^{\pi} \log(2\sin u)\,du$

For $x \in (0, \pi)$, the *Lobachevsky function*

$$f(x) = \int_0^x \log(2\sin u)\,du = -\lim_{\delta \to 0^+} \int_{\delta}^{x} \log(2\sin u)\,du$$

arises in the computation of volumes in hyperbolic space. Showing that $f(\pi) = 0$, so that f can be extended continuously (and π-periodically) to $\mathbb{R}$ via $f(x) = -\int_0^x \log|2\sin u|\,du$, takes some work [e.g. 1, 2]. Here is an elementary approach.

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It is a neat fact [3,4,5] that the product of the lengths of the chords drawn from a point on the unit circle to $n - 1$ other equally-spaced points, is n. (In Figure 1, $n = 8$.) That is,

$$\prod_{k=1}^{n-1} 2 \sin\left(\frac{k\pi}{n}\right) = n.$$

Therefore

$$\sum_{k=1}^{n-1} \log\left(2 \sin\left(\frac{k\pi}{n}\right)\right)\frac{\pi}{n} = \pi \frac{\log n}{n}.$$

Since the integrand is monotonic near 0 and near π, and of the same sign in those two places, the left-hand side qualifies as a Riemann sum for the improper integral from 0 to π (improper at *both* ends). Here, the right-hand side tends to 0 as $n \to \infty$, and therefore

$$\int_0^{\pi} \log(2 \sin u)\, du = 0. \qquad (*)$$

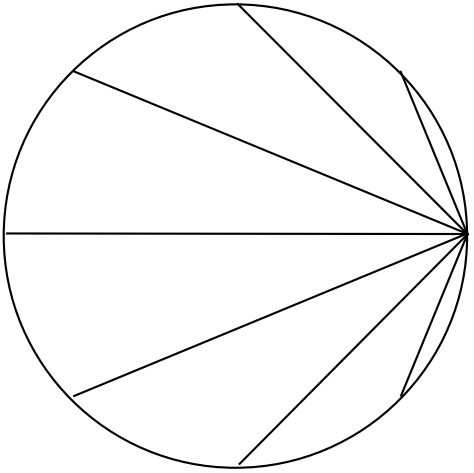


FIGURE 1

A consequence is the following, which appears in many comprehensive tables of integrals.

Corollary: $\int_0^{\pi/2} \log(\sin u)\, du = \int_0^{\pi/2} \log(\cos u)\, du = -\frac{\pi}{2} \log 2.$

Proof: Set $I = \int_0^{\pi/2} \log(\sin u)\, du$. That $I = \int_0^{\pi/2} \log(\cos u)\, du$ is verified using the change of variable $u = \pi/2 - t$. Now $(*)$ gives

$$-\pi \log 2 = \int_0^{\pi} \log(\sin u)\, du,$$

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and then the change of variable $u = 2x$ gives

$$-\frac{\pi}{2}\log 2 = \int_0^{\pi/2} \log(\sin 2x)\,dx = \int_0^{\pi/2} \log(2\sin x\cos x)\,dx$$

$$= \frac{\pi}{2}\log 2 + \int_0^{\pi/2} \log(\sin x)\,dx + \int_0^{\pi/2} \log(\cos x)\,dx$$

$$= \frac{\pi}{2}\log 2 + 2I.$$

Solving for I yields the desired result.

Remark:

The function $g(x) = 2f(\frac{1}{2}x)$ often goes by the name of Clausen's integral.

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103.07 More on slowly convergent, or divergent, series

The negative answer to the question of whether a series with positive terms can converge, or diverge, 'most slowly' was considered in [1]. In fact, the following proposition is well-known in the real analysis literature (see [2, p.47] and [3, Exercises 11, 12, pp.79-80]). Here we offer a straightforward proof by means of a neat construction based on the difference of two squares.

Proposition: Let S denote the sum of a series $\sum_{n=1}^{\infty} a_n$, where $a_n > 0$, with $S = \infty$ if the series is divergent. We then have the following:

(i) If $S = \infty$ then there is a sequence (b_n), with $b_n > 0$, such that

$$\sum_{n=1}^{\infty} b_n = \infty \text{ and } \lim_{n\to\infty} \frac{b_n}{a_n} = 0.$$

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