The Journal of Symbolic Logic
Volume 91, Number 1, March 2026

A METRIC SET THEORY WITH A UNIVERSAL SET

JAMES E. HANSON

Abstract. Motivated by ideas from the model theory of metric structures, we introduce a metric set theory, MSE, which takes bounded quantification as primitive and consists of a natural metric extensionality axiom (the distance between two sets is the Hausdorff distance between their extensions) and an approximate, non-deterministic form of full comprehension (for any real-valued formula $\varphi(x, y)$, tuple of parameters a, and $r < s$, there is a set containing the class $\{x : \varphi(x, a) \leq r\}$ and contained in the class $\{x : \varphi(x, a) < s\}$). We show that MSE is sufficient to develop classical mathematics after the addition of an appropriate axiom of infinity. We then construct canonical representatives of well-order types and prove that ultrametric models of MSE always contain externally ill-founded ordinals, conjecturing that this is true of all models. To establish several independence results and, in particular, consistency, we construct a variety of models, including pseudo-finite models and models containing arbitrarily large standard ordinals. Finally, we discuss how to formalize MSE in either continuous logic or Łukasiewicz logic.

§1. Introduction. Ever since the discovery of the inconsistency of full comprehension principles at the turn of the last century, there have been various efforts to rescue the idea and formulate systems in which the entire domain of discourse is meaningfully represented as an element of the domain of discourse itself.

Neo-naïve set theories commonly take one of two approaches to repairing full comprehension. One is to weaken the comprehension principle while maintaining full classical logic, and the other is to weaken the underlying logic while maintaining the full comprehension principle. Extensionality is often weakened or abandoned entirely. While there have been many investigations into such theories, there are approximately three in particular we will be occasionally comparing to ours: Quine's New Foundations, NF, and Jensen's "slight (?) modification" thereof, NFU; the positive topological set theory GPK^+, studied most prominently by Esser; and Cantor–Łukasiewicz set theory, originally isolated by Skolem but named by Hájek, which consists simply of the full comprehension scheme interpreted in the $[0, 1]$-valued Łukasiewicz predicate logic. $\mathsf{NF(U)}$ and GPK^+ fall under the first approach mentioned above, and Cantor–Łukasiewicz set theory, abbreviated $\mathrm{C}\text{Ł}_0$ by Hájek, falls under the second. NF and GPK^+ have full extensionality, but $\mathrm{C}\text{Ł}_0$ is entirely inconsistent with it and NFU weakens it by allowing urelements. To keep this introduction short, we will point the reader to [18] for an overview of $\mathsf{NF(U)}$ and GPK^+ and to [17, Chapter 4.5] for an overview of Cantor–Łukasiewicz set theory.

Received February 10, 2023.
2020 *Mathematics Subject Classification.* Primary 03E70, 03B50, 03C66.
Key words and phrases. metric set theory.

© The Author(s), 2025. Published by Cambridge University Press on behalf of The Association for Symbolic Logic. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited.

0022-4812/26/9101-0005
DOI:10.1017/jsl.2025.10147

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

In particular, Hájek's result [17, Theorem 4.17] that $\mathrm{C}\text{Ł}_0$ has no ω-models will be relevant, in that we will find a similar real-valued failure of induction in our theory (Theorem 5.10), although for us the failure may occur at arbitrarily large ordinals (Theorem 7.15), rather than necessarily at ω.

In this article, we introduce a new set theory, MSE, which takes a combined approach to repairing comprehension: We weaken the comprehension scheme *and* weaken (or, more charitably, generalize) the underlying logic by working in a real-valued logic. For the sake of presentation, we will work in an ad hoc formalism, heavily based on first-order continuous logic, that capitalizes on the fact that our theory is a theory of sets that are actually sets of elements, rather than arbitrary $[0,1]$-valued predicates on some domain. In particular, our formalism will take bounded quantification as primitive, which is not possible to do in continuous logic, introduced in its modern form in [25].

Models of MSE are triples $(M, d, