ON TWO-DIMENSIONAL WAVES IN AN ELASTIC HALF-SPACE

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ABSTRACT. Two-dimensional elastic waves in a half-space $0 \leqslant r < \infty$, $0 \leqslant \theta \leqslant \pi$ are examined under the assumption of dynamic similarity, so that the stresses depend only on r/t, θ. Analytic solutions are given for constant surface traction on $\theta = 0$, $0 < r/t < V$, where V is constant, the rest of the surface being unloaded, and for a concentrated load at $r = 0$.

Numerical results are quoted for the particular case $V \to \infty$, corresponding to a load on half the bounding plane.

Introduction. From the time of Lord Rayleigh's discovery of elastic surface waves (4) the problem of elastic waves in a half-space has attracted the attention of many writers. Notable contributions are the classical paper of Lamb (2) in which he considered the waves generated in a half-space by point and line sources in the surface, and the recent extensions of his work by Lapwood (3) and Sauter (5). Sherwood (7) has also given a solution of this type of problem. Each of these writers used some variation of the technique of Fourier analysis, building up general solutions from the special case of a simple harmonic wave. More or less complicated methods were then used to evaluate the repeated integrals in the complex plane which resulted from the superposition of solutions to satisfy given boundary conditions.

A different type of problem, suggested by the needs of road constructors rather than the observations of seismologists, has been studied by Sneddon (8) and Cole and Huth (1), who found quasi-steady solutions for the elastic waves generated by concentrated loads moving over the surface of the half-space with uniform speed.

Both the above-mentioned methods are useful for studying the nature of the elastic waves which already exist in a material, but neither is entirely appropriate for the examination of transient effects. Suppose a load is applied to a part of the surface of a material initially at rest. Then one would expect, at least in the early stages, that the motion of the material would extend only over a finite region, bounded by wave fronts emanating from the loaded part. To describe such a motion in terms of simple harmonic waves, extant in the whole of space, but with zero resultant outside of the wave front, is to invite mathematical complication. On the other hand, the quasi-steady solutions are obviously inapplicable. These considerations suggest the use of some other method of reducing the wave problem to one involving only two independent variables.

In the present paper the method chosen is that of dynamic similarity, in which the components of stress, strain and material velocity depend only on the variables x/t, y/t, where x, y are Cartesian coordinates and t the time. Solutions with this symmetry can be superimposed to fit more general boundary conditions and have two advantages over the harmonic type of solution. First, the disturbance emanating from a given bounded region is bounded at any time by clearly defined wave fronts, which are easily predicted on physical grounds, and the physical meaning of the

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mathematical solution is much more easily grasped. Secondly, no complex inversion theorems are needed, the numerical solution for general boundary conditions depends only on simple quadrature.

The specification of the basic problem studied is given in § 1, where the notation is defined. The results are summarized in § 14 for the most important case, and some comments are made in § 15.

1. *Specification of the problem.* Consider the half-space specified in cylindrical polar coordinates (r, θ, z) by $0 < \theta < \pi$, $r > 0$, initially at rest and subject to given boundary conditions, depending on the time, on the surface $\theta = 0, \pi$. Assume linear elasticity with Lamé constants λ, μ.

Let the displacement vector have components $(u_r, u_\theta, 0)$, and call the non-vanishing stress components σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$, σ_{zz}, where σ_{zz} is not used in the analysis but may be determined if necessary from the condition of plane strain. Assume that the velocity and stress components are initially all zero, and are always independent of z.

There are two particular sets of boundary conditions which may be expected to give rise to solutions with dynamic similarity, in the sense that the stress and velocity components will depend only on θ, and $r/t = \omega$.

I. A constant surface traction, $\sigma_{\theta\theta} = -P$, $\sigma_{r\theta} = S$ is applied over the part $0 < r < Vt$ of the surface, $\theta = 0$, where V is a constant of the dimensions of velocity, and the rest of the surface of the half-space is free from stress.

II. A concentrated line load, with direct and tangential components increasing linearly with time, is applied for $t > 0$ to the surface over the line $r = 0$.

These two problems are subsequently referred to as problem I and problem II.

2. *The equations of motion.* Choose scalar and vector potentials Φ and Ψ such that

$$u_r = \frac{\partial \Phi}{\partial r} + \frac{1}{r}\frac{\partial \Psi}{\partial \theta}, \quad u_\theta = \frac{1}{r}\frac{\partial \Phi}{\partial \theta} - \frac{\partial \Psi}{\partial r}. \tag{1}$$

The stress components are then

$$\sigma_{rr} = \lambda \nabla^2 \Phi + 2\mu \left(\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial^2 \Psi}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \Psi}{\partial \theta} \right), \tag{2}$$

$$\sigma_{r\theta} = \mu \left\{ \frac{2}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta} - \frac{2}{r^2}\frac{\partial \Phi}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2} - \frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} \right\}, \tag{3}$$

$$\sigma_{\theta\theta} = \lambda \nabla^2 \Phi + 2\mu \left(\frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial \Psi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \Psi}{\partial r \partial \theta} \right), \tag{4}$$

and the equations of motion reduce to

$$\frac{1}{c_1^2}\frac{\partial^2 \Phi}{\partial t^2} = \nabla^2 \Phi \quad \text{with} \quad \rho c_1^2 = \lambda + 2\mu \tag{5}$$

and

$$\frac{1}{c_2^2}\frac{\partial^2 \Psi}{\partial t^2} = \nabla^2 \Psi \quad \text{with} \quad \rho c_2^2 = \mu, \tag{6}$$

where ρ is the density of the material and

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

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Now look for solutions in which

$$\partial^2\Phi/\partial t^2 = \phi, \quad \partial^2\Psi/\partial t^2 = \psi \tag{7}$$

depend only on $\omega = r/t$ and θ. Then (5) and (6), differentiated twice with respect to t, reduce to

$$\omega^2\left(\frac{\omega^2}{c_1^2}-1\right)\frac{\partial^2\phi}{\partial\omega^2}+\omega\left(\frac{2\omega^2}{c_1^2}-1\right)\frac{\partial\phi}{\partial\omega}-\frac{\partial^2\phi}{\partial\theta^2} = 0 \tag{8}$$

and

$$\omega^2\left(\frac{\omega^2}{c_2^2}-1\right)\frac{\partial^2\psi}{\partial\omega^2}+\omega\left(\frac{2\omega^2}{c_2^2}-1\right)\frac{\partial\psi}{\partial\omega}-\frac{\partial^2\psi}{\partial\theta^2} = 0. \tag{9}$$

The treatment of (8) and (9) follows closely that of the 'cone-field' equations of linearized supersonic flow in fluid dynamics (cf. Ward (9)).

In (8), for $\omega > c_1$, write

$$\omega = c_1 \sec\alpha_1, \tag{10}$$

so that

$$\frac{\partial^2\phi}{\partial\alpha_1^2}-\frac{\partial^2\phi}{\partial\theta^2} = 0, \tag{11}$$

with the characteristic form of solution

$$\phi = f_1(\alpha_1-\theta)+g_1(\alpha_1+\theta). \tag{12}$$

The equation is then hyperbolic, with characteristic lines, $\alpha_1 \pm \theta =$ constant, given by the tangents to the circle $\omega = c_1$ in a plane with polar coordinates (ω, θ) (Fig. 1). As usual in a problem of mixed hyperbolic-elliptic type, the parts of the characteristic line on opposite sides of its point of contact with the envelope, in this case the circle $\omega = c_1$, are to be reckoned as distinct and of different families.

Again, for $\omega < c_1$, put

$$\omega = c_1 \operatorname{sech}(-\beta_1), \tag{13}$$

where the minus sign is included to make $\beta_1 \to -\infty$ as $\omega \to 0$. Then (8) reduces to

$$\frac{\partial^2\phi}{\partial\beta_1^2}+\frac{\partial^2\phi}{\partial\theta^2} = 0, \tag{14}$$

with the solution

$$w_1 = \phi + i\chi = w_1(\beta_1 + i\theta), \tag{15}$$

where χ is an auxiliary variable defined to complete the complex potential w_1.

Equation (9) is treated similarly, giving

$$\psi = f_2(\alpha_2-\theta)+g_2(\alpha_2+\theta) \quad (\omega > c_2), \tag{16}$$

where

$$\omega = c_2 \sec\alpha_2, \tag{17}$$

and

$$w_2 = \psi + i\kappa = w_2(\beta_2 + i\theta) \quad (\omega < c_2) \tag{18}$$

with

$$\omega = c_2 \operatorname{sech}(-\beta_2). \tag{19}$$

To express the boundary conditions in terms of ϕ and ψ, differentiate the equations (2)–(4) twice with respect to time. Then, for example

$$t^2\frac{\partial^2\sigma_{rr}}{\partial t^2} = (\lambda+2\mu)\frac{\partial^2\phi}{\partial\omega^2}+\lambda\left(\frac{1}{\omega}\frac{\partial\phi}{\partial\omega}+\frac{1}{\omega^2}\frac{\partial^2\phi}{\partial\theta^2}\right)+2\mu\frac{\partial}{\partial\omega}\left(\frac{1}{\omega}\frac{\partial\psi}{\partial\theta}\right). \tag{20}$$

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Use of (8) and (9) then leads to

$$\frac{t^2}{\mu}\frac{\partial^2\sigma_{rr}}{\partial t^2} = 2\frac{\partial^2\phi}{\partial\omega^2}+\left(\frac{\omega^2}{c_2^2}-\frac{2\omega^2}{c_1^2}\right)\frac{\partial^2\phi}{\partial\omega^2}+2\left(\frac{\omega}{c_2^2}-\frac{2\omega}{c_1^2}\right)\frac{\partial\phi}{\partial\omega}+2\frac{\partial}{\partial\omega}\left(\frac{1}{\omega}\frac{\partial\psi}{\partial\theta}\right)$$

$$= \frac{\partial}{\partial\omega}\left\{\left(2+\frac{\omega^2}{c_2^2}-\frac{2\omega^2}{c_1^2}\right)\frac{\partial\phi}{\partial\omega}+\frac{2}{\omega}\frac{\partial\psi}{\partial\theta}\right\}; \tag{21}$$

but $\partial/\partial t = -(r/t^2)\,\partial/\partial\omega$ so (21) may be integrated to give

$$\frac{r}{\mu}\frac{\partial\sigma_{rr}}{\partial t} = -\left(2+\frac{\omega^2}{c_2^2}-\frac{2\omega^2}{c_1^2}\right)\frac{\partial\phi}{\partial\omega}-\frac{2}{\omega}\frac{\partial\psi}{\partial\theta}, \tag{22}$$

provided that $\omega^2(\partial\phi/\partial\omega)$, $\partial\psi/\omega\partial\theta \to 0$ as $\omega\to\infty$, uniformly in θ. Similarly (3) and (4) lead to

$$\frac{r}{\mu}\frac{\partial\sigma_{\theta\theta}}{\partial t} = \left(2-\frac{\omega^2}{c_2^2}\right)\frac{\partial\phi}{\partial\omega}+\frac{2}{\omega}\frac{\partial\psi}{\partial\theta} \tag{23}$$

and

$$\frac{r}{\mu}\frac{\partial\sigma_{r\theta}}{\partial t} = -\frac{2}{\omega}\frac{\partial\phi}{\partial\theta}+\left(2-\frac{\omega^2}{c_2^2}\right)\frac{\partial\psi}{\partial\omega}, \tag{24}$$

under corresponding conditions.

3. *Discontinuities on plane wave fronts.* Since the equations of motion are linear, it is necessary to examine the possibility of a solution discontinuous across a plane wave front corresponding to a characteristic of the equations. To avoid undue algebraic complication, use Cartesian coordinates and assume discontinuities in stress components across a plane $x = ct$, such that

$$\sigma_{xx} = \sigma_{xy} = \sigma_{yy} = u_x = u_y = 0 \quad (x > ct)$$

with

$$\sigma_{xx} = \delta_1, \quad \sigma_{xy} = \delta_2, \quad \sigma_{yy} = \delta_3 \quad (x < ct).$$

The equations of elasticity give

$$\delta_1 = (\lambda+2\mu)\frac{\partial^2\Phi}{\partial x^2}+\lambda\frac{\partial^2\Phi}{\partial y^2}+2\mu\frac{\partial^2\Psi}{\partial x\partial y} \quad (x < ct), \tag{25}$$

$$\delta_2 = \mu\left(2\frac{\partial^2\Phi}{\partial x\partial y}+\frac{\partial^2\Psi}{\partial y^2}-\frac{\partial^2\Psi}{\partial x^2}\right) \quad (x < ct) \tag{26}$$

and

$$\delta_3 = \lambda\frac{\partial^2\Phi}{\partial x^2}+(\lambda+2\mu)\frac{\partial^2\Phi}{\partial y^2}-2\mu\frac{\partial^2\Psi}{\partial x\partial y} \quad (x < ct); \tag{27}$$

and conservation of momentum leads to

$$\rho c\left(\frac{\partial^2\Phi}{\partial x\partial t}+\frac{\partial^2\Psi}{\partial y\partial t}\right) = -\delta_1 \quad (x < ct) \tag{28}$$

and

$$\rho c\left(\frac{\partial^2\Phi}{\partial y\partial t}-\frac{\partial^2\Psi}{\partial x\partial t}\right) = -\delta_2 \quad (x < ct). \tag{29}$$

The conditions of continuity of displacement in time and space lead to

$$\frac{\partial^2\Phi}{\partial x\partial y}+\frac{\partial^2\Psi}{\partial y^2} = \frac{\partial^2\Phi}{\partial y^2}-\frac{\partial^2\Psi}{\partial x\partial y} = 0 \quad (x < ct), \tag{30}$$

$$\frac{\partial^2\Phi}{\partial x\partial t}+\frac{\partial^2\Psi}{\partial y\partial t}+c\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Psi}{\partial x\partial y}\right) = 0 \quad (x < ct) \tag{31}$$

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and
$$\frac{\partial^2\Phi}{\partial y\,\partial t}-\frac{\partial^2\Psi}{\partial x\,\partial t}+c\left(\frac{\partial^2\Phi}{\partial y\,\partial x}-\frac{\partial^2\Psi}{\partial x^2}\right)=0 \quad (x<ct). \tag{32}$$

Equations (25), (28), (30) and (31) imply

$$\rho c^2\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Psi}{\partial x\,\partial y}\right)=\delta_1=(\lambda+2\mu)\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Psi}{\partial x\,\partial y}\right), \tag{33}$$

giving $\delta_1 = 0$ or $c = c_1$. Similarly, from (26), (28), (29) and (32)

$$\rho c^2\left(\frac{\partial^2\Phi}{\partial y\,\partial x}-\frac{\partial^2\Psi}{\partial x^2}\right)=\delta_2=\mu\left(\frac{\partial^2\Phi}{\partial x\,\partial y}-\frac{\partial^2\Psi}{\partial x^2}\right), \tag{34}$$

with $\delta_2 = 0$ or $c = c_2$.

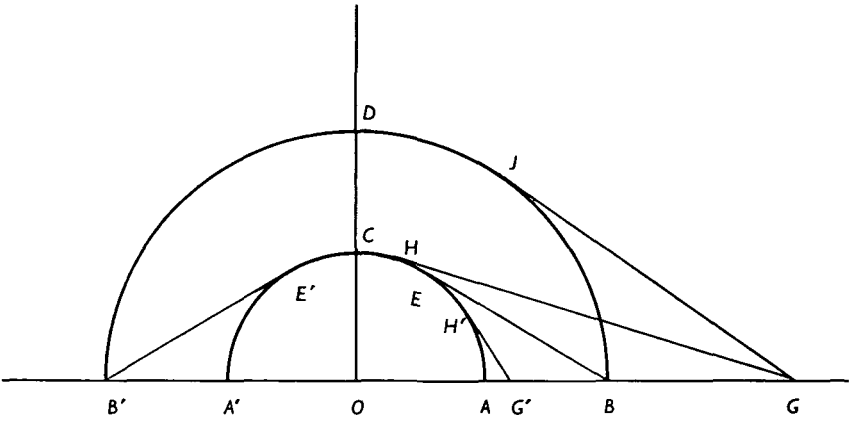


Fig. 1. The (ω, θ) plane.

Consider first a longitudinal wave, $\delta_2 = 0$, $c = c_1$. Then from (33), (34), (5) and the condition that the displacement is continuous at $x = c_1 t$, the values of the potentials for $x < c_1 t$ are

$$\Phi=\frac{\delta_1}{2\rho c_1^2}(x-c_1t)^2, \quad \Psi=0. \tag{35}$$

Similar reasoning shows that for a transverse wave $\delta_1 = \delta_3 = 0$, $c = c_2$,

$$\Phi=0, \quad \Psi=\frac{-\delta_2}{2\rho c_2^2}(x-c_2t)^2 \quad (x<c_2t). \tag{36}$$

Equations (35) and (36) represent the basic discontinuous solutions of the equations in terms of Cartesian coordinates, and may be applied across any characteristics of the equations (5) and (6) by a simple rotation of axes. Note that a longitudinal wave corresponds to a simple discontinuity in $\phi = \partial^2\Phi/\partial t^2$, and a transverse wave to a simple discontinuity in ψ.

4. *The solution for* $\omega > c_1$. It is convenient to explain the method of solution by reference to a diagram in the (ω, θ) plane (Fig. 1) where O is the origin, ACA' is the semicircle $\omega = c_2$, $0 \leqslant \theta \leqslant \pi$, and BDB' a semicircle $\omega = c_1$, $0 \leqslant \theta \leqslant \pi$. The lines BE, $B'E'$ are tangents to ACA' from B, B', respectively, and are the characteristics of

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(9) which form the boundary of the von Schmidt wave. For $V > c_1$, take the point G as $\theta = 0$, $\omega = V$, corresponding to the boundary conditions of problem I, and the tangents GJ, GH as the characteristics of equations (8) and (9) which pass through G. For $c_2 < V < c_1$, the corresponding point G' defines one relevant characteristic $G'H'$. The boundary conditions are

$$\sigma_{\theta\theta} = -P, \quad \sigma_{r\theta} = S; \qquad \theta = 0, \quad \omega < V. \tag{37}$$

The initial conditions, of zero stress and displacement as $t \to 0+$, correspond to conditions as $\omega \to \infty$, and, with the boundary conditions that the surface is free from traction except for $\theta = 0$, $0 < \omega < V$, determine $\phi = \psi = 0$, outside of the figure $GB'DJG$, if $V > c_1$. The conditions (37) are then satisfied on BG if $\rho\phi = \delta_1$, constant in the region BJG and $\rho\psi = \delta_2$, constant in the region $GHEB$, where by (35) and (36), with the usual formulae for rotation of Cartesian axes,

$$\frac{c_2^2\delta_1}{V^2} = \frac{(2 - V^2c_2^{-2})P + 2S(V^2c_2^{-2} - 1)^{\frac{1}{2}}}{(2 - V^2c_2^{-2})^2 + 4(V^2c_2^{-2} - 1)^{\frac{1}{2}}(V^2c_1^{-2} - 1)^{\frac{1}{2}}}, \tag{38}$$

$$\frac{c_2^2\delta_2}{V^2} = \frac{2(V^2c_1^{-2} - 1)^{\frac{1}{2}}P + (2 - V^2c_2^{-2})S}{(2 - V^2c_2^{-2})^2 + 4(V^2c_2^{-2} - 1)^{\frac{1}{2}}(V^2c_1^{-2} - 1)^{\frac{1}{2}}}. \tag{39}$$

In problem I for the case $V < c_1$, $\phi \equiv 0$ for $\omega > c_1$ and $\psi \equiv 0$ outside of $BECE'B'$. The results of later sections will show, however, that the formal solutions for w_1, w_2 deduced on the assumption that $V > c_1$ are in fact valid without restriction on V.

In problem II, the entire surface, apart from the singular point O, is free from traction and ϕ, ψ are therefore zero everywhere outside of the circle $\omega = c_1$.

5. *The von Schmidt waves.* The key to the analytic solution of the problems is provided by the conditions in the von Schmidt waves, the regions ABE, $A'B'E'$. (These waves are sometimes referred to as head waves.) The arguments applied to both regions are similar, and it suffices to consider ABE. Then in ABE, the equation (9) for ψ is hyperbolic, and the condition $\psi = 0$, $\omega \to \infty$, with the boundary conditions on $\theta = 0$, shows that the function f_2 of (16) vanishes throughout the region, and that

$$\psi = g_2(\alpha_2 + \theta) \quad \text{in} \quad ABE. \tag{40}$$

It follows from (40) that $\partial\psi/\partial\alpha_2 - \partial\psi/\partial\theta = 0$, and from (17) that

$$\frac{\partial\psi}{\partial\theta} - \omega\left(\frac{\omega^2}{c_2^2} - 1\right)^{\frac{1}{2}}\frac{\partial\psi}{\partial\omega} = 0. \tag{41}$$

Now on AB the boundary conditions of constant traction give $(\partial\sigma_{r\theta}/\partial t) = (\partial\sigma_{\theta\theta}/\partial t) = 0$, so (23) and (24) with (41) imply

$$\left(2 - \frac{\omega^2}{c_2^2}\right)\frac{\partial\phi}{\partial\omega} = -\frac{2}{\omega}\frac{\partial\psi}{\partial\theta} = -2\left(\frac{\omega^2}{c_2^2} - 1\right)^{\frac{1}{2}}\frac{\partial\psi}{\partial\omega}$$
$$= \frac{4}{\omega}\left(\frac{\omega^2}{c_2^2} - 2\right)^{\frac{1}{2}}\left(\frac{\omega^2}{c_2^2} - 2\right)^{-1}\frac{\partial\phi}{\partial\theta} \quad (\theta = 0;\ c_2 < \omega < c_1).$$

But for $\omega < c_1$, $\partial\phi/\partial\theta = -\partial\chi/\partial\beta_1 = -\omega(1 - \omega^2/c_1^2)^{\frac{1}{2}}\partial\chi/\partial\omega$, so

$$\left(\frac{\omega^2}{c_2^2} - 2\right)^2\frac{\partial\phi}{\partial\omega} - 4\left(\frac{\omega^2}{c_2^2} - 1\right)^{\frac{1}{2}}\left(1 - \frac{\omega^2}{c_1^2}\right)^{\frac{1}{2}}\frac{\partial\chi}{\partial\omega} = 0 \quad (\theta = 0;\ c_2 < \omega < c_1). \tag{42}$$

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The left-hand side of (42) is the real part of the expression

$$\left\{\left(\frac{\omega^2}{c_2^2}-2\right)^2+4i\left(\frac{\omega^2}{c_2^2}-1\right)^{\frac{1}{2}}\left(1-\frac{\omega^2}{c_1^2}\right)^{\frac{1}{2}}\right\}\frac{\partial w_1}{\partial \omega}, \tag{43}$$

where, by the definition (15), $w_1=\phi+i\chi$; and the function (43) is therefore purely imaginary for $\theta=0$, $c_2<\omega<c_1$.

Now introduce a new variable

$$\zeta_1=\xi_1+i\eta_1=\operatorname{sech}(-\beta_1-i\theta), \tag{44}$$

so that the inside of the semicircle $B'OBDB'$ is mapped on the half-plane $\eta_1>0$. Then the line $B'OB$ corresponds to $\eta_1=0$, $-1<\xi_1<1$, and the arcs BD, DB' to $\eta_1=0$, $1<\xi_1<\infty$ and $-\infty<\xi_1<-1$, respectively. Also, from (13) and (44), the relation

$$\zeta_1=\omega/c_1$$

holds for points which, in the (ω,θ) plane, lie on $B'B$.

