Ergod. Th. & Dynam. Sys. (2018), **38**, 1745–1767 © *Cambridge University Press, 2016*
doi:10.1017/etds.2016.97

On density of ergodic measures and generic points

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(*Received* 30 *October* 2015 *and accepted in revised form* 9 *August* 2016)

Abstract. We introduce two properties of dynamical systems on Polish metric spaces: closeability and linkability. We show that they imply density of ergodic measures in the space of invariant probability measures and the existence of a generic point for every invariant measure. In the compact case, it follows from our conditions that the set of invariant measures is either a singleton of a measure concentrated on a periodic orbit or the Poulsen simplex. We provide examples showing that closability and linkability are independent properties. Our theory applies to systems with the periodic specification property, irreducible Markov chains over a countable alphabet, certain coded systems including β-shifts and S-gap shifts, C^1-generic diffeomorphisms of a compact manifold M and certain geodesic flows of a complete connected negatively curved manifold.

1. *Introduction*

We study the space of invariant measures $\mathcal{M}_T(X)$ of a dynamical system (X, T) given by a continuous onto map $T : X \to X$ acting on a Polish space. We introduce two properties (*linkability* and *closability*) related to a subset of the set $\mathrm{Per}(T)$ of periodic points of T. We show that if (X, T) is closable with respect to a linkable set $K \subset \mathrm{Per}(T)$, then ergodic measures $\mathcal{M}_T^e(X)$ are dense in $\mathcal{M}_T(X)$ and every invariant measure has a generic point. Moreover, we identify typical properties of measures in $\mathcal{M}_T(X)$. We postpone the precise statement of our main result (Theorem 6.1) until we introduce all necessary notation.

Recall that if X is compact, then $\mathcal{M}_T(X)$ equipped with the weak* topology is a Choquet simplex and $\mathcal{M}_T^e(X)$ is the set of extreme points of $\mathcal{M}_T(X)$. Ville [**31**] showed that ergodic measures are dense among invariant measures of the binary full shift and so

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gave the first example of a simplex eventually named after Poulsen [**24**]. Lindenstrauss Olsen, and Sternfeld [**20**] showed that this remarkable simplex is unique up to an affine homeomorphism and therefore one may refer to it as *the* Poulsen simplex. Furthermore, every Choquet simplex appears as a face of the Poulsen simplex. On the other hand, Downarowicz [**11**] proved that each Choquet simplex is affinely homeomorphic to $\mathcal{M}_T(X)$ for some minimal dynamical system (X, T). Hence a non-trivial $\mathcal{M}_T(X)$ with $\mathcal{M}_T^e(X)$ dense is, in some sense, the 'biggest' possible simplex of invariant measure for a compact dynamical system. It is natural to ask for which dynamical systems (X, T) is the set of invariant measures the Poulsen simplex.

This question was studied by Parthasarathy [**21**] and Sigmund [**26**, **27**]. Since then the specification property (or its generalization) was the main tool for such investigations. Roughly speaking, the *periodic specification property* says that given any number of orbit segments there is $y \in \mathrm{Per}(T)$ which ε-shadows each segment and spends M iterations between the consecutive segments, where M depends only on ε. This property holds, for example, for topologically mixing shifts of finite type [**10**], topologically mixing interval maps [**5**] and axiom A diffeomorphisms restricted to a basic set [**4**]. However, it fails for general diffeomorphisms and many symbolic systems, such as 'most' β-shifts or S-gap shifts. Known extensions of [**26**, **27**] usually replace specification by a property implying a periodic variant of the *approximate product property* introduced by Pfister and Sullivan [**23**] (see [**18**] for discussion and references). Yet, our approach is still more general (see §4.4) and applies to many coded systems (including *all* β-shifts and S-gap shifts), C^1 generic diffeomorphisms of a compact manifold and some systems on non-necessarily compact Polish spaces (such as irreducible Markov chains over a countable alphabet or certain geodesic flows of a complete connected negatively curved manifold). In particular, it applies to examples which cannot be covered by any method used so far (see §4.4).

The goal of this paper is to identify properties that are strong enough to yield the desired results, yet simple enough to be readily checked in many situations. For example, our hypotheses are easy to verify for β-shifts and S-gap shifts. In contrast, a proof of density of ergodic measures from [**23**] seems to be more involved and does not work at all for some S-gap shifts.

In brief outline, here is the main idea of our approach. We observe that the proof of density of ergodic measures in $\mathcal{M}_T(X)$ breaks naturally into two independent steps: in the first step we establish density of periodic measures among ergodic measures and in the second step we prove density of periodic measures in their convex hull. Closability allows us to carry out the first step (see Theorem 4.11) and linkability helps with the second one (see Theorem 4.25), but neither of these properties implies the other (see §4). Roughly speaking, *K-closability* means that, for every ergodic measure, there is a generic point x which satisfies that for any $\varepsilon > 0$ there are an infinite number of n so that $x, T(x), \ldots, T^{n-1}(x)$ can be ε-shadowed by $y \in K$ with a period close to n. *Linkability of K* means that T restricted to K satisfies a variant of the specification property.

Similar ideas were independently used by Abdenur *et al* [**2**], Coudene and Schapira [**9**] and Sun and Tian [**28**]. Our main theorem unifies and extends these results, and can be applied beyond the framework of [**2**, **9**, **28**] (see Remark 4.16 and §7).

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The paper is organized as follows. Section 2 contains preliminaries on dynamical systems. Section 3 recalls concepts from symbolic dynamics. Sections 4 introduces and discusses notions of closability and linkability. In §5 we study the existence of generic points. Our main result is stated and proved in §6. Section 7 discusses various applications.

2. *Preliminaries*

For basics of ergodic theory and topological dynamics we refer the reader to [**10**, **32**].

2.1. *Standing assumptions.* Throughout this paper, X is a complete separable metric space with complete metric ρ and $T : X \to X$ is a continuous onto map. By a *dynamical system* we mean a pair (X, T). Let $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2.2. *Topological concepts.* Recall that a subset of a topological space is *of first category* (*residual*) if it is a countable union of closed nowhere dense sets (countable intersection of open and dense sets). By a *continuum* we mean a *non-empty, compact and connected set*. By $\overline{A}$ we denote the closure of A.

2.3. *Choquet theory.* Let K be a non-empty metrizable convex compact subset of a locally convex topological vector space. By ext K we denote the set of extreme points of K. We say that K is a *Choquet simplex* if every point of K is the barycenter of a unique probability measure supported on ext K (see [**25**, p. 174]. A *Poulsen simplex* is a Choquet simplex K_P such that $\overline{\text{ext } K_P} = K_P$ and ext $K_P \neq K_P$. The Poulsen simplex is unique up to affine homeomorphism (see [**20**] for more details).

2.4. *Topological dynamics.* We say that $x \in X$ is *periodic* for T and $k \in \mathbb{N}$ is a *period* for x if $T^k(x) = x$. The least possible period is the *minimal period* of x, $\text{mp}(x)$. We write $\text{Per}(T)$ for the set of periodic points of T. We say that T is *transitive* if for every non-empty open set $U, V \subset X$ there is $n > 0$ such that $U \cap T^{-n}(V) \neq \emptyset$. A map T is *topologically mixing* if there is $N \in \mathbb{N}$ such that, for all $n \geq N$, $U \cap T^{-n}(V) \neq \emptyset$. A dynamical system (Y, S) is a *factor* of (X, T) through $\Phi : X \to Y$ if Φ is a continuous onto map such that $\Phi \circ T = S \circ \Phi$.

2.5. *Specification.* Let $N \in \mathbb{N}_0$ and $I \subset \mathbb{N}_0$ be such that $I = \mathbb{N}_0 \cap \bigcup_{i=1}^{k} [a_i, b_i]$ for some integers $0 = a_1 \leq b_1 < \cdots < a_k \leq b_k$ and $k \in \mathbb{N}$. A map $\xi : I \to X$ is an *N-spaced specification* for T if $a_{j+1} - b_j \geq N$ for $j = 1, \ldots, k-1$ and $\xi(t) = T^{t-s}(\xi(s))$ for any $s, t \in \mathbb{N}_0$ such that $a_i \leq s \leq t \leq b_i$ for some $i = 1, \ldots, k$. A specification $\xi : I \to X$ is *ε-traced* by $z \in X$ if $\rho(\xi_j, T^j(z)) < \varepsilon$ for $j \in I$. We say that T has the *periodic specification property* if, for every $\varepsilon > 0$, there is $N = N(\varepsilon) \in \mathbb{N}$ such that every N-spaced specification $\xi : I \to X$ is ε-traced by $z \in X$ with $T^p(z) = z$, where $p = \max I + N$.

2.6. *Invariant measures.* Let $\mathcal{M}(X)$ be the set of all Borel probability measures on X. The weak* topology on $\mathcal{M}(X)$ is the weakest topology on $\mathcal{M}(X)$ such that the map $\mathcal{M}(X) \ni \mu \mapsto \int_X \varphi \, d\mu \in \mathbb{R}$ is continuous for each bounded continuous function $\varphi : X \to \mathbb{R}$ (see [**3**, Ch. 15]). It is well known that $\mathcal{M}(X)$ is a Polish space if and only if X is

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(see [**3**, Theorem 15.15]). Furthermore, the formula

$$\vec{d}(\mu, \nu) = \inf\{\varepsilon > 0 : \mu(A) \le \nu(A^\varepsilon) + \varepsilon, \text{ for every Borel set } A \subset X\},$$

where $\mu, \nu \in \mathcal{M}(X)$ and $A^\varepsilon = \{x \in X : \rho(x, A) < \varepsilon\}$ denotes the ε-neighborhood of A gives a complete metric† for the weak* topology on $\mathcal{M}(X)$ (see [**14**, Corollary 11.5.5]). By $\mathcal{M}_T(X)$ ($\mathcal{M}_T^e(X)$, $\mathcal{M}_T^{\mathrm{mix}}(X)$) we denote the set of all T-invariant (ergodic, strongly mixing) measures. Given a Borel set $B \subset X$, we write $\mathcal{M}_T^+(B) = \{\mu \in \mathcal{M}_T(X) : B \subset \operatorname{supp} \mu\}$. In particular, $\mathcal{M}_T^+(X)$ is the set of measures with full support. Recall that if X is compact, then $\mathcal{M}_T(X)$ is a non-empty Choquet simplex (see [**32**, §6.2]). In a non-compact setting, $\mathcal{M}_T(X)$ can be empty (see [**16**]), but it is always the closure of the convex hull of $\mathcal{M}_T^e(X)$ (see [**15**] for a very general variant of the ergodic decomposition theorem which implies this).

2.7. *Empirical measures and generic points.* Given $x \in X$ and $n \ge 1$ the *nth empirical measure of* x, $\mathfrak{m}(x, n) \in \mathcal{M}(X)$, is defined by

$$\mathfrak{m}(x, n)(A) = \frac{1}{n}\#\{0 \le j < n : T^j(x) \in A\} \quad \text{for each Borel set } A \subset X.$$

We will frequently use the following facts, the proof of which is straightforward.

LEMMA 2.1. *Let* $k \in \mathbb{N}_0$, $m, n \in \mathbb{N}$, $x \in X$, $\mu_1, \mu_2 \in \mathcal{M}(X)$ *and* $\alpha, \beta \in [0, 1]$.