Now AB is part of $B'B$, so (42) is satisfied if

$$\frac{1}{c_1}\frac{dw_1}{d\zeta_1}=\frac{iF(\zeta_1)}{(m^2\zeta_1^2-2)^2+4i(m^2\zeta_1^2-1)^{\frac{1}{2}}(1-\zeta_1^2)^{\frac{1}{2}}} \quad (\eta_1=0;\ 1/m<\xi_1<1), \tag{45}$$

where $F(\zeta_1)$ is a function of ζ_1 real when ζ_1 is real and $1/m<\xi_1<1$, where m is defined by

$$m=c_1/c_2, \tag{46}$$

and where the surds are real on the line considered.

By exactly similar arguments, it can be shown that the boundary conditions on $B'A'$, and the characteristic condition in $B'A'E'$, can be satisfied by taking w_1 of the form (45), but with $F(\zeta_1)$ real on the real axis $-1<\xi_1<-1/m$. Then the principle of analytic continuation allows (45) to be used to define w_1 in the upper half of the ζ-plane.

6. *The conditions for* $\omega<c_2$. The next step is to determine the form of the function $w_2(\beta_2+i\theta)$ in the semicircle $\omega<c_2$, $0<\theta<\pi$, by using the boundary conditions on $A'A$ and the expression (45).

The functions $w_1=\phi+i\chi$, $w_2=\psi+i\kappa$ are analytic functions of the complex variables $(\beta_1+i\theta)$ and $(\beta_2+i\theta)$, respectively, so

$$\frac{\partial\phi}{\partial\theta}=-\frac{\partial\chi}{\partial\beta_1}, \quad \frac{\partial\psi}{\partial\theta}=-\frac{\partial\kappa}{\partial\beta_2}.$$

Use these conditions to eliminate the θ-derivatives from (23) and (24), set

$$\partial\sigma_{\theta\theta}/\partial t=\partial\sigma_{r\theta}/\partial t=0 \quad \text{with} \quad \theta=0,\pi; \quad 0<\omega<c_2,$$

and replace β_1 and β_2 by ω through (13) and (19). Then

$$(2-\omega^2/c_2^2)(\partial\phi/\partial\omega)-2(1-\omega^2/c_2^2)^{\frac{1}{2}}(\partial\kappa/\partial\omega)=0 \quad (\theta=0,\pi;\ \omega<c_2), \tag{47}$$

and

$$(2-\omega^2/c_2^2)(\partial\psi/\partial\omega)+2(1-\omega^2/c_1^2)^{\frac{1}{2}}(\partial\chi/\partial\omega)=0 \quad (\theta=0,\pi;\ \omega<c_2). \tag{48}$$

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Now the derivatives of ϕ, χ may be obtained from (45) and the derivatives of ψ, κ defined, on $\theta = 0, \pi$, by (47) and (48). Let

$$\zeta_2 = \xi_2 + i\eta_2 = \operatorname{sech}(-\beta_2 - i\theta) \tag{49}$$

so that the interior of the semicircle $\omega < c_2$, $0 < \theta < \pi$ is mapped on the upper half of the ζ_2-plane; then for $\theta = 0, \pi$, $\omega \leqslant c_2$, $\zeta_2 = \pm \omega/c_2$. The function $w_2(\zeta_2)$ can now be constructed on the section of the real axis where the boundary conditions hold, and extended by analytic continuation to the half-plane $\eta_2 > 0$.

The above process implies certain restrictions on the function $F(\zeta_1)$ of (45). The mapping of the ζ_1-plane on the ζ_2-plane is not a conformal mapping, yet both $w_1(\zeta_1)$ and $w_2(\zeta_2)$ must be analytic functions of their arguments. In particular, the real and imaginary parts of $dw_1/d\zeta_1$ on the line $(-1/m, 1/m)$ are multiplied by different factors to give the imaginary and real parts of $dw_2/d\zeta_2$. It follows that $dw_1/d\zeta_1$ must be the sum of analytic functions each of which is either entirely real or entirely imaginary on the line $(-1/m, 1/m)$ and that irrational factors with singular points $\pm 1/m$ must be either of the form $(m^2\zeta_1^2 - 1)^{\frac{1}{2}}$ or of the form $\log(m\zeta_1 - 1) - \log(m\zeta_1 + 1)$. In fact the occurrence of the logarithmic factor would violate a later condition, and this possibility is therefore ignored.

Again, there is no mathematical or physical reason to expect a singularity when $\omega^2/c_2^2 = 2$, so (45) may be rewritten as

$$\frac{1}{c_1}\frac{dw_1}{d\zeta_1} = \frac{i(m^2\zeta_1^2 - 2)\,F_1(\zeta_1) + 2i(m^2\zeta_1^2 - 1)^{\frac{1}{2}}\,F_2(\zeta_1)}{(m^2\zeta_1^2 - 2)^2 + 4i(m^2\zeta_1^2 - 1)^{\frac{1}{2}}(1 - \zeta_1^2)^{\frac{1}{2}}}, \tag{50}$$

where F_1, F_2 are both real on the real axis, $-1 < \zeta_1 < 1$. The expression for $dw_2/d\zeta_2$ deduced from (47) and (48) is then

$$\frac{1}{c_2}\frac{dw_2}{d\zeta_2} = \frac{2(1 - \zeta_2^2/m^2)^{\frac{1}{2}}\,F_1(\zeta_2/m) - (2 - \zeta_2^2)\,iF_2(\zeta_2/m)}{(\zeta_2^2 - 2)^2 - 4(1 - \zeta_2^2)^{\frac{1}{2}}(1 - \zeta_2^2/m^2)^{\frac{1}{2}}}. \tag{51}$$

Finally, the solution of § 4 shows that $\partial\phi/\partial\theta = 0$ on BDB', except for the singular point J, and hence that $\partial\phi/\partial\xi_1 = 0$ for $\eta_1 = 0$, $-\infty < \xi_1 < -1$ and $1 < \xi_1 < \infty$. It is this condition which excludes the logarithmic singularity. It follows that $F_1(\zeta_1)$ and $F_2(\zeta_1)$ are functions analytic in the upper half-plane $\eta_1 > 0$, real on the whole real axis, with no singularities except poles on the real axis. They are therefore rational functions of ζ_1.

7. *Singularities of $dw_1/d\zeta_1$, $dw_2/d\zeta_2$.* The expressions (50) and (51) now contain two unknown functions, $F_1(\zeta_1)$, $F_2(\zeta_1)$ each of which is a rational function of ζ_1 with real coefficients. The complete specification of these functions depends on the following conditions.

(i) No singularity in the potentials ϕ, ψ is to be expected at D or C (Fig. 1). But D corresponds to the point at infinity in the ζ_1-plane (and C to the point at infinity in the ζ_2-plane). The point at infinity is therefore an ordinary point of the first-order differential equation (50) and

$$F_1(\zeta_1) = O(1), \quad F_2(\zeta_1) = O(1) \quad (\zeta_1 \to \infty). \tag{52}$$

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The condition (52) is sufficient to ensure that the point at infinity is also an ordinary point of (51).

Now a rational function bounded at infinity may be specified in terms of its poles, and in the present case these must all lie on the real axis. It is therefore sufficient to examine the significance of real poles of $F_1(\zeta_1)$, $F_2(\zeta_2)$.

(ii) Consider first the possibility of a multiple pole. Then a multiple pole of $dw_1/d\zeta_1$ implies a pole of $w_1(\zeta_1)$, and the type of stress singularity associated with a concentrated force. Only in problem II is a concentrated force envisaged, so in problem I multiple poles of $dw_1/d\zeta_1$ must be excluded. In problem II there may be a double pole of $dw_1/d\zeta_1$ at the origin, $\zeta_1 = 0$ (see § 8). Note that the term

$$D = (2-m^2\zeta_1^2)^2 + 4i(m^2\zeta_1^2-1)^{\frac{1}{2}}(1-\zeta_1^2)^{\frac{1}{2}} \tag{53}$$

has a double zero at the origin, so even in problem II, $F_1(\zeta_1)$, $F_2(\zeta_1)$ are finite at $\zeta_1 = 0$, and have only simple poles elsewhere.

(iii) Finally, consider a simple pole of $dw_1/d\zeta_1$. Then it will be shown in §§ 9, 10 that a simple pole on the real axis corresponds, in general, to a discontinuity in the value of $\sigma_{\theta\theta}$ or $\sigma_{r\theta}$ on the (physical) surface $\theta = 0, \pi$. The boundary conditions exclude such discontinuities except at the points $r = 0$, ($\zeta_1 = 0, \zeta_2 = 0$) or $r = Vt$ (which corresponds to $\zeta_1 = V/c_1$, $\zeta_2 = V/c_2$).

It follows from these arguments that the most general forms for F_1, F_2 are

$$F_1(\zeta_1) = \frac{L\zeta_1}{\zeta_1 - V/c_1}, \quad F_2(\zeta_1) = \frac{M\zeta_1}{\zeta_1 - V/c_1} \quad \text{(problem I)}, \tag{54}$$

or

$$F_1(\zeta_1) = L', \quad F_2(\zeta_1) = M' \quad \text{(problem II)}, \tag{54a}$$

where L, M, L', M' are real constants. Note that (54a) is the form taken by (54) as $V \to 0$.

8. *Analysis near the origin.* Assume that, as $\zeta_1 \to 0$,

$$F_1(\zeta_1) = L' + O(\zeta_1), \quad F_2(\zeta_1) = M' + O(\zeta_1).$$

Then (50), (51) lead to

$$\frac{1}{c_1}\frac{dw_1}{d\zeta_1} = \frac{-2iL'-2M'}{2(1-m^2)\zeta_1^2} + O\left(\frac{1}{\zeta_1}\right),$$

$$\frac{1}{c_2}\frac{dw_2}{d\zeta_2} = \frac{m^2(L'-iM')}{(1-m^2)\zeta_2^2} + O\left(\frac{1}{\zeta_2}\right).$$

Now near the origin ($\omega \to 0$),

$$d\zeta_1/d(\beta_1 + i\theta) = \zeta_1(1-\zeta_1^2)^{\frac{1}{2}}, \quad d\omega/d\beta_1 = \omega(1-\omega^2/c_1^2)^{\frac{1}{2}}$$

and

$$\zeta_1 = \frac{\omega\, e^{i\theta}}{c_1}\left(1 - \frac{\omega^2}{2c_1^2}e^{i\theta}\sin\theta\right),$$

with similar expressions for ζ_2; so from (22) and (23),

$$\frac{r}{\mu}\frac{\partial\sigma_{rr}}{\partial t} = -(m^2-1)M'\sin\theta + O\left(\frac{\omega}{c_1}\right),$$

$$\frac{r}{\mu}\frac{\partial\sigma_{r\theta}}{\partial t} = (m^2-1)L'\sin\theta + O\left(\frac{\omega}{c_1}\right).$$

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The force applied to the surface of a semi-cylinder, $r = \epsilon$, $\epsilon \to 0$ has components $-St$, Pt per unit length, tangential and normal to the plane $\theta = 0, \pi$, where

$$S = \tfrac{1}{2}\pi\mu(m^2-1)\,L', \quad P = \tfrac{1}{2}\pi\mu(m^2-1)\,M'. \tag{55}$$

Thus $L' = M' = 0$ in problem I, and (55) apply to problem II.

9. *A pole for* $\zeta_1 < m^{-1}$. Next assume that, for real A, B, V, as $\zeta_1 \to V/c_1$,

$$\frac{F_1(\zeta_1)}{D(\zeta_1)} = \frac{AV/c_1}{\zeta_1 - V/c_1} + O(1), \quad \frac{F_2(\zeta_1)}{D(\zeta_1)} = \frac{BV/c_1}{\zeta_1 - V/c_1} + O(1), \tag{56}$$

with $V < c_2$. This corresponds to a simple pole at $\omega = V$. Then (50) and (51) give

$$\frac{1}{V}\frac{dw_1}{d\zeta_1} = \frac{iA(V^2/c_2^2-2) - 2B(1-V^2/c_2^2)^{\frac{1}{2}}}{\zeta_1 - V/c_1} + O(\zeta_1 - V/c_1),$$

$$\frac{1}{V}\frac{dw_2}{d\zeta_2} = \frac{2A(1-V^2/c_1^2)^{\frac{1}{2}} - iB(2-V^2/c_2^2)}{\zeta_2 - V/c_2} + O(\zeta_2 - V/c_2).$$

Put $\omega = V + \epsilon_1$, $\theta = \epsilon_2$, and use

$$\zeta_1 = (\omega/c_1)\{\cos\theta + i(1-\omega^2/c_1^2)^{\frac{1}{2}}\sin\theta\}^{-1}.$$

Then, treating ϵ_1, ϵ_2 as small and neglecting squares,

$$\zeta_1 - V/c_1 = c_1^{-1}\{\epsilon_1 + iV(1-V^2/c_1^2)^{\frac{1}{2}}\epsilon_2\}$$

and

$$\frac{1}{c_1}\frac{dw_1}{d(\beta_1 + i\theta)} = \frac{\zeta_1(1-\zeta_1^2)^{\frac{1}{2}}}{c_1}\frac{dw_1}{d\zeta_1}$$

$$= \frac{-V^2}{c_1^2}\left(1 - \frac{V^2}{c_1^2}\right)^{\frac{1}{2}}\left\{\frac{iA(2-V^2/c_2^2) + 2B(1-V^2/c_2^2)^{\frac{1}{2}}}{\epsilon_1 + iV(1-V^2/c_1^2)^{\frac{1}{2}}\epsilon_2}\right\}.$$

Separation into real and imaginary parts gives

$$\frac{1}{c_1}\frac{\partial\phi}{\partial\omega} = \frac{(1-\omega^2/c_1^2)^{-\frac{1}{2}}}{c_1\omega}\frac{\partial\phi}{\partial\beta_1} = \frac{-2BV\epsilon_1(1-V^2/c_2^2)^{\frac{1}{2}} + AV^2\epsilon_2(V^2/c_2^2-2)(1-V^2/c_1^2)^{\frac{1}{2}}}{\epsilon_1^2 + V^2\epsilon_2^2(1-V^2/c_1^2)}, \tag{57}$$

$$\frac{1}{\omega c_1}\frac{\partial\phi}{\partial\theta} = \frac{-AV\epsilon_1(V^2/c_2^2-2)(1-V^2/c_1^2)^{\frac{1}{2}} - 2BV^2\epsilon_2(1-V^2/c_2^2)^{\frac{1}{2}}(1-V^2/c_1^2)}{\epsilon_1^2 + V^2\epsilon_2^2(1-V^2/c_1^2)}. \tag{58}$$

Similar analysis gives

$$\frac{1}{c_1}\frac{\partial\psi}{\partial\omega} = \frac{2AV\epsilon_1(1-V^2/c_1^2)^{\frac{1}{2}} + BV^2\epsilon_2(V^2/c_2^2-2)(1-V^2/c_2^2)^{\frac{1}{2}}}{\epsilon_1^2 + V^2\epsilon_2^2(1-V^2/c_2^2)},$$

$$\frac{1}{\omega c_1}\frac{\partial\psi}{\partial\theta} = \frac{2AV^2\epsilon_2(1-V^2/c_2^2)(1-V^2/c_1^2)^{\frac{1}{2}} - BV\epsilon_1(V^2/c_2^2-2)(1-V^2/c_2^2)^{\frac{1}{2}}}{\epsilon_1^2 + V^2\epsilon_2^2(1-V^2/c_2^2)}.$$