(1) *If* $0 \le k < n \le m$, *then*

$$\vec{d}(\mathfrak{m}(x, m), \mathfrak{m}(T^k(x), n-k)) \le \frac{1}{m}(m - n + k).$$

(2) *For* $j = 1, 2$, $\vec{d}(\alpha\mu_1 + (1-\alpha)\mu_2, \mu_j) \le \vec{d}(\mu_1, \mu_2)$.

(3) *If* $\nu_1, \nu_2 \in \mathcal{M}(X)$ *and* $\delta = \max\{\vec{d}(\mu_j, \nu_j) : j = 1, 2\}$, *then*

$$\vec{d}(\alpha\mu_1 + (1-\alpha)\mu_2, \beta\nu_1 + (1-\beta)\nu_2) \le |\alpha - \beta| + \delta.$$

(4) *If* $d_1 = \vec{d}(\mathfrak{m}(x, m), \mu_1)$ *and* $d_2 = \vec{d}(\mathfrak{m}(T^m(x), n), \mu_2)$, *then*

$$\vec{d}\left(\mathfrak{m}(x, m+n), \frac{m}{m+n}\mu_1 + \frac{n}{m+n}\mu_2\right) \le \max\{d_1, d_2\}.$$

Let $V_T(x)$ be the set of accumulation points of the sequence $(\mathfrak{m}(x, n))_{n=1}^\infty$. It is a closed connected subset of $\mathcal{M}_T(X)$ (see [**10**, Proposition 3.8]). We say that $x \in X$ is *generic* for $\mu \in \mathcal{M}_T(X)$ if $V_T(x) = \{\mu\}$. By the ergodic theorem, every ergodic measure has a generic point. This is well known for compact spaces, but less well known for Polish spaces and therefore we provide a proof.

LEMMA 2.2. *Every ergodic* T*-invariant measure has a generic point.*

Proof. Let μ be an ergodic T-invariant Borel probability measure. By [**3**, Corollary 3.41], there is a metric d, compatible with the topology on X and so that the metric space (X, d) is totally bounded. In particular, the completion $(\hat{X}, \hat{d})$ of (X, d) is a compact metric space.

† Despite the apparent lack of symmetry in the definition of $\vec{d}$, it is indeed a metric on $\mathcal{M}(X)$, see [**14**, Theorem 11.3.1].

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By [**3**, Lemma 3.11], each bounded d-uniformly continuous real function f on X has the unique uniformly continuous extension $\hat{f}$ to $\hat{X}$. Furthermore, the map $f \mapsto \hat{f}$ is an isometry in the uniform metric given by the sup-norm $\|\cdot\|_\infty$. By [**3**, Theorem 9.14], the space of real continuous maps on $\hat{X}$ is separable, so there is a countable dense subset D in the space of d-uniformly continuous real functions f on X with the sup-norm. Now, by the ergodic theorem applied to μ, for every $\psi \in D$ there is a set of full measure subset $X_\psi \subset X$ such that, for each $x \in X_\psi$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(T^j(x)) = \lim_{n\to\infty} \int_X \psi \, d\mathfrak{m}(x, n) = \int_X \psi \, d\mu. \tag{1}$$

Let $X_\mu = \bigcap\{X_\psi : \psi \in D\}$. Since D is countable, $\mu(X_\mu) = 1$ and, for each $x \in X_\mu$, (1) holds for all $\psi \in D$. By [**3**, Theorem 15.3] this means that for every $x \in X_\mu$ the sequence $(\mathfrak{m}(x, n))_{n=1}^\infty$ weak* converges to μ, so x is a generic point for μ. □

A point $x \in X$ is *quasiregular* if it is generic with respect to some $\mu \in \mathcal{M}_T(X)$. A point $x \in X$ has *maximal oscillation* if $V_T(x) = \mathcal{M}_T(X)$.

2.8. *CO-measures.* Let $\mathcal{M}_T^{\mathrm{co}}(X)$ be the set of all CO-measures. We follow [**10**, Definition 21.7] and call an ergodic measure concentrated on the orbit of a periodic orbit $x \in \mathrm{Per}(T)$ the *CO-measure*† of x, and denote it by

$$\gamma(x) = \mathfrak{m}(x, k) \quad \text{where } \mathrm{mp}(x) = k.$$

Given $K \subset \mathrm{Per}(T)$, we write $\mathcal{M}_T^{\mathrm{co}}(K)$ for the set of CO-measures of points in K.

2.9. *Measure center.* The *measure center* of T, denoted by $C_{\mathcal{M}}(T)$ is the smallest closed set such that $\mu(C_{\mathcal{M}}(T)) = 1$ for every $\mu \in \mathcal{M}_T(X)$. Then $x \in C_{\mathcal{M}}(T)$ if and only if, for every neighborhood $U \subset X$ of x, there is $\mu \in \mathcal{M}_T(X)$ with $\mu(U) > 0$. The set $C_{\mathcal{M}}(T)$ is non-empty, closed and T-invariant.

3. *Symbolic dynamics and examples*

We recall some concepts from symbolic dynamics. For more details see [**19**].

3.1. *Shift spaces.* Let $\mathcal{A}$ be an at most countable set of *symbols* with discrete topology. Let $\Omega = \mathcal{A}^{\mathbb{N}}$ be a product topological spaces. It is a Polish space, which is compact if $\mathcal{A}$ is finite. Let $\sigma : \Omega \to \Omega$ be the continuous onto map given by $\sigma(\omega)_i = \omega_{i+1}$. A *shift space* over $\mathcal{A}$ is a σ-invariant and closed subset of Ω.

3.2. *Words and languages.* A *word* over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. We write u^∞ to denote the periodic sequence which is the *concatenation* of an infinite number of copies of a word u. We say that a word $u = u_1, \ldots, u_n$ *occurs* in $\omega = (\omega_i)_{i=1}^\infty$ if there is $k \in \mathbb{N}$ such that $\omega_{k+j-1} = u_j$ for all $j = 1, \ldots, n$. The collection $\mathcal{L}(X)$ of all words occurring in points from a shift space X is the *language of* X.

† Presumably, CO stands for *(C)losed (O)rbit*.

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3.3. *Directed countable graphs.* Let $G = (V, E)$ be a *directed graph* with at most a countable set of *vertices* V and at most a countable set of *edges* E. We allow multiple edges between vertices. Each edge $e \in E$ joins an *initial vertex* $\mathrm{i}(e) \in V$ with a *terminal vertex* $\mathrm{t}(e) \in V$. A *path* of length k from $u \in V$ to $w \in V$ is a finite sequence $\pi = (e_1, e_2, \ldots, e_k)$ of edges of G such that $\mathrm{i}(e_1) = u$, $\mathrm{t}(e_k) = w$ and $\mathrm{i}(e_{i+1}) = \mathrm{t}(e_i)$ for $1 \le i \le k-1$. A *loop* is a path $\pi = (e_1, e_2, \ldots, e_k)$ such that $\mathrm{i}(e_1) = \mathrm{t}(e_k)$. We say that $G = (V, E)$ is *irreducible* if, for every $u, v \in V$, there is a path on G from u to v.

3.4. *Topological Markov chains.* An irreducible *topological Markov chain* given by the connected directed graph $G = (V, E)$ is a shift space over $\mathcal{A} = E$ which consists of all sequences $(e_1, e_2, \ldots)$ of edges of G such that $\mathrm{i}(e_{i+1}) = \mathrm{t}(e_i)$ for each $i \ge 1$.

3.5. *Labeled graphs and coded systems.* A function $\Theta : E \to \mathcal{A}$ is a *labeling* of edges of G by symbols from a finite $\mathcal{A}$. We call the pair (G, Θ) a *labeled graph*. A labeled graph (G, Θ) is *right-resolving*, if, for each vertex v of G, the edges starting at v carry different labels. We extend Θ to paths in G in a natural way. The closure of the set of labels of infinite paths on an irreducible right-resolving labeled graph (G, Θ) defines a shift space $X_{(G,\Theta)} \subset \Omega$. We say that the shift space $X_{(G,\Theta)}$ is *presented* by (G, Θ). A shift space which can be presented in this way is called a *coded system* (see [**19**, p. 450], [**22**]). Observe that a label $w = \Theta(\pi)$ of any loop $\pi = (e_1, e_2, \ldots, e_k)$ on G gives us a periodic point $x = w^\infty$ in $X_{(G,\Theta)}$. We say that π *presents* x. Let $\mathrm{Per}(G, \Theta) \subset \mathrm{Per}(X_{(G,\Theta)})$ be the set of all periodic points of $X_{(G,\Theta)}$ presented by loops on G. Note that a shift space presented by an irreducible labeled graph is always transitive.

3.6. *S-gap shifts.* Let $S \subset \mathbb{N}$ and write $S = \{n_1, n_2, \ldots\}$ with $n_i < n_{i+1}$ for $i < |S|$ ($|S|$ may be ∞). The *S-gap shift* X_S is the set of all binary sequences such that between any two successive 1s, the number of 0s belongs to S. By [**17**, Example 3.4], X_S has the specification property if and only if $\sup_i (n_{i+1} - n_i) < \infty$ and $\gcd\{n + 1 : n \in S\} = 1$ and X_S is topologically mixing if and only if $\gcd\{n + 1 : n \in S\} = 1$. Every S-gap shift is coded and presented by the directed graph $\Gamma_S = (V, E_S)$, whose vertices are v_j for $0 \le j \le |S|$, and there is an edge $v_i \to v_j$ in E_S if and only if $j = i + 1$ or $i \in S$ and $j = 0$. We define labeling Θ_S which labels each edge $v_i \to v_{i+1}$ with 0 and all other edges with 1.

3.7. *β-shifts.* Fix $\beta > 1$. Let $d_\beta = (d'_j)_{j=1}^\infty$ be the β-expansion of 1 given by $d'_j = \lfloor \beta \cdot T_\beta^{j-1}(1) \rfloor$, where $T_\beta(x) = \beta x - \lfloor \beta x \rfloor = \beta x \mod 1$. We say that d_β is *finite* if there exists $k \in \mathbb{N}$ such that $d'_j = 0$ for all $j > k$ and, in this case, we define $\widehat{d}_\beta = (d_j)_{j=1}^\infty$ by $\widehat{d}_\beta = \bar{w}^\infty$, where $\bar{w} = d'_1 \ldots d'_{k-1}(d'_k - 1)$. Otherwise, we let $\widehat{d}_\beta = d_\beta$. Let $(\Gamma_\beta, \Theta_\beta)$ be a labeled graph whose vertices are $v_0, v_1, v_2, \ldots,$, and where the edges and their labels are defined as follows (see Figure 1).

(1) For each $i = 0, 1, 2, \ldots$, there is an edge $v_i \to v_{i+1}$ labeled by d_{i+1}.
(2) If $d_{i+1} > 0$, then there are d_{i+1} edges $v_i \to v_0$ labeled by $0, 1, \ldots, d_{i+1} - 1$.

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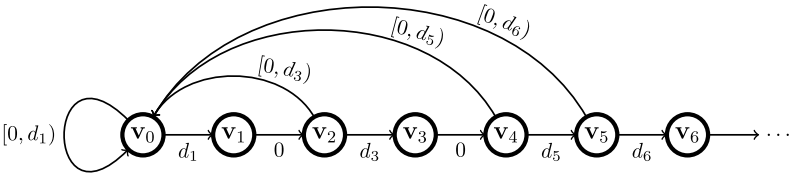


FIGURE 1. The graph presenting a β-shift (an edge labeled by $[0, d_j)$ means d_j edges labeled by $0, 1, \ldots, d_j - 1$).

The *β-shift* X_β is a coded system presented by $(\Gamma_\beta, \Theta_\beta)$. Recall that, for every $w \in \mathcal{L}(X_\beta)$, there is a path in Γ_β starting at v_0 and labeled by w (see [**7**, **23**, **29**]).

4. *Closability and linkability*

We introduce and study the notions of *closability* and *linkability*. We also discuss their independence and their relations to the notions of entropy density and intrinsic ergodicity.

4.1. *Closability.* Closability is a more general version of a property guaranteed by (what it is frequently referred to) the closing lemma, which roughly states that every piece of orbit that comes back close to its initial point can be approximated by a periodic orbit.

Definition 4.1. An (n, ε)-*Bowen ball* around $x \in X$ is the set

$$B(x, n, \varepsilon) = \{y \in X : \rho(T^j(y), T^j(x)) < \varepsilon \text{ for } j = 0, 1, \ldots, n-1\}.$$

Definition 4.2. A point $x \in X$ is *closable with respect to a non-empty set* $K \subset \mathrm{Per}(T)$, or simply *$K$-closable*, if, for every $\varepsilon > 0$ and $N > 0$, there exist $p = p(x, \varepsilon, N)$, $q = q(x, \varepsilon, N) \in \mathbb{N}$ such that there is $y \in B(x, p, \varepsilon) \cap K$ satisfying $T^q(y) = y$ and $N \le p \le q \le (1+\varepsilon)p$.

Note that $K \neq \emptyset$ is required in the definition of K-closability.

LEMMA 4.3. *If $x \in X$, $\varepsilon > 0$ and $p, q \in \mathbb{N}$ satisfy $p \le q \le (1+\varepsilon)p$, then, for every $y \in B(x, p, \varepsilon)$, $\vec{d}(\mathfrak{m}(y, q), \mathfrak{m}(x, p)) \le \varepsilon$.*

Proof. Take any Borel set $A \subset X$. For every $y \in B(x, p, \varepsilon)$

$$\begin{aligned}\mathfrak{m}(y, q)(A) &= \frac{1}{q}|\{0 \le j < q : T^j(y) \in A\}| \\ &\le \frac{1}{p}|\{0 \le j < p : T^j(x) \in A^\varepsilon\}| + \frac{1}{p}|\{p \le j < q : T^j(y) \in A\}| \\ &\le \mathfrak{m}(x, p)(A^\varepsilon) + \varepsilon,\end{aligned}$$

because $T^j(y) \in A$ implies $T^j(x) \in A^\varepsilon$ for $j = 0, 1, \ldots, p-1$ and $1/q \le 1/p$. Therefore $\mathfrak{m}(y, q)(A) \le \mathfrak{m}(x, p)(A^\varepsilon) + \varepsilon$. Hence, $\vec{d}(\mathfrak{m}(y, q), \mathfrak{m}(x, p)) \le \varepsilon$. □

COROLLARY 4.4. *Let $x \in X$ be K-closable. If $p, q \in \mathbb{N}$ and $y \in K$ are chosen for some $N \in \mathbb{N}$ and $\varepsilon > 0$, as in Definition 4.2, then $\vec{d}(\gamma(y), \mathfrak{m}(x, p)) < \varepsilon$.*

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Definition 4.5. A measure $\mu \in \mathcal{M}_T(X)$ is *K-closable* if some generic point of μ is K-closable. A dynamical system (X, T) is *K-closable* if every $\mu \in \mathcal{M}_T^e(X)$ is K-closable. If $K = \mathrm{Per}(T)$, then we say that a point (a measure, a dynamical system) is *closable*.

LEMMA 4.6. *Let (Y, S) be a factor of (X, T) through a uniformly continuous $\Phi : X \to Y$. If $K \subset \mathrm{Per}(T)$ and $\mu \in \mathcal{M}_T(X)$ is K-closable, then the push-forward $\Phi_*(\mu) \in \mathcal{M}_S(Y)$ is $\Phi(K)$-closable. In particular, if (X, T) is K-closable and $\Phi_* : \mathcal{M}_T(X) \to \mathcal{M}_S(Y)$ is surjective, then (Y, S) is $\Phi(K)$-closable.*

Proof. Recall that $\Phi_* : \mathcal{M}_T(X) \to \mathcal{M}_S(Y)$ and $\nu = \Phi_*(\mu)$ if and only if $\nu(B) = \mu(\Phi^{-1}(B))$ for every Borel set $B \subset Y$. Then Φ_*-image of an ergodic measure is ergodic and Φ maps generic points onto generic points. It is also easy to see that if $x \in X$ is K-closable, then $\Phi(x)$ is $\Phi(K)$-closable. □

Remark 4.7. If X is not necessarily compact, then a uniformly continuous factor $\Phi : X \to Y$ from (X, T) onto (Y, S) need not induce a surjective map $\Phi_* : \mathcal{M}_T(X) \to \mathcal{M}_S(Y)$. The image of a K-closable system through such a factor map need not to be $\Phi(K)$-closable. For example, consider the topological Markov chain (X, T) defined by the graph $\Gamma' = (V', E')$ introduced in the proof of Proposition 4.26 and a uniformly continuous factor map Φ induced by the labeling Θ'' from the proof of Proposition 4.27 (see Figure 3). The topological Markov chain (X, T) is closable (that is, $\mathrm{Per}(T)$-closable), but the factor system X'' is not $\Phi(\mathrm{Per}(T))$-closable, because the measure concentrated on 0^∞ is an isolated point of the set of CO-measures.

PROPOSITION 4.8. *If (X, T) has the periodic specification property, then every point in X is closable.*

Proof. Let $x \in X$, $N \in \mathbb{N}$ and $\varepsilon > 0$. Let $M = M(\varepsilon)$ be provided by the specification property. Take $p \in \mathbb{N}$ such that $p > N$ and $M < p\varepsilon$. Use specification to find $z \in X$ such that $z \in B(x, p, \varepsilon)$ and $T^{p+M}(z) = z$, which completes the proof. □

The converse is not true, because many β-shifts and S-gap shifts do not have the specification property (see [**5**] and [**17**, Example 3.4]).

PROPOSITION 4.9. *Every β-shift X_β is closable with respect to* $\mathrm{Per}(\Gamma_\beta, \Theta_\beta)$.

Proof. Fix $\beta > 1$. Let $\mu \in \mathcal{M}_\sigma^e(X_\beta)$. Either μ is concentrated on 0^∞, or there exists a symbol $j \in \{1, \ldots, \lfloor \beta \rfloor\}$ such that $\mu([j]) > 0$, where $[j]$ is the set of all $x \in X_\beta$ beginning with j. In the former case, μ is clearly closable as $0^\infty \in \mathrm{Per}(\Gamma_\beta, \Theta_\beta)$. In the latter case, fix a μ-generic point $x = (x_i)_{i=1}^\infty$. Given $k > 0$, let $\pi_x^{(k)} = (e_1^{(k)}, \ldots, e_k^{(k)})$ be a path in Γ_β starting at v_0 and labeled by $x_1 x_2 \ldots x_k$. Note that $x_k = j$ for an infinite number of k and then the path $\pi^{(k)}(x)$ ends with an edge labeled by $j > 0$. Therefore, either $\pi_x^{(k)}$ is a loop labeled by $w^{(k)} = x_1 x_2 \ldots x_k$, or we may replace the edge $e_k^{(k)}$ in $\pi_x^{(k)}$ by an edge $\bar{e}_k^{(k)} = (v_l \to v_0)$ labeled by 0 which gives a loop $\bar{\pi}_k^{(k)} = (e_1^{(k)}, \ldots, e_{k-1}^{(k)}, \bar{e}_k^{(k)})$ labeled by $w^{(k)} = x_1 \ldots x_{k-1} 0$. We have found an infinite number of k so that the CO-measure of $(w^{(k)})^\infty \in \mathrm{Per}(\Gamma_\beta, \Theta_\beta)$ is close to the $k-1$-empirical measure of x. This implies that x is $\mathrm{Per}(\Gamma_\beta, \Theta_\beta)$-closable. □

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The proof of the next result is analogous, and hence it is omitted.

PROPOSITION 4.10. *Every S-gap shift X_S is closable with respect to* $\mathrm{Per}(X_S)\setminus\{0^\infty\}$.

We can now state and prove the first main result of this section.

THEOREM 4.11. *If (X, T) is K-closable for some $K \subset \mathrm{Per}(T)$, then $\mathcal{M}_T^{\mathrm{co}}(K)$ is dense in $\mathcal{M}_T^e(X)$.*

Proof. Let $\mu \in \mathcal{M}_T^e(X)$. Fix $\varepsilon > 0$. Let x be a closable μ-generic point. Let $N > 0$ be such that $\vec{d}(\mu, \mathfrak{m}(x, n)) < \varepsilon/2$ for $n \geq N$. Use K-closability for N and $\varepsilon/2$ to find $p, q \in \mathbb{N}$ with $N \leq p \leq q \leq (1 + \varepsilon/2)p$ and a periodic point $z \in B(x, p, \varepsilon/2) \cap K$ such that $T^q(z) = z$. Then

$$\vec{d}(\gamma(z), \mu) \leq \vec{d}(\mathfrak{m}(z, q), \mathfrak{m}(x, p)) + \vec{d}(\mathfrak{m}(x, p), \mu) \leq \varepsilon,$$

because $\vec{d}(\mathfrak{m}(z, q), \mathfrak{m}(x, p)) \leq \varepsilon/2$, by Corollary 4.4. This concludes the proof. □

4.2. *Linkability.*

Definition 4.12. A set $K \subset \mathrm{Per}(T)$ is *linkable* if, for every $y_1, y_2 \in K$, $\varepsilon > 0$ and $\lambda \in [0, 1]$, there exist $p_1, p_2, q_1, q_2 \in \mathbb{N}$ and $z \in K$ with $T^{q_2}(z) = z$ satisfying:

(1) $\lambda - \varepsilon \leq p_1/(p_1 + p_2) \leq \lambda + \varepsilon$;

(2) $p_j \leq q_j - q_{j-1} \leq (1 + \varepsilon)p_j$ and $T^{q_{j-1}}(z) \in B(y_j, p_j, \varepsilon)$ for $q_0 = 0$, $j = 1, 2$.

Note that if K is linkable, then so is $\bigcup_{n=0}^{\infty} T^n(K) \subset \mathrm{Per}(T)$, so, henceforth, we will assume that K is T-invariant. Furthermore, we show that $p_1, p_2 \in \mathbb{N}$ can be chosen to be divisible by a fixed $N \in \mathbb{N}$.