Now substitute in (23), write

$$\mu^{-1}\omega^2\partial\sigma_{\theta\theta}/\partial\epsilon_1 = -r(\mu c_1)^{-1}\partial\sigma_{\theta\theta}/\partial t$$

and integrate with respect to ϵ_1. Then for small ϵ_1, ϵ_2,

$$\frac{V^2}{\mu}\sigma_{\theta\theta} = AV\left[\left(2-\frac{V^2}{c_2^2}\right)^2\tan^{-1}\frac{\epsilon_1}{V\epsilon_2(1-V^2/c_1^2)^{\frac{1}{2}}} - 4\left(1-\frac{V^2}{c_2^2}\right)^{\frac{1}{2}}\left(1-\frac{V^2}{c_1^2}\right)^{\frac{1}{2}}\tan^{-1}\frac{\epsilon_1}{V\epsilon_2(1-V^2/c_2^2)^{\frac{1}{2}}}\right]$$

$$+ BV\left(1-\frac{V^2}{c_2^2}\right)^{\frac{1}{2}}(2-V^2/c_2^2)\log\left[\frac{\epsilon_1^2 + V^2\epsilon_2^2(1-V^2/c_1^2)}{\epsilon_1^2 + V^2\epsilon_2^2(1-V^2/c_2^2)}\right]. \tag{59}$$

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Now compare the values as $\theta = \epsilon_2 \to 0$ with ϵ_1 positive and negative. Then as ω increases through the value V the surface stress $\sigma_{\theta\theta}$ has a discontinuity

$$\frac{\pi\mu A}{V}\left\{\left(2-\frac{V^2}{c_2^2}\right)^2 - 4\left(1-\frac{V^2}{c_1^2}\right)^{\frac{1}{2}}\left(1-\frac{V^2}{c_2^2}\right)^{\frac{1}{2}}\right\}. \tag{60}$$

The corresponding treatment of (24) shows a discontinuity in $\sigma_{r\theta}$ of

$$\frac{\pi\mu B}{V}\left\{\left(2-\frac{V^2}{c_2^2}\right)^2 - 4\left(1-\frac{V^2}{c_1^2}\right)^{\frac{1}{2}}\left(1-\frac{V^2}{c_2^2}\right)^{\frac{1}{2}}\right\}. \tag{61}$$

Note that for σ_{rr} the analogous expression to (59) contains the two logarithmic terms, which are there combined, with different coefficients; so $\sigma_{rr} \to \infty$ as $\epsilon_1, \epsilon_2 \to 0$, unless $M = 0$. This is not unexpected from static theory. Note also that there is no discontinuity in $\sigma_{\theta\theta}$ or $\sigma_{r\theta}$ if V/c_1 is a solution of $D(\zeta_1) = 0$, that is if V is the velocity appropriate to a Rayleigh wave on the free surface. This point is further discussed in § 12.

10. *The effect of poles for* $\zeta_1 \geqslant m^{-1}$. It is convenient next to look at the effect of a pole $\zeta_1 = (V/c_1) > 1$. For real $\zeta_1 > 1$, $D(\zeta_1)$ is real. Also the point $\zeta_1 = V/c_1$ corresponds to the point J of Fig. 1 and the corresponding point $\zeta_2 = V/c_2$ to H. At these points the solution of § 4 shows that ϕ, ψ should have discontinuities given by δ_1, δ_2 of (38) and (39), and it will now be verified that this will be so if $\zeta_1 = V/c_1$ is a simple pole of $F_1(\zeta_1)$ and $F_2(\zeta_1)$.

For let $F_1(\zeta_1)$, $F_2(\zeta_2)$ have the forms given in (56) but with $V > c_1$. Then the discontinuities in ϕ, ψ are δ_1, δ_2 if the residues of $(dw_1/d\zeta_1)$, $(dw_2/d\zeta_2)$ are

$$iV\{(V^2/c_2^2-2)A - 2B(V^2/c_2^2-1)^{\frac{1}{2}}\} = \delta_1/(\pi i) \tag{62}$$

and

$$iV\{2(V^2/c_1^2-1)^{\frac{1}{2}}A - (2-V^2/c_2^2)B\} = \delta_2/(\pi i). \tag{63}$$

The argument can be reversed to show that the poles of $F_1(\zeta_1)$, $F_2(\zeta_1)$ correspond to discontinuous surface values of $\sigma_{\theta\theta}$, $\sigma_{r\theta}$.

The treatment of a pole for $m^{-1} < \zeta_1 < 1$ is a little more complicated. The relevant arguments are as follows. Assume forms

$$F_1(\zeta_1) = \frac{LV/c_1}{\zeta_1 - V/c_1} + O(1), \quad F_2(\zeta_1) = \frac{MV/c_1}{\zeta_1 - V/c_0} + O(1). \tag{64}$$

Then (50) and (51) give

$$\frac{1}{V}\frac{dw_1}{d\zeta_1} = \frac{iL(V^2/c_2^2-2) - 2iM(V^2/c_2^2-1)^{\frac{1}{2}}}{(\zeta_1 - V/c_1)\{(2-V^2/c_2^2)^2 + 4i(1-V^2/c_1^2)^{\frac{1}{2}}(V^2/c_2^2-1)^{\frac{1}{2}}\}} + O(1) \tag{65}$$

and

$$\frac{1}{V}\frac{dw_2}{d\zeta_2} = \frac{2L(1-V^2/c_1^2)^{\frac{1}{2}} - iM(2-V^2/c_2^2)}{(\zeta_2 - V/c_2)\{(2-V^2/c_2^2)^2 + 4i(1-V^2/c_1^2)^{\frac{1}{2}}(V^2/c_2^2-1)^{\frac{1}{2}}\}} + O(1). \tag{66}$$

Except for the slight complication that the denominator in (65) is complex, the treatment of the equation is the same as in § 9. The singularity of (65) contributes terms in $\log\{\epsilon_1^2 + V^2\epsilon_2^2(1-V^2/c_1^2)\}$ and $\tan^{-1}\{\epsilon_1(1-V^2/c_1^2)^{-\frac{1}{2}}V^{-1}\epsilon_2^{-1}\}$ to $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ near G' of Fig. 1. The singularity of (66), however, corresponds to the point H' and must be

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treated differently. Let $\omega = c_2$, $\theta = \sec^{-1} V/c_2 + \epsilon$, for small real ϵ, be a point near H'. Then it corresponds to

$$\zeta_2 = (V/c_2) + \epsilon(V^2/c_2^2 - 1)^{\frac{1}{2}} + O(\epsilon^2).$$

Substitute in (66) and integrate; then

$$\frac{\pi\mu}{V^2} w_2 = \frac{\{2L(1 - V^2/c_1^2)^{\frac{1}{2}} - Mi(2 - V^2/c_2^2)\} \log \epsilon}{\{(2 - V^2/c_2^2)^2 + 4i(1 - V^2/c_1^2)^{\frac{1}{2}} (V^2/c_2^2 - 1)\}^{\frac{1}{2}}} + O(\epsilon) \quad (\epsilon > 0)$$

with $\log \epsilon$ replaced by $i\pi + \log(-\epsilon)$ for $\epsilon < 0$. Now near G' set $\omega = V + \epsilon_1$, $\theta = \epsilon_2$ as before. Then $\psi = \psi(\epsilon)$ where $\epsilon = \epsilon_2 + \epsilon_1 V^{-1}(V^2/c_2^2 - 1)^{-\frac{1}{2}}$. For given L, M the discontinuities in $\sigma_{\theta\theta}$, $\sigma_{r\theta}$ may now be obtained as in § 9.

The above arguments do not apply immediately to simple poles of $F_1(\zeta_1)$, $F_2(\zeta_1)$ at $\zeta_1 = m^{-1}$ (the point A of Fig. 1) or $\zeta_1 = 1$ (the point B). In these cases a slight complication arises from the fact that different techniques must be used according as the singularity is approached from one side or the other. However, the arguments are in principle similar to those already given, and yield similar results.

11. *The final determination of the solution.* It has now been shown that any pole of $F_1(\zeta_1)$ or $F_2(\zeta_2)$ corresponds to a discontinuity in the surface tractions on the boundary of the half-space, and therefore that only such poles may be allowed as correspond to given discontinuities.