LEMMA 4.13. *Let $r \in \mathbb{N}$, $\lambda \in [0, 1]$ and $\varepsilon > 0$. If $K \subset \mathrm{Per}(T)$ is linkable, $z \in K$, $y_1, y_2 \in K$, then there are $p_1, p_2, q_1, q_2 \in \mathbb{N}$ such that the conditions from Definition 4.12 are satisfied and r divides p_j for $j = 1, 2$. In particular, one may assume that $T^{p_j}(y_j) = y_j$ for $j = 1, 2$.*

Proof. For any $M \in \mathbb{N}$ one can find $\lambda' \in [0, 1]$ and $\varepsilon' > 0$ so that $\lambda - \varepsilon < \lambda' - \varepsilon'$, $\lambda' + \varepsilon' < \lambda + \varepsilon$, and there is no rational number with denominator smaller than M between $\lambda' - \varepsilon'$ and $\lambda' + \varepsilon'$. Use Definition 4.12 to find p_1', p_2' and q_1', q_2' for $y_1, y_2 \in K$, $\lambda' \in [0, 1]$ and $\varepsilon' > 0$. Note that $p_1' + p_2' \geq M$. Therefore, if M is sufficiently large, then replacing p_1' and p_2' by some multiples p_1 and p_2 of r will not affect the other conditions for original ε and λ. We can pick p_1 and p_2 which are both multiples of r and minimal periods of y_1 and y_2. Then $T^{p_j}(y_j) = y_j$ for $j = 1, 2$. □

Remark 4.14. The *barycenter property* [**2**, Definition 4.5] is implied by linkability, but it is unclear if the converse is true. Recall that $K \subset \mathrm{Per}(T)$ has the *barycenter property* if, for any $y_1, y_2 \in K$, any $\lambda \in [0, 1]$ and $\varepsilon > 0$, there exist $x \in K$ and pairwise disjoint $I, J \subset \mathbb{N} \cap [0, \mathrm{mp}(x))$ such that

$$\lambda - \varepsilon < \#I/\,\mathrm{mp}(x) < \lambda + \varepsilon \quad \text{and} \quad 1 - \lambda - \varepsilon < \#J/\,\mathrm{mp}(x) < 1 - \lambda + \varepsilon, \tag{2}$$

$$\rho_I^T(x, y_1) < \varepsilon \quad \text{and} \quad \rho_J^T(x, y_2) < \varepsilon, \tag{3}$$

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where $\rho_L^T(z, w) = \max\{\rho(T^m(z), T^m(w)) : m \in L\}$. Note that there is no further assumption on I and J. On the other hand, Definition 4.12 requires that I and J are intervals of consecutive integers. Because of this, the barycenter property can not replace linkability in the proof of existence of generic points.

Definition 4.15. We say that a set $K \subset \mathrm{Per}(T)$ is *strongly linkable* provided that, for every $y_1, y_2 \in K$ and every $\varepsilon > 0$, there exists an integer $N = N(y_1, y_2, \varepsilon)$ such that, for any integers $p_1, p_2 \geq N$ with $T^{p_j}(y_j) = y_j$ for $j = 1, 2$, there are $z \in K$ and integers $q_1 \leq q_2$ satisfying $T^{q_2}(z) = z$ and

$$p_1 \leq q_1 \leq (1+\varepsilon)p_1 \quad \text{and} \quad z \in B(y_1, p_1, \varepsilon),$$
$$p_2 \leq q_2 - q_1 \leq (1+\varepsilon)p_2 \quad \text{and} \quad T^{q_1}(z) \in B(y_2, p_2, \varepsilon).$$

Remark 4.16. Strong linkability is in the spirit of the property defined by [**7**, Definition 2.1] (we provisionally refer to it as *condition CT*), which applies to shift spaces where the specification property holds on some collection of 'good' words. However, strong linkability and the condition CT are independent. Indeed, condition CT implies intrinsic ergodicity [**7**, Theorem C], while Proposition 4.28 below shows that strong linkability and closability cannot guarantee that. On the other hand, one can verify that the shift space from Proposition 4.26 satisfies [**7**, Definition 2.1], but the simplex of invariant measure is not Poulsen.

Remark 4.17. Let $K \subset \mathrm{Per}(T)$ be linkable. If $z \in K$ and $p_1, p_2, q_1, q_2 \in \mathbb{N}$ are chosen for some $y_1, y_2 \in \mathbb{N}$, $\lambda \in [0, 1]$ and $\varepsilon > 0$, as in Definition 4.15, with $T^{p_j}(y_j) = y_j$ for $j = 1, 2$, then, using Lemma 4.3,

$$\vec{d}(\mathfrak{m}(z, q_1), \gamma(y_1)) < \varepsilon \quad \text{and} \quad \vec{d}(\mathfrak{m}(T^{q_1}(z), q_2 - q_1), \gamma(y_2)) < \varepsilon.$$

It follows from the above and Lemma 2.1(4) that $\gamma(z)$ is close to the convex combination $(q_1/q_2)\gamma(y_1) + (1 - q_1/q_2)\gamma(y_2)$: more precisely,

$$\vec{d}\left(\mathfrak{m}(z, q_2), \frac{q_1}{q_2}\gamma(y_1) + \frac{q_2 - q_1}{q_2}\gamma(y_2)\right) \leq \varepsilon. \tag{4}$$

Also

$$\frac{p_1}{(1+\varepsilon)(p_1+p_2)} \leq \frac{q_1}{q_2} \leq \frac{(1+\varepsilon)p_1}{p_1+p_2}$$

and, subtracting $p_1/(p_1 + p_2)$, we conclude that $|q_1/q_2 - p_1/(p_1 + p_2)| < \varepsilon$. Now, the last inequality together with (4) and Lemma 2.1(4) gives

$$\vec{d}\left(\mathfrak{m}(z, q_2), \frac{p_1}{p_1+p_2}\gamma(y_1) + \frac{p_2}{p_1+p_2}\gamma(y_2)\right) \leq 2\varepsilon. \tag{5}$$

By Lemma 2.1(3), the inequality $|p_1/(p_1 + p_2) - \lambda| < \varepsilon$ implies that

$$\vec{d}\left(\lambda\gamma(y_1) + (1-\lambda)\gamma(y_2), \frac{p_1}{p_1+p_2}\gamma(y_1) + \frac{p_2}{p_1+p_2}\gamma(y_2)\right) \leq \varepsilon,$$

and therefore $\vec{d}(\mathfrak{m}(z, q_2), \lambda\gamma(y_1) + (1 - \lambda)\gamma(y_2)) \leq 3\varepsilon$. Hence, $\gamma(z)$ is close to $\lambda\gamma(y_1) + (1 - \lambda)\gamma(y_2)$.

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The following two lemmas are easy consequences of the above definitions.

LEMMA 4.18. *Let (Y, S) be a factor of (X, T) through a uniformly continuous $\Phi : X \to Y$. If $K \subset \mathrm{Per}(T)$ is (strongly) linkable, then so is $\Phi(K) \subset \mathrm{Per}(S)$.*

LEMMA 4.19. *If $K \subset \mathrm{Per}(T)$ is (strongly) linkable and every point in $\mathrm{Per}(T)$ is K-closable, then $\mathrm{Per}(T)$ is (strongly) linkable.*

The Dyck shift (X_D, σ) is a topologically transitive example such that there are linkable sets $K_1, K_2 \subset \mathrm{Per}(\sigma)$ with $K_1 \cap K_2 \neq \emptyset$ and $K_1 \cup K_2$ is not linkable (see [**6**]).

PROPOSITION 4.20. *If (X, T) has the periodic specification property, then $\mathrm{Per}(T)$ is strongly linkable.*

Proof. Let $y_1, y_2 \in \mathrm{Per}(T)$ and $m_j = \mathrm{mp}(y_j)$ for $j = 1, 2$. Fix $\varepsilon > 0$ and, using specification, choose $M = M(\varepsilon)$. Let $n \in \mathbb{N}$ be such that $M < \varepsilon m_j n$ for $j = 1, 2$. We set $N = N(y_1, y_2, \varepsilon) = m_1 m_2 n$. Pick $p_1, p_2 \in \mathbb{N}$ with $p_j \geq N$ and $T^{p_j}(y_j) = y_j$ for $j = 1, 2$. Set $q_0 = 0$, $q_1 = p_1 + M - 1$ and $q_2 = q_1 + p_2 + M$. Let z be a point ε-tracing the specification $\xi : I \to X$, where $I = \{0, \ldots, p_1 - 1\} \cup \{q_1, \ldots, q_1 + p_2 - 1\}$ and $\xi(k) = T^{k-q_{j-1}}(y_j)$ for $q_{j-1} \leq k \leq q_{j-1} + p_j - 1$, $j = 1, 2$. Then $p_j \leq q_j - q_{j-1} \leq (1 + \varepsilon)p_j$ and $T^{q_{j-1}}(z) \in B(y_j, p_j, \varepsilon)$ for $j = 1, 2$. □

PROPOSITION 4.21. *If X is a coded system presented by a labeled graph (G, Θ), then the set of all periodic points presented by loops in G passing through a fixed vertex is strongly linkable.*

Proof. Note that these loops can be freely concatenated. □

Observe that Proposition 4.21, together with Propositions 4.9, 4.10, and Lemma 4.19, yields the following results.

PROPOSITION 4.22. *Every β-shift X_β is strongly linkable.*

PROPOSITION 4.23. *Every S-gap shift X_S is strongly linkable.*

Let $\mathrm{conv}(A)$ be the *convex hull* of $A \subset \mathcal{M}(X)$, that is, the smallest convex subset of $\mathcal{M}(X)$ containing A. By $\mathrm{cch}(A)$ we denote the *closed convex hull* of $A \subset \mathcal{M}(X)$. The closure of the convex hull equals the closed convex hull, $\overline{\mathrm{conv}(A)} = \mathrm{cch}(A)$ (see [**25**, Theorem 5.2]). With this notation we note the following lemma whose proof we omit.

LEMMA 4.24. *Let $L \subset K \subset \mathcal{M}(X)$. If L is dense in $\{\lambda\mu_1 + (1 - \lambda)\mu_2 : \lambda \in [0, 1], \mu_1, \mu_2 \in K\}$, then L is dense in $\mathrm{conv}(K)$, and hence $\overline{L} = \overline{\mathrm{conv}(K)} = \mathrm{cch}(K)$.*

We are ready to present our second main result in this section.

THEOREM 4.25. *If $K \subset \mathrm{Per}(T)$ is linkable, then $\overline{\mathcal{M}_T^{\mathrm{co}}(K)} = \mathrm{cch}(\mathcal{M}_T^{\mathrm{co}}(K))$.*

Proof. Take $\gamma(y_1), \gamma(y_2) \in \mathcal{M}_T^{\mathrm{co}}(K)$ supported on orbits of $y_1, y_2 \in K$, respectively. Fix $\lambda \in [0, 1]$ and $\varepsilon > 0$. By Lemma 4.24, it is enough to find $z \in K$ such that

$$\vec{d}(\gamma(z), \lambda\gamma(y_1) + (1 - \lambda)\gamma(y_2)) < \varepsilon.$$

But the point z provided by linkability is exactly such a point, by Remark 4.17. □

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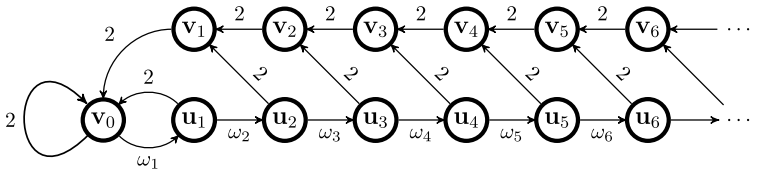


FIGURE 2. Labeled graph presenting X'.

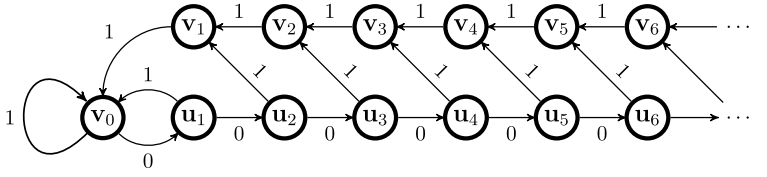


FIGURE 3. Labeled graph presenting X''.

4.3. *Independence of linkability and closability.*

PROPOSITION 4.26. *There is a shift space X' which is not* Per(X')*-closable, but* Per(X') *is strongly linkable.*

Proof. Let $V = \{v_0, v_1, v_2, \ldots\}$ and $U = \{u_1, u_2, u_3, \ldots\}$ be disjoint. Let $\Gamma' = (V', E')$ be a directed graph, with $V' = V \cup U$ and $E' = E_f \cup E_b \cup E_s \cup \{v_0 \to v_0\}$, where $E_s = \{u_{i+1} \to v_i : i \geq 0\}$, $E_b = \{v_{i+1} \to v_i : i \geq 0\}$ and

$$E_f = \{v_0 \to u_1\} \cup \{u_i \to u_{i+1} : i \geq 1\}.$$

Fix $(\omega_i)_{i=1}^{\infty} \in Y$, where Y be is an aperiodic and strictly ergodic shift space over $\{0, 1\}$. Let $\Theta' : E' \to \{0, 1, 2\}$ be given by $\Theta'(e) = \omega_i$ if $\mathrm{t}(e) = u_i$, for $i \in \mathbb{N}$, and by $\Theta'(e) = 2$, otherwise.

Consider the coded system X' presented by (Γ', Θ') (see Figure 2). Obviously, Per(X') is linkable (we can freely concatenate periodic orbits as they all pass through v_0). On the other hand, Y is a subsystem of X', and the unique invariant measure of (Y, σ) is neither a CO-measure, nor closable, as the measure of the set of all points in X' starting with two is at least $1/2$ for every CO-measure. □

PROPOSITION 4.27. *There is a shift space X'' which is* Per(X'')*-closable, but* Per(X'') *is not linkable.*

Proof. We follow the notation from the proof of Proposition 4.26. Consider the graph Γ' in the proof of Proposition 4.26. Let $\Theta''(e) = 0$ for $e \in E_f$ and $\Theta''(e) = 1$ for $e \in E_b \cup E_s \cup \{v_0 \to v_0\}$. Let X'' be the coded system presented (Γ', Θ'') (see Figure 3). Let $\mu_0 = \gamma(0^{\infty})$ and $\mu_1 = \gamma(1^{\infty})$ be the point masses concentrated on fixed points of X''. It is easy to see that $(2/3)\mu_0 + (1/3)\mu_1$ is not approximable by CO-measures, and hence Per(X'') is not linkable. On the other hand, X'' is closable, as a recurrent point $x \in X''$ is either fixed or the path labeled by x must pass through v_0 an infinite number of times. □

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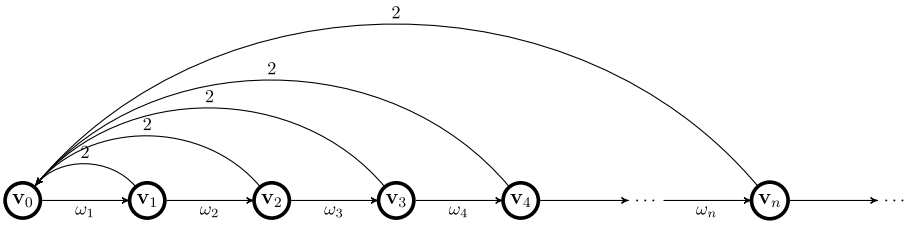


FIGURE 4. Graph (Γ, Θ) used in the proofs of Propositions 4.28 and 4.29.

4.4. *Relations with intrinsic ergodicity and entropy density.* We prove that closability and strong linkability do not imply intrinsic ergodicity, or entropy density. Both examples are inspired by [**22**]. The second example also proves that our assumptions of closability and linkability are strictly weaker than the g-almost product property and the approximate product property from [**23**].

PROPOSITION 4.28. *There is a shift space X which is closable and linkable, but has multiple measures of maximal entropy.*

Proof. Let Y be a strictly ergodic shift over $\{0, 1\}$ with topological entropy $h_{\text{top}}(Y) = \log((1+\sqrt{5})/2)$. We construct a coded system X with the desired properties. To analyze measures of maximal entropy of X, we apply the results of [**30**].

Let $\Gamma = (V, E)$ be a directed graph, whose vertices are labeled $v_0, v_1, v_2, \ldots$, $E = E_f \cup E_b$, where $E_b = \{v_i \to v_0 : i \geq 1\}$ and $E_f = \{v_{i-1} \to v_i : i \geq 1\}$ (see Figure 4). It is easy to see that Γ is strongly positive recurrent, that is,

$$\limsup_{n\to\infty} \frac{1}{n} \log l_n < \lim_{n\to\infty} \frac{1}{n} \log r_n,$$

where r_n is the number of distinct paths of length $n \geq 0$ from v_0 to v_0 in Γ and l_n is the number of paths of length n starting and ending at v_0 and not visiting v_0 in between (the limit on the right hand side exists since $r_{k+l} \leq r_k \cdot r_l$ for all $l, k \geq 1$). Fix $\omega = (\omega_i)_{i\in\mathbb{N}} \in Y$ and define a labeling $\Theta : E \to \{0, 1, 2\}$ by

$$\Theta(e) = \begin{cases} \omega_i & \text{if } e = v_{i-1} \to v_i \text{ for some } i \in \mathbb{N}, \\ 2 & \text{if } e = v_i \to v_0 \text{ for some } i \in \mathbb{N}. \end{cases}$$

Let X be the coded system presented by (Γ, Θ).

Note that (Γ, Θ) is right-resolving. It is also *follower separated*, that is, if $u, v \in V$ and $u \neq v$, then the sets of labels of paths starting at u and v, respectively, are different. Finally, 2 is a *magic word* of (Γ, Θ) because all paths with label 2 end at v_0. This implies that the theory from [**30**] applies to X. Given $w \in \mathcal{L}(X)$, let $F(w) = \{u \in \mathcal{L}(X) : wu \in \mathcal{L}(X)\}$ and $P(w) = \{u \in \mathcal{L}(X) : uw \in \mathcal{L}(X)\}$. The *Markov boundary* of X (see [**30**, pp. 1236–7]) is the shift space $\partial_M X$ with the language

$$\partial_M \mathcal{L}(X) = \{w \in \mathcal{L}(X) : \#\{F(xw) : x \in P(w)\} = \infty\}.$$

It is easy to see that here $\partial_M X = Y$. Let h_Γ denote the *Gurevič entropy* of Γ defined by $h_\Gamma = -\log R$, where R is the radius of convergence of $\sum r_n z^n$. It is easy to see

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that $h_\Gamma = h_{\text{top}}(\partial_M X)$. Since Γ is strongly positively recurrent, we conclude, by [**30**, Theorem 7.4(b1)], that there are two ergodic measures of maximal entropy supported, respectively, on X and on $\partial_M \mathcal{L}(X)$. Hence X is not intrinsically ergodic. On the other hand, X is $\text{Per}(X)$-closable and $\text{Per}(X)$ is strongly linkable. □

Recall that the ergodic measures are *entropy dense* if, for every $\mu \in \mathcal{M}_T(X)$, every neighborhood U of μ in $\mathcal{M}_T(X)$ and $\varepsilon > 0$, there is $\nu \in U \cap \mathcal{M}_T^e(X)$ such that $h_\mu(T) - \varepsilon < h_\nu(T)$. In particular, ergodic measures are dense. Climenhaga asked [**6**] whether density of ergodic measures implies entropy density. A counterexample is given in Proposition 4.29. Using [**12**, **13**], one can construct† a minimal system with a dense, but not entropy dense, set of ergodic measures.

PROPOSITION 4.29. *There is a shift space X which is closable and linkable, but ergodic measures are not entropy dense in $\mathcal{M}_\sigma(X)$.*

Proof. Let Y be a strictly ergodic shift over $\{0, 1\}$ with $h_{\text{top}}(Y) > \log((1+\sqrt{5})/2)$. We repeat the construction from the proof of Proposition 4.28 and we conclude that X is closable, linkable and $h_{\text{top}}(X) = h_{\text{top}}(\partial_M X) > h_\Gamma$. By [**30**, Theorem 7.4(c)], X has the unique measure of maximal entropy μ_Y with $\text{supp}\,\mu_Y = Y = \partial_M X$ and $h_\mu(\sigma) < h_\Gamma$ for every $\mu \in \mathcal{M}_\sigma^e(X) \backslash \{\mu_Y\}$. In particular, $\mathcal{M}_\sigma^e(X)$ is dense, but not entropy dense in $\mathcal{M}_\sigma(X)$. □

5. *Generic points*

In general, a non-ergodic invariant measure may have no generic points at all (see Proposition 5.4 and the end of this section). We show that this is not the case when (X, T) is closable with respect to a linkable $K \subset \text{Per}(T)$.

LEMMA 5.1. *If $K \subset \text{Per}(T)$ is linkable and $\mathcal{M}_T^{\text{co}}(K)$ is dense in $\mathcal{M}_T^e(X)$, then K is dense in $C_\mathcal{M}(T)$.*

Proof. Take any open set $U \subset X$ such that $U \cap C_\mathcal{M}(T) \neq \emptyset$ or, equivalently, such that $\mu(U) > 0$ for some $\mu \in \mathcal{M}_T(X)$. By Theorems 4.11 and 4.25 there is a sequence $(\gamma(x_n))_{n=1}^\infty \subset \mathcal{M}_T^{\text{co}}(K)$ such that $\gamma(x_n) \to \mu$ as $n \to \infty$. Hence (see [**32**, p. 149])

$$0 < \mu(U) \leq \liminf_{n\to\infty} \gamma(x_n)(U).$$

Thus, the orbit of x_n meets U for sufficiently big n: that is, $K \cap U \cap C_\mathcal{M}(T) \neq \emptyset$. □

THEOREM 5.2. *If $K \subset \text{Per}(T)$ is linkable and if $\mathcal{M}_T^{\text{co}}(K)$ is dense in $\mathcal{M}_T^e(X)$, then, for every $\mu \in \mathcal{M}_T(X)$, the set of μ-generic points is dense in $C_\mathcal{M}(T)$.*

Proof. Fix $\mu \in \mathcal{M}_T(X)$ and an open set $U \subset X$ such that $U \cap C_\mathcal{M}(T) \neq \emptyset$. By Theorems 4.11 and 4.25, there is a sequence $(x_n)_{n=1}^\infty \subset K$ such that $\mu_n = \gamma(x_n) \to \mu$ as $n \to \infty$. Using Lemma 5.1, K is dense in $C_\mathcal{M}(T)$. Therefore we can find $x_0 \in K$ and $0 < \varepsilon_0 < 1$ such that the closed ball $\overline{B}(x_0, \varepsilon_0)$ is contained in U. Let $M_n = \text{mp}(x_n)$ for

† We are grateful to Tomasz Downarowicz for pointing this out to us.