In problem II, then, $F_1(\zeta_1)$, $F_2(\zeta_1)$ are constant and, from (55),

$$F_1(\zeta_1) = L' = 2S\{\pi\mu(m^2 - 1)\}^{-1}, \tag{67}$$

$$F_2(\zeta_2) = M' = 2P\{\pi\mu(m^2 - 1)\}^{-1}. \tag{68}$$

Equations (67) and (68) with (50), (51) and (40) determine the potentials ϕ, ψ everywhere and the stresses follow from (23), (24) and (25).

In problem I, the forms must be

$$F_1(\zeta_1) = \frac{LV/c_1}{\zeta_1 - V/c_1}, \quad F_2(\zeta_1) = \frac{MV/c_1}{\zeta_1 - V/c_1}, \tag{69}$$

as was stated in (54), and now from (56), (60) and (61),

$$L = AD(V/c_1) = PV/(\pi\mu), \quad M = -SV/(\pi\mu), \tag{70}$$

provided that $V < c_2$, $D(V/c_1) \neq 0$. But for $V > c_1$ (62) and (63) again lead to (70), and the complete analysis for $c_2 \leqslant V \leqslant c_1$ (summarized above) gives the same form again.

So for all V such that $D(V/c_1) \neq 0$, the solutions in the elliptic regions are

$$\frac{\pi\mu}{V}\frac{dw_1}{d\zeta_1} = \frac{\{iP(m^2\zeta_1^2 - 2) - 2iS(m^2\zeta_1^2 - 1)^{\frac{1}{2}}\} c_1\zeta_1}{(\zeta_1 - V/c_1)\{(m^2\zeta_1^2 - 2)^2 + 4i(m^2\zeta_1^2 - 1)^{\frac{1}{2}}(1 - \zeta_1^2)^{\frac{1}{2}}\}}, \tag{71}$$

$$\frac{\pi\mu}{V}\frac{dw_2}{d\zeta_2} = \frac{\{2P(1 - \zeta_2^2/m^2)^{\frac{1}{2}} - S(2 - \zeta_2^2)\} c_2\zeta_2}{(\zeta_2 - V/c_2)\{(\zeta_2^2 - 2)^2 - 4(1 - \zeta_2^2)^{\frac{1}{2}}(1 - \zeta_2^2/m^2)^{\frac{1}{2}}\}}. \tag{72}$$

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12. *The Rayleigh wave.* Note that in both (71) and (72) the denominator has a simple zero when ω is a solution of

$$\left(\frac{\omega^2}{c_2^2}-2\right)^2-4\left(1-\frac{\omega^2}{c_2^2}\right)^{\frac{1}{2}}\left(1-\frac{\omega^2}{c_1^2}\right)^{\frac{1}{2}}=0, \tag{73}$$

but from (60) and (61), the surface tractions are continuous at the corresponding points, $r=\pm\omega t$, $\theta=0$. If the applied shear stress is non-zero, the dilatation is infinite at the point. For any S or P either the horizontal or vertical velocity component is infinite at the same points.

Now (73) is the equation for the velocity of Rayleigh waves, and Lamb's solution (2) indicates the presence of precisely this type of singularity. It is acceptable on physical grounds because the associated energy is finite.

Secondly, observe that if in (71), (72) V is the Rayleigh wave velocity, the expressions on the right have double poles as $\zeta_1\to V/c_1$, $\zeta_2\to V/c_2$ and (for example) the dilatation is infinite like a simple pole. Such a singularity implies infinite energy, and must be rejected. The present solution, then, like those of Sneddon (8) and Cole and Huth (1) breaks down for a load travelling at the speed of the Rayleigh wave. This is not surprising, because there is then some kind of resonance, and one would not expect a dynamically similar solution.

13. *The conditions on the surface.* For the discussion of conditions on the surface of the half-space it is convenient to use Cartesian coordinates, so that the surface is $y=0$. The stress components σ_{yy}, σ_{xy} are given by the boundary conditions. The remaining component, σ_{xx}, can be found by integration of (71). For

$$\phi=\partial^2\Phi/\partial t^2=c_1^2\nabla^2\Phi=(\lambda+2\mu)(\sigma_{xx}+\sigma_{yy})/\{2\rho(\lambda+\mu)\} \tag{74}$$

and from (64)

$$\phi(x/t)=\Re\int_{-\infty}^{x/t}\left(\frac{dw_1}{d\zeta_1}\right)d\zeta_1\quad(y=0), \tag{75}$$

where the integral is taken along the line $\eta=0$, indented to pass anticlockwise round the singular points. The integral in (75) can be expressed in terms of known elliptic integrals, but the algebra is so cumbersome that it is easier to integrate (75) numerically. When the applied force is normal to the surface, so that $S=M=0$, the dilatation is finite everywhere except at the origin, where it tends to infinity like $\log r$ in problem I and like r^{-1} in II. An applied force tangential to the surface gives a logarithmic infinity of dilatation at the surface wave and also, if $V<c_2$, at the outer edge of the loaded area. Corresponding singularities are already well known to occur in static loading problems in two dimensions at the edges of a load, and the singularity at the surface wave is implied in the solution found by Lamb (2).

The velocity components on the surface are a little more troublesome. Equations (71) and (72) must be separated into real and imaginary parts to give $\partial\phi/\partial\omega$, $\partial\phi/\partial\theta$, $\partial\psi/\partial\omega$, and $\partial\psi/\partial\theta$. Then since

$$\frac{\partial^2 u_r}{\partial t^2}=\frac{\partial\phi}{\partial r}+\frac{1}{r}\frac{\partial\psi}{\partial\theta}=\frac{1}{t}\left(\frac{\partial\phi}{\partial\omega}+\frac{1}{\omega}\frac{\partial\psi}{\partial\theta}\right)=-\frac{r}{t^2}\frac{\partial}{\partial\omega}\left(\frac{\partial u_r}{\partial t}\right),$$

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the value of $\dot{u}_r = \dot{u}_x$ must be found from

$$\partial \dot{u}_r/\partial\omega = \omega^{-1}\partial\phi/\partial\omega + \omega^{-2}\partial\psi/\partial\theta \tag{76}$$

and $\dot{u}_\theta = \dot{u}_y$ from

$$\partial \dot{u}_\theta/\partial\omega = \omega^{-2}\partial\phi/\partial\theta - \omega^{-1}\partial\psi/\partial\omega. \tag{77}$$

Equations (70) and (71) must then be integrated from $\omega = c_1$, with suitable treatment at the singularities.

Again it is in principle possible to express the integrals in terms of known elliptic integrals, but so much algebra is involved that it is easier, and more accurate, to perform the quadratures numerically.

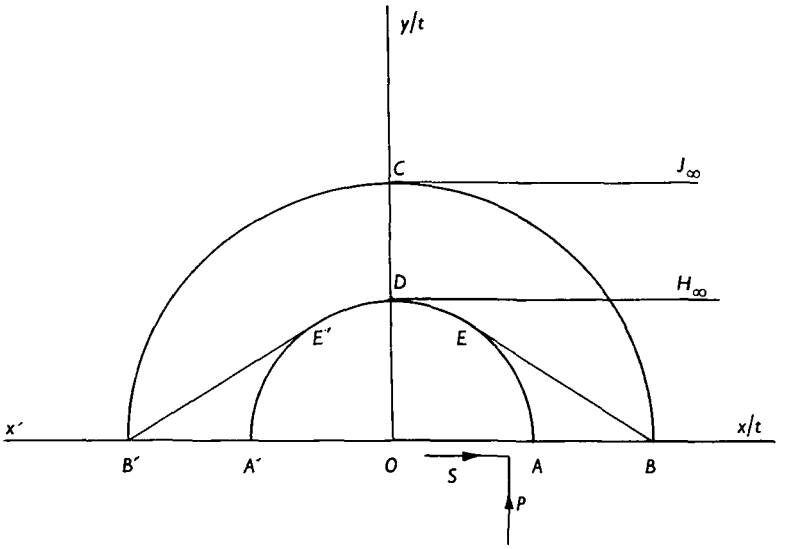


Fig. 2. Step-function loading.