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$n \in \mathbb{N}_0$. Let $(\varepsilon_n)_{n=1}^{\infty}$ be a sequence with $\varepsilon_1 < \varepsilon_0$ and $\varepsilon_n \searrow 0$ as $n \to \infty$. Without loss of generality, we may assume that, for each $n \in \mathbb{N}$,

$$\vec{d}(\mu, \mu_n) \le \varepsilon_n \quad \text{and} \quad \vec{d}(\mu_n, \mu_{n+1}) \le \varepsilon_n. \tag{6}$$

We will inductively construct $z_0, z_1, \ldots \in K$ and auxiliary sequences $(p_1^{(n)}, p_2^{(n)})_{n=0}^{\infty}$ and $(q_1^{(n)}, q_2^{(n)})_{n=0}^{\infty}$ in $\mathbb{N} \times \mathbb{N}$.

Set $z_0 = x_0$, $p_1^{(0)} = q_1^{(0)} = p_2^{(0)} = M_0$, $q_2^{(0)} = 2M_0$. Assume that we have defined $z_0, z_1, \ldots, z_{n-1} \in K$, $(p_1^{(j)}, p_2^{(j)})_{j=0}^{n-1}$ and $(q_1^{(j)}, q_2^{(j)})_{j=0}^{n-1}$ for some $n \in \mathbb{N}$. Let $\varepsilon < 2^{-n}\varepsilon_n$ and $\lambda \in [0, 1]$ be such that $|\lambda - p_1^{(n)}/(p_1^{(n)} + p_2^{(n)})| < \varepsilon$ implies that

$$(p_1^{(n)}(1 + 2^{-n}\varepsilon_n) + 2^n M_n)/p_2^{(n)} \le 2^{-n}. \tag{7}$$

Let N_n be some multiple of $2^n \cdot M_n$ and $q_2^{(n-1)}$. For example, let

$$N_n = 2^{2n} \cdot M_n \cdot N_n \cdot q_2^{(n-1)}.$$

We use the linking property to find $z_n \in K$ and integers $p_1^{(n)}, p_2^{(n)}, q_1^{(n)} \le q_2^{(n)}$ for $y_1 = z_{n-1}$, $y_2 = x_n$ and λ, ε, as above, so that Definition 4.15 is fulfilled. In particular, $\gamma(z_n) = \mathfrak{m}(z_n, q_2^{(n)})$. We assume that N_n divides $p_j^{(n)}$ for $j = 1, 2$. It implies $q_2^{(n-1)}$ divides $p_1^{(n)}$ and M_n divides $p_2^{(n)}$, that is,

$$p_1^{(n)} = k_n \cdot q_2^{(n-1)} \quad \text{for some } k_n \in \mathbb{N}, \qquad p_2^{(n)} = l_n \cdot M_n \quad \text{for some } l_n \in \mathbb{N}. \tag{8}$$

This completes the induction step and the whole construction.

In what follows, we write $r^{(n)} = q_2^{(n)} - q_1^{(n)}$ and $y_n = T^{q_1^{(n)}}(z_n)$. Define

$$\nu_n = \mathfrak{m}(y_n, r^{(n)}) = \mathfrak{m}(T^{q_1^{(n)}}(z_n), q_2^{(n)} - q_1^{(n)}).$$

With this notation,

$$p_1^{(n)} \le q_1^{(n)} \le (1 + 2^{-n}\varepsilon_n)p_1^{(n)}, \tag{9a}$$

$$p_2^{(n)} \le r^{(n)} \le (1 + 2^{-n}\varepsilon_n)p_2^{(n)}, \tag{9b}$$

$$z_n \in B(z_{n-1}, p_1^{(n)}, 2^{-n}\varepsilon_n), \tag{9c}$$

$$y_n \in B(x_n, p_2^{(n)}, 2^{-n}\varepsilon_n). \tag{9d}$$

Note that, by repeated application of (9*c*) for any $n = 0, 1, \ldots$ and $k \in \mathbb{N}$,

$$\max_{j=0,1,\ldots,q_2^{(n)}-1} \rho(T^j(z_n), T^j(z_{n+k})) \le \max_{j=0,1,\ldots,q_2^{(n)}-1} \sum_{i=1}^{k} \rho(T^j(z_{n+i-1}), T^j(z_{n+i}))$$
$$\le \frac{1}{2^n} \sum_{i=1}^{k} \frac{\varepsilon_{n+i}}{2^i} < \frac{\varepsilon_{n+1}}{2^n}.$$

It follows that $(z_n)_{n=0}^{\infty}$ is a Cauchy sequence, and hence it converges to some $\bar{z} \in \overline{B}(z_0, \varepsilon_0)$. Moreover, $\bar{z} \in B(z_n, q_2^{(n)}, 2^{-n}\varepsilon_{n+1})$. Therefore

$$\vec{d}(\mathfrak{m}(\bar{z}, j), \mathfrak{m}(z_n, j)) \le 2^{-n}\varepsilon_{n+1} \quad \text{for } j = 0, 1, \ldots, q_2^{(n)} - 1.$$

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We claim that $\bar{z}$ is a generic point for μ. To this end, we must show that $\mathfrak{m}(\bar{z}, n) \to \mu$ as $n \to \infty$. Let

$$\Delta = \mathbb{N}_0 \setminus \left(\bigcup_{n=0}^{\infty} ((p_1^{(n)}, q_1^{(n)}) \cup (p_2^{(n)}, q_2^{(n)})) \right),$$

where (a, b) denotes an open interval and we agree that $(a, b) = \emptyset$ if $a = b$. It is easy to see that

$$d(\Delta) = \lim_{n\to\infty} \frac{|\Delta \cap \{0, 1, \ldots, n-1\}|}{n} = 1$$

and that $\bar{z}$ is a generic point for μ if and only if $\{\mathfrak{m}(\bar{z}, n) : n \in \Delta\}$ converges to μ. To show that the latter condition holds, it suffices to prove that

$$\lim_{n\to\infty} \mathfrak{m}(z_n, q_2^{(n)}) = \lim_{n\to\infty} \gamma(z_n) = \mu, \tag{10}$$

$$\vec{d}(\mathfrak{m}(z_n, q_2^{(n-1)} + j), \gamma(z_n)) \le E(n), \quad j = 1, \ldots, p_1^{(n)} - q_2^{(n-1)}, \tag{11}$$

$$\vec{d}(\mathfrak{m}(z_n, q_1^{(n)} + j), \gamma(z_n)) \le E(n), \quad j = 1, \ldots, p_2^{(n)}, \tag{12}$$

for some function $E : \mathbb{N} \to (0, \infty)$ with $E(n) \to 0$ as $n \to \infty$.

First, we prove (10). Note that inequalities (7) and conditions (9*a*)–(9*b*) imply that, for every $n \in \mathbb{N}$,

$$\frac{q_1^{(n)}}{q_2^{(n)}} < \frac{q_1^{(n)} + 2^n M_n}{q_2^{(n)}} \le \frac{p_1^{(n)}(1 + 2^{-n}\varepsilon_n) + 2^n M_n}{p_2^{(n)}} \le 2^{-n}. \tag{13}$$

It follows, from (13) and Lemma 2.1(1), that

$$\vec{d}(\mathfrak{m}(z_n, q_2^{(n)}), \mathfrak{m}(y_n, r^{(n)})) = \vec{d}(\gamma(z_n), \nu_n) \le 2^{-n}. \tag{14}$$

From (9*b*) and (9*d*) (see Remark 4.17), we deduce that

$$\vec{d}(\mathfrak{m}(y_n, r^{(n)}), \mathfrak{m}(x_n, p_2^{(n)})) = \vec{d}(\nu_n, \mu_n) \le 2^{-n}. \tag{15}$$

Then (14) combined with (15) gives, for every $n \in \mathbb{N}$,

$$\vec{d}(\gamma(z_n), \mu_n) \le 2^{-n+1}. \tag{16}$$

We have proved that (10) holds.

We proceed to show (11). Recall that we consider $\mathfrak{m}(z_n, q_2^{(n-1)} + j)$ for $j = 1, 2, \ldots, p_1^{(n)} - q_2^{(n-1)}$. On account of (9*c*) for $j = 1, 2, \ldots, p_1^{(n)}$,

$$\vec{d}(\mathfrak{m}(z_n, q_2^{(n-1)} + j), \mathfrak{m}(z_{n-1}, q_2^{(n-1)} + j)) \le 2^{-n}\varepsilon_n.$$

By (8), we may write $q_2^{(n-1)} + j = kq_2^{(n-1)} + s$, where $k \in \{1, \ldots, k_n\}$ and $0 \le s < q_2^{(n-1)}$. If $s < q_1^{(n-1)}$, then (13) and Lemma 2.1(1) give

$$\begin{aligned} &\vec{d}(\mathfrak{m}(z_{n-1}, kq_2^{(n-1)} + s), \mathfrak{m}(z_{n-1}, kq_2^{(n-1)})) \\ &\quad = \vec{d}(\mathfrak{m}(z_{n-1}, kq_2^{(n-1)} + s), \gamma(z_{n-1})) < q_1^{(n-1)}/q_2^{(n-1)} < 2^{-n+1}. \end{aligned} \tag{17}$$

Suppose that $q_1^{(n-1)} \le s \le p_2^{(n-1)}$. Using (8), we may find $t = 0, \ldots, l_{n-1}$ and $0 \le i < M_{n-1}$ such that $g = kq_2^{(n-1)} + q_1^{(n-1)}$, $m = tM_{n-1}$ and $i = s - q_1^{(n-1)} - m$. From (13) and Lemma 2.1(1), we obtain

$$\vec{d}(\mathfrak{m}(z_{n-1}, kq_2^{(n-1)} + s), \mathfrak{m}(z_{n-1}, g + m)) \le i/q_2^{(n-1)} < 2^{-n+1}. \tag{18}$$

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Note that $T^g(z_{n-1}) = y_{n-1}$. Hence $\mathfrak{m}(T^g(z_{n-1}), M) = \mathfrak{m}(y_{n-1}, M)$. We can regard $\mathfrak{m}(z_{n-1}, g+m)$ as a convex combination of the measures $\mathfrak{m}(z_{n-1}, g)$ and $\mathfrak{m}(T^g(z_{n-1}), m)$. Writing $\alpha = g/(g+m)$ leads to

$$\mathfrak{m}(z_{n-1}, g+m) = \alpha\, \mathfrak{m}(z_{n-1}, g) + (1-\alpha)\, \mathfrak{m}(y_{n-1}, m). \tag{19}$$

Since $g = kq_2^{(n-1)} + q_1^{(n-1)}$ and $q_2^{(n-1)}$ is a period of z_{n-1}, we conclude, from Lemma 2.1(1) and (13), that

$$\vec{d}(\mathfrak{m}(z_{n-1}, g), \gamma(z_{n-1})) \le q_1^{(n-1)}/q_2^{(n-1)} \le 2^{-n+1}. \tag{20}$$

The definition of z_{n-1} (see (9d) and Remark 4.17) yields

$$\vec{d}(\mathfrak{m}(y_{n-1}, m), \gamma(x_{n-1})) \le \varepsilon_{n-1}/2^{n-1}. \tag{21}$$

Combining (19), (20), (21) and Lemma 2.1(2), we deduce that

$$\vec{d}(\mathfrak{m}(z_{n-1}, g+m), \alpha\gamma(z_{n-1}) + (1-\alpha)\gamma(x_{n-1})) \le 2^{-n+1}.$$

From this, (16) and Lemma 2.1(2), we get

$$\begin{aligned}
&\vec{d}(\mathfrak{m}(z_{n-1}, g+m), \gamma(z_{n-1})) \\
&\quad\le \vec{d}(\mathfrak{m}(z_{n-1}, g+m), \alpha\gamma(z_{n-1}) + (1-\alpha)\gamma(x_{n-1})) \\
&\qquad + \vec{d}(\alpha\gamma(z_{n-1}) + (1-\alpha)\gamma(x_{n-1}), \gamma(z_{n-1})) \le 2^{-n+1} + 2^{-n+2}.
\end{aligned} \tag{22}$$

We can now combine (17), (18) and (22) to assert that $E(n) = 2^{-n+3}$ is a function $E : \mathbb{N} \to (0, \infty)$, as desired, and this finishes the proof of (11).

The proof of (12) is similar so we will only sketch it. Recall that we consider $\mathfrak{m}(z_n, q_1^{(n)} + j)$ for $j = 1, 2, \ldots, p_2^{(n)}$. Again the difference between $\mathfrak{m}(z_n, q_1^{(n)} + j)$ and $\mathfrak{m}(z_n, q_1^{(n)} + j + i)$ is negligible provided $0 \le i < M_n$. Hence we may consider only $j = \ell \cdot M_n$ for $\ell = 1, \ldots, \ell_n$. By our construction, $\mathfrak{m}(z_n, q_1^{(n)})$ is close to $\gamma(z_{n-1})$ and $\mathfrak{m}(y_n, \ell M_n)$ is close to $\gamma(x_n)$. Moreover, by (6) and (10), we see that $\vec{d}(\gamma(z_{n-1}), \gamma(x_n))$ tends to zero. But $\mathfrak{m}(z_n, q_1^{(n)} + \ell M_n)$ is a convex combination of $\gamma(z_{n-1})$ and $\gamma(x_n)$ and hence $\vec{d}(\mathfrak{m}(z_n, q_1^{(n)} + \ell M_n), \gamma(x_n)) \to 0$ as $n \to \infty$. □

The following result generalizes [**10**, Proposition 21.14].

COROLLARY 5.3. *If $K \subset \mathrm{Per}(T)$ is linkable and $\mathcal{M}_T^{\mathrm{co}}(K)$ is dense in $\mathcal{M}_T^e(X)$, then, for every continuum $V \subset \mathcal{M}_T(X)$, there is $x \in X$ such that $V_T(x) = V$.*

Proof. Take a continuum $V \subset \mathcal{M}_T(X)$. As $\{\gamma(x) : x \in K\}$ is dense in $\mathcal{M}_T(X)$, there is a sequence $(\mu_n)_{n\in\mathbb{N}} \subset \{\gamma(x) : x \in K\}$ such that V is the set of all its accumulation points and $\vec{d}(\mu_n, \mu_{n+1}) \to 0$ as $n \to \infty$. Then we can repeat the same construction as in the proof of Theorem 5.2. □

PROPOSITION 5.4. *There exists a topologically mixing shift space X with $\mathcal{M}_\sigma^e(X) = \{\mu_1, \mu_2\}$ such that every $x \in X$ is a generic point for μ_j for some $j = 1, 2$. Hence, every $\mu \in \mathcal{M}_\sigma(X) \setminus \mathcal{M}_\sigma^e(X)$ has no generic points.*

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Proof. Let Y be an aperiodic strictly ergodic shift over $\{0, 1\}$. Hence there is $K > 0$ such that $0^k \notin \mathcal{L}(Y)$ for all $k \geq K$. Let $\mathcal{L}'_k$ be the set of words w over $\{0, 1\}$ such that 1 appears exactly k times in w. Define $\mathcal{U} = \{u \in \{0, 1\}^n : n \in \mathbb{N} \text{ and } u_1 = u_n = 1\}$ and $\mathcal{L}''_k = \mathcal{L}'_k \cap \mathcal{U}$. Let $L_0 = \{0^n : n \in \mathbb{N}\}$, $L_1^* = \mathcal{L}''_1$ and $L_1 = \mathcal{L}'_1$. Assume that we have defined L_j and L_j^* for $1 \leq j \leq n$. Then $w \in L_{n+1}^*$ if and only if $w \in \mathcal{L}(Y) \cap \mathcal{L}''_{n+1}$ or $w = u0^k v$, where $u \in L_s^*$, $v \in L_t^*$, $s + t = n + 1$ and $k > K$ is such that $s + t \leq \log_2 k$. Let $L_{n+1} = \{0^a u 0^b : u \in L_{n+1}^* \text{ and } a, b \in \mathbb{N}_0\}$. Define

$$X = \{(x_j)_{j=1}^{\infty} \in \{0, 1\}^{\mathbb{N}} : x_n, \ldots, x_{n+k-1} \in L_k \cup L_0 \text{ for every } n, k \in \mathbb{N}\}.$$

Then X is easily seen to be a shift space and $Y \subset X$. Note that if $u, v \in \mathcal{L}(Y)$, then $u0^k v \in \mathcal{L}(Y)$ for all k large enough. Hence X is topologically mixing. Let μ_1 be the unique invariant measure on Y and let μ_2 be the CO-measure of 0^{∞}. We claim that every $x \in X$ is generic either for μ_1 or μ_2. If $x = u0^{\infty}$ for some $u \in \mathcal{L}(X)$, or if $x = vy$ for some $v \in \mathcal{L}(X)$ and $y \in Y$, then we have finished. Otherwise, $x = x_1x_2x_3 \ldots = 0^{l(0)}u_1 0^{l(1)}u_2 0^{l(2)}u_3 \ldots$, where $l(0) \geq 0$, $u_j \in \mathcal{L}(Y) \cap \mathcal{U}$ and $l(j) \geq K$ for each $j \geq 1$. But the definition of Y implies that the number of ones in $x_1, \ldots, x_n$ decreases exponentially fast as $n \to \infty$. Hence $x_j = 0$ for all j in a set of asymptotic density one. This means that x is a generic point for μ_2. □

6. *Main theorem*

THEOREM 6.1. *Let $T : X \to X$ be a continuous map on a Polish space X. Assume that $K \subset \mathrm{Per}(T)$ is linkable and that $\mathcal{M}_T^{\mathrm{co}}(K)$ is dense in $\mathcal{M}_T^e(X)$.*

(1) *The set $\mathcal{M}_T^{\mathrm{co}}(K)$ is dense in $\mathcal{M}_T(X)$.*
(2) *The set $\mathcal{M}_T^e(X)$ is residual in $\mathcal{M}_T(X)$.*
(3) *The set $\mathcal{M}_T^e(X) \cap \mathcal{M}_T^+(C_{\mathcal{M}}(T))$ is residual in $\mathcal{M}_T(X)$.*
(4) *The set $\mathcal{M}_T^{mix}(X)$ is of first category in $\mathcal{M}_T(X)$.*
(5) *The restriction $T|_{C_{\mathcal{M}}(T)}$ is topologically transitive.*
(6) *The set K is dense in $C_{\mathcal{M}}(T)$.*
(7) *If $\mathcal{M}_T(X)$ is not a singleton, then the set of all non-atomic measures is residual in $\mathcal{M}_T(X)$.*
(8) *For every continuum $V \subset \mathcal{M}_T(X)$, there is a dense set $D \subset C_{\mathcal{M}}(T)$ such that $V_T(x) = V$ for every $x \in D$. In particular, every $\mu \in \mathcal{M}_T(X)$ has a generic point.*
(9) *The set of points with maximal oscillation is residual in $C_{\mathcal{M}}(T)$.*
(10) *If $\mathcal{M}_T(X)$ is non-trivial, then the set of quasiregular points is of first category.*

Moreover, if X is compact, then:

(11) *$\mathcal{M}_T^e(X)$ is arcwise connected; and*
(12) *either $\mathcal{M}_T(X)$ is a singleton of a CO-measure or $\mathcal{M}_T(X)$ is the Poulsen simplex.*

We postpone the proof until the end of this section, as it is a compilation of previous results and some results we are now going to formulate.

The following corollary is an immediate consequence of Theorem 4.11.

COROLLARY 6.2. *Let $T : X \to X$ be a continuous map on a Polish space X. Assume that (X, T) is closable with respect to a linkable set $K \subset \mathrm{Per}(T)$. Then the conclusions (1)–(12) of Theorem 6.1 hold.*

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For the rest of this section, we assume that $K \subset \mathrm{Per}(T)$ is linkable and $\mathcal{M}_T^{\mathrm{co}}(K)$ is dense in $\mathcal{M}_T^e(X)$.

LEMMA 6.3. *The set $\mathcal{M}_T^+(C_{\mathcal{M}}(T))$ is residual in $\mathcal{M}_T(X)$.*

Proof. Let $\mathcal{U} = (U_n)_{n=1}^{\infty}$ be a countable base for the topology on X. Let $\mathcal{V}$ be a collection of $V \in \mathcal{U}$ such that $V \cap C_{\mathcal{M}}(T) \neq \emptyset$. We may write $\mathcal{V} = (V_n)_{n=1}^{\infty}$ (repeating some V an infinite number of times if necessary). For each $n \in \mathbb{N}$, there is $\mu_n \in \mathcal{M}_T(X)$ such that $\mu_n(V_n) > 0$. Let $\mu^* = \sum_{n\in\mathbb{N}} 2^{-n-1}\mu_n$. Clearly, $\mu^* \in \mathcal{M}_T(X)$ and $\operatorname{supp}\mu^* = C_{\mathcal{M}}(T)$. Let $D_n = \{\mu \in \mathcal{M}_T(X) : \mu(V_n) = 0\}$. Then D_n is closed and nowhere dense since $\mu_\varepsilon = \varepsilon\mu^* + (1-\varepsilon)\mu \notin D_n$ for every $\mu \in D_n$ and $\varepsilon \in (0, 1)$. Thus $D = \bigcup_{n\in\mathbb{N}} D_n$ is nowhere dense and hence $\mathcal{M}_T(X)\backslash D = \{\mu \in \mathcal{M}_T(X) : \operatorname{supp}\mu = C_{\mathcal{M}}(T)\}$ is residual. □

PROPOSITION 6.4. *If $\mathcal{M}_T^e(X)$ is dense in $\mathcal{M}_T(X)$, then $T|_{C_{\mathcal{M}}(T)}$ is transitive.*

Proof. By Lemma 6.3, $\mathcal{M}_T^+(C_{\mathcal{M}}(T))$ is residual in $\mathcal{M}_T(X)$. By our assumption, $\mathcal{M}_T^e(X)$ is also residual. Hence there exists an ergodic measure with support $C_{\mathcal{M}}(T)$. It is well known that a dynamical system restricted to a support of an ergodic measure is transitive. □

The converse of Proposition 6.4 is not true (take a minimal system with exactly two ergodic measures).

Let $\mathcal{M}_T^{\mathrm{co}\,k}(K)$ denote the set of CO-measures supported by periodic points from K with minimal periods at most $k \in \mathbb{N}$.