14. *The solution for step-loading.* The most important case of the above analysis is the case of step-loading, which is given by $V \to \infty$. This corresponds to a material in $y > 0$, initially at rest. The applied loads are surface tractions

$$\sigma_{yy} = -PQ(x)\,Q(t), \quad \sigma_{xy} = SQ(x)\,Q(t),$$

where Q is defined by

$$Q(\Lambda) = \begin{cases} 0 & \text{for} \quad \Lambda < 0, \\ 1 & \text{for} \quad \Lambda > 0. \end{cases} \tag{78}$$

The plane $\omega = r/t$, θ, which corresponds to the plane of motion for any fixed $t > 0$, is illustrated in Fig. 2. The constant uniform loads P, S are applied to the part of the surface indicated by the heavy line. Above $X'B'CJ$ the material is at rest. The line CJ represents a plane wave front, moving perpendicular to itself with velocity c_1, across which σ_{yy} changes from O to $-P$, with a corresponding jump $P/(c_1\rho)$ in the velocity component $\dot{u}_y$. Similarly DH is a plane wave of velocity c_2 across which σ_{xy} changes to S and $\dot{u}_x$ to $-S/(c_2\rho)$. These values persist to the right of the arc CB and the curve DEB, respectively.

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The scalar potential in the semicircle BCB' is given by $\phi = \Re(w_1)$, where

$$\frac{\pi\mu}{c_1^2}\frac{dw_1}{d\zeta_1} = \frac{-\{iP(m^2\zeta_1^2-2)-2iS(m^2\zeta_1^2-1)^{\frac{1}{2}}\}\zeta_1}{(m^2\zeta_1^2-2)^2+4i(m^2\zeta_1^2-1)^{\frac{1}{2}}(1-\zeta_1^2)^{\frac{1}{2}}}, \tag{79}$$

$\zeta_1 = \operatorname{sech}(\beta_1+i\theta)$, $-\beta_1 = \operatorname{sech}^{-1}(r/c_1t)$ and the radicals are positive when ζ_1^2 is real and lies between $m^{-1} = c_2/c_1$ and unity. The vector potential in the semicircle ADA' is given by $\psi = \Re(w_2)$, where

$$\frac{\pi\mu}{c_2^2}\frac{dw_2}{d\zeta_2} = \frac{-\{2P(1-\zeta_2^2/m^2)^{\frac{1}{2}}-S(2-\zeta_2^2)\}\zeta_2}{(\zeta_2^2-2)^2-4(1-\zeta_2^2)^{\frac{1}{2}}(1-\zeta_2^2/m^2)^{\frac{1}{2}}}, \tag{80}$$

$\zeta_2 = \operatorname{sech}(\beta_2+i\theta)$, $-\beta_2 = \operatorname{sech}^{-1}(r/c_2t)$, the radicals being positive for ζ_2^2 real, less than unity. In the regions ABE, $A'B'E'$, the vector potential at any point is the same as that at the point of contact of a tangent from that point to the semicircle ADA', as given by (80). The stress components follow from (22), (23) and (24). Table 1 gives the surface values of σ_{xx} for $P = 1$, $S = 0$, $\theta = 0$ by way of illustration, with $\lambda = \mu$.

Note that any function $G(x,t)$ which is continuous, differentiable with respect to both variables, and zero for sufficiently large negative values of x and t, may be expressed in terms of step-functions by

$$G(x,t) = \int_{\xi=-\infty}^{\infty}\int_{\tau=-\infty}^{\infty} Q(x-\xi)\,Q(t-\tau)\,\frac{\partial^2 G(\xi,\tau)}{\partial\xi\,\partial\tau}\,d\xi\,d\tau$$

(with obvious extensions when G has discontinuities or discontinuous derivatives). The state of a material, initially at rest, when subjected to stresses on the boundary which vary in a general manner with x and t may therefore be deduced by integration from the results for the step-function load.

15. *Comments.* (i) The solution here presented shows clearly how finite loading of the surface of a material initially at rest can give rise to Rayleigh waves of large amplitude (infinite in the linearized problem (see § 12)).

(ii) For the problem specified in § 1 which contains a velocity parameter V, the solution has the surprising property that in the subsonic regions, the potentials are given in the same analytic form for all values of V. It is not necessary to solve separately for the various cases $V < c_2$, $c_2 < V < c_1$, $V > c_1$.

(iii) Again for the problems specified in § 1, which have dynamic similarity, the solution for the potentials is given simply in terms of elliptic integrals. For more general loading programmes, results may be deduced by superposition, involving at worst two integrations in real variables.

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Table 1. *Surface values*, $V \to \infty$, $P = 1$, $S = 0$, $\theta = 0$

ω/c_1	σ_{xx}	$\frac{(\lambda+2\mu)\,\dot{u}_x}{c_1}$	$\frac{(\lambda+2\mu)\,\dot{u}_y}{c_1}$
0·00	0·371	0·072	0·194
0·02	0·371	0·072	0·194
0·04	0·371	0·072	0·194
0·06	0·371	0·072	0·194
0·08	0·371	0·072	0·195
0·10	0·371	0·072	0·196
0·12	0·371	0·072	0·197
0·14	0·371	0·072	0·199
0·16	0·371	0·072	0·202
0·18	0·371	0·072	0·205
0·20	0·371	0·072	0·210
0·22	0·371	0·072	0·215
0·24	0·371	0·072	0·223
0·26	0·371	0·072	0·231
0·28	0·371	0·072	0·242
0·30	0·371	0·072	0·255
0·32	0·371	0·072	0·271
0·34	0·371	0·072	0·291
0·36	0·371	0·072	0·315
0·38	0·371	0·072	0·344
0·40	0·371	0·072	0·381
0·42	0·371	0·072	0·428
0·44	0·371	0·072	0·488
0·46	0·371	0·072	0·569
0·48	0·371	0·072	0·686
0·50	0·371	0·072	0·878
0·52	0·371	0·072	1·403
0·54	0·329	−0·025	1·388
0·56	0·329	−0·025	0·773
0·58	0·329	−0·025	0·806
0·60	0·329	−0·033	0·853
0·62	0·334	−0·045	0·898
0·64	0·339	−0·056	0·904
0·66	0·352	−0·066	0·919
0·68	0·359	−0·075	0·929
0·70	0·366	−0·083	0·938
0·72	0·371	−0·089	0·942
0·74	0·376	−0·095	0·944
0·76	0·379	−0·100	0·946
0·78	0·382	−0·103	0·947
0·80	0·384	−0·105	0·947
0·82	0·385	−0·106	0·947
0·84	0·385	−0·104	0·947
0·86	0·385	−0·101	0·948
0·88	0·383	−0·096	0·949
0·90	0·380	−0·088	0·952
0·92	0·370	−0·077	0·956
0·94	0·361	−0·062	0·963
0·96	0·353	−0·041	0·974
0·98	0·340	−0·015	0·990
1·00	0·333	−0·000	1·000

The surface wave is at $c/c_1 = 0{\cdot}531$.

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REFERENCES

(1) Cole, J. D. and Huth, J. H. Elastic stresses in a half-space under steadily moving loads. *Congress of Applied Mathematics* (Brussels, 1956).
(2) Lamb, H. *Phil. Trans.* A, 203 (1904), 1–42.
(3) Lapwood, E. R. *Phil. Trans.* A, 242 (1949), 63–100.
(4) Rayleigh, Lord. *Proc. Lond. Math. Soc.* 17 (1885), 4–11.
(5) Sauter, F. *Z. Angew. Math. Mech.* 30 (1950).
(6) von Schmidt, O. *Phys. Z.* 39 (1938).
(7) Sherwood, J. W. C. *Proc. Phys. Soc.* 71 (1958), 207–19.
(8) Sneddon, I. N. *Rend. Circ. math. Palermo*, 2 (1952), 57–62.
(9) Ward, G. N. *The linearized theory of steady high-speed flow*, ch. vii (Cambridge, 1955).

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