PROPOSITION 6.5. *Either $\mathcal{M}_T(X)$ consists of a single CO-measure or, for each $k \in \mathbb{N}$, the set $\{\gamma(x) \in \mathcal{M}_T(X) : x \in K\}\backslash\, \mathcal{M}_T^{\mathrm{co}\,k}(X)$ is dense in $\mathcal{M}_T(X)$.*

Proof. Suppose that there is $U \subset \mathcal{M}_T(X)$ open such that $\mathcal{K}^{\mathrm{co}\,k} = U \cap \mathcal{M}_T(K) \subset \mathcal{M}_T^{\mathrm{co}\,k}(X)$ is residual in U. By Lemma 6.3, the set $\mathcal{M}_T^+(C_{\mathcal{M}}(T))$ is also residual in U. Therefore, if $\mu \in U \cap \mathcal{M}_T^+(C_{\mathcal{M}}(T)) \cap \mathcal{K}^{\mathrm{co}\,k}$, then μ is a measure concentrated on a periodic orbit of length at most k with $\operatorname{supp}\mu = C_{\mathcal{M}}(T)$. Hence we conclude that $\mathcal{M}_T(X) = \{\mu\}$. □

Now we can prove our main result. To avoid repetition, we are going to refer the reader to [**10**] for some of the details. Note that although the proofs in [**10**] are presented in the setting of a compact metric space, they are based around the Baire category argument. Thus we can apply them in our context once we replace the references to the specification property by our results.

Proof of Theorem 6.1. By Theorem 4.11, K-closability implies that $\mathcal{M}_T^{\mathrm{co}}(K)$ is dense in $\mathcal{M}_T^e(X)$. Hence (1) follows from Theorem 4.25. Since $\mathcal{M}_T^e(X)$ is always a G_δ subset of $\mathcal{M}_T(X)$ (see [**21**, Theorem 2.1]), (1) implies (2). Lemma 6.3 and (2) imply (3). Item (4) can be shown analogously to [**10**, Proposition 21.13], because, in [**10**, Proposition 21.13], the specification property is needed only to apply [**10**, Proposition 21.8], which, in our case, follows from Proposition 6.5. We have proved (5) in Proposition 6.4 and (6) in Lemma 5.1. The proof of (7) is the same as the proof of [**10**, Proposition 21.10]

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with reference to [**10**, Proposition 21.13] replaced by application of Proposition 6.5. Theorem 5.2 and Corollary 5.3 lead to (8). Item (9) can be shown analogously to [**10**, Proposition 21.18] (using Theorem 5.2 instead of [**10**, Proposition 21.15]). Further, (10) is a consequence of (9).

Finally, (1) and [**20**, §3] yield (11) and (12). □

Remark 6.6. Note that some of our assertions are stated for T restricted to the measure center $C_{\mathcal{M}}(T)$. This is because the behavior of T outside $C_{\mathcal{M}}(T)$ is negligible from the *measure-theoretic* point of view, even though $C_{\mathcal{M}}(T)$ may be *topologically* negligible (nowhere dense) in X. If (X, T) has periodic specification, then $C_{\mathcal{M}}(T) = X$ and (X, T) is topologically mixing, but, in general, our hypotheses do not guarantee that. We can only prove that $T|_{C_{\mathcal{M}}(T)}$ is topologically transitive and any non-mixing S-gap shift shows that this cannot be improved. Proposition 6.7 below shows that $C_{\mathcal{M}}(T)$ can be topologically negligible (nowhere dense) in X even for topologically mixing systems. Note that Theorem 6.1 applies to this example.

PROPOSITION 6.7. *Let X be the full shift over $\{1, 2\}$. There exists a topologically mixing shift space Y over $\{0, 1, 2\}$ such that $X \subset Y$ and $C_{\mathcal{M}}(Y) = X$.*

Proof. Let $|w|_0$ denote the number of zeros appearing in a word w. Define

$$Y = \{(x_j)_{j=1}^{\infty} \in \{0, 1, 2\}^{\mathbb{N}} : |x_n \dots x_{n+k}|_0 \le \log_2(k+1) \text{ for every } n, k \in \mathbb{N}\}.$$

Let $[0] = \{(x_j)_{j=1}^{\infty} \in Y : x_1 = 0\}$. Then $\mu([0]) = 0$ for every $\mu \in \mathcal{M}_\sigma(Y)$. This implies that $C_{\mathcal{M}}(Y) = X$. Take $u, v \in \mathcal{L}(Y)$. There is k_0 such that, for all $k \ge k_0$, $u1^k v \in \mathcal{L}(Y)$, and hence Y is mixing. □

7. *Applications*

Theorem 6.1 applies to the following examples and so provides a uniform approach and yields the following new results (the existence of generic points and generic properties of measures):

- a system with periodic specification property (by Propositions 4.8 and 4.20);
- β-shifts (by Propositions 4.9 and 4.22); and
- S-gap shifts (by Propositions 4.10 and 4.23).

Theorem 6.1 also applies to many coded systems. Furthermore, as explained in Remarks 4.14, 4.16, 7.2 and in §4.4, our hypotheses are indeed much weaker than in previous works.

We discuss two more families of dynamical systems to which Theorem 6.1 can be applied. All arguments required to check that linkability and density of periodic measures hold are already given in the cited papers. Therefore we only sketch the proofs, since a detailed exposition would make the present paper much longer without introducing anything new. As a result, we get a bit more than the *ad hoc* methods used in the cited papers.

7.1. *C^1-generic diffeomorphisms.*

The following theorem combines Theorem 6.1 and [**2**, Theorem 3.5].

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Let M be a compact boundaryless manifold and denote by $\mathrm{Diff}^1(M)$ the space of C^1-diffeomorphisms of M endowed with the usual C^1-topology. A property is said to hold *C^1-generically* if it holds for every f in a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$. We say that a compact f-invariant set $\Lambda \subset M$ is *isolated* if there is an open set $U \supset \Lambda$ such that $\Lambda = \bigcap\{f^n(U) : n \in \mathbb{Z}\}$. A transitive set is *trivial* if it is a single periodic orbit. For further concepts see [**2**].

THEOREM 7.1. *Let $\Lambda \subset M$ be an isolated non-trivial transitive set of a C^1-generic diffeomorphism $f \in \mathrm{Diff}^1(M)$.*

(1) *The set $\mathcal{M}_f^{\mathrm{co}}(\Lambda)$ is dense in $\mathcal{M}_f(\Lambda)$. Hence, $\mathcal{M}_f(\Lambda)$ is the Poulsen simplex and $\mathcal{M}_f^e(\Lambda)$ is arcwise connected.*

(2) *There is a residual set $\mathcal{N} \subset \mathcal{M}_f(\Lambda)$ such that every $\mu \in \mathcal{N}$ satisfies:*
   (a) *μ is ergodic, but not mixing;*
   (b) $\operatorname{supp} \mu = \Lambda$*;*
   (c) $h_\mu(f) = 0$*;*
   (d) *for μ-almost every $x \in \Lambda$, the Oseledets splitting coincides with the finest dominated splitting over Λ; and*
   (e) *μ is non-uniformly hyperbolic.*

(3) *If $V \subset \mathcal{M}_f(\Lambda)$ is a continuum, then there is $D \subset \Lambda$ such that $\overline{D} = \Lambda$ and $V_T(x) = V$ for all $x \in D$. Hence, each $\mu \in \mathcal{M}_f(\Lambda)$ has a generic point.*

(4) *The set of points having maximal oscillation is residual in Λ.*

(5) *The set of quasiregular points is of first category in Λ.*

Proof. For a proof of (c)–(e), see [**2**]. Note that the density of periodic (ergodic) measures and (b) also follows from [**2**], but our proof is new. To apply Theorem 6.1, we need to check that linkability and approximability hold C^1-generically. Because any isolated non-trivial transitive set of a C^1-generic diffeomorphism f is a relative homoclinic class, it is enough to show that if $V \subset M$ is an open set and $\mathcal{O}$ is a hyperbolic periodic orbit of f, then the set of periodic points belonging to the relative homoclinic class of $\mathcal{O}$ with respect to V is linkable. This follows from [**2**, p. 23, lines 3–7]) and [**1**, §3.2], as outlined in the proof of [**2**, Proposition 4.8]. Density of $\mathcal{M}_f^{\mathrm{co}}(\Lambda)$ in $\mathcal{M}_f(\Lambda)$ follows from Mañé's ergodic general density theorem (see [**2**, Theorem 3.8 and Proposition 4.1]). □

We point out that items (a), (3)–(5) are new. We remark that the proof of Theorem 6.1 (and hence Theorem 7.1(3)–(5)) does not work if we assume the barycenter property from [**2**] instead of linkability (see Remark 4.14).

Remark 7.2. Another barycenter notion is defined in [**28**]. It implies Theorem 7.1(4)–(5). It also implies that, for every continuum $V \subset \mathcal{M}_T(X)$, there is $x \in X$ such that $V \subset V_T(x)$, but it is too weak to yield the reverse inclusion.

7.2. *Flows.* Let us indicate how our results can be adapted to flows, although we refrain from giving full details. Note that, for flows, the topological and ergodic properties are slightly more technical to state and to verify, but all properties of discrete dynamical systems considered above do have their natural counterparts for flows. This also includes *closability* and *linkability*.

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Consider a continuous flow $\varphi : \mathbb{R} \times X \to X$ on a complete separable metric space X. One says that *the closing lemma holds* in an open set $U \subset X$ if, for every $\varepsilon > 0$, there exist $\delta > 0$ and $t_0 > 0$ such that, for all $x \in U$ and $t \geq t_0$ with $\rho(x, \varphi_t(x)) < \delta$ and $\varphi_t(x) \in U$, there are $x_0 \in X$ and $l > 0$ with $|l - t| < \varepsilon$, $\varphi_l(x_0) = x_0$, and $\rho(\varphi_s(x), \varphi_s(x_0)) < \varepsilon$ for $0 < s < \min\{t, l\}$. If the flow *satisfies the closing lemma*, that is, if every point in X has a neighborhood in which the closing lemma holds, then one can conclude that it is closable. The flow has a *local product structure* if, for every $\varepsilon > 0$, there is $\delta > 0$ such that, for every $x, y \in X$ with $d(x, y) \leq \delta$, there is a real number t with $|t| \leq \varepsilon$ and a point z satisfying $d(\varphi_s(z), \varphi_s(x)) \leq \varepsilon$ and $d(\varphi_{-s}(z), \varphi_{t-s}(y)) \leq \varepsilon$ for all $s \geq 0$. If the flow admits a local product structure and satisfies the closing lemma, then it has the linkability property (to show this, one proceeds as in the proof of [**8**, Lemma 1]).

A setting in which the above properties (local product structure and closing lemma) were verified is, for example, studied in [**9**]. There φ is the geodesic flow on the unit tangent bundle of a negatively curved, connected, complete (but not necessarily compact) Riemannian manifold and it is assumed that it has a non-empty non-wandering set. Thus, in such a setting, corresponding statements from Theorem 6.1 can extend [**9**, Theorems 1.1 and 4.2] to provide new results about generic points.

Acknowledgements. We are grateful to Christian Bonatti, Keith Burns, Vaughn Climenhaga, Sylvain Crovisier, Lorenzo J. Díaz, Martha Łącka, Piotr Oprocha, Paulo Varandas and Benjamin Weiss for numerous discussions and remarks related to this paper. DK would also like to thank Bus Rapid System of the beautiful city of Rio de Janeiro for providing a lot of time and (a somehow limited) space to develop the material presented here. KG has been supported in part by CNPq (Brazil) and by EU Marie-Curie IRSES Brazilian-European partnership in Dynamical Systems (FP7-PEOPLE-2012-IRSES 318999 BREUDS). DK was supported by the National Science Centre (NCN) grant 2013/08/A/ST1/00275.

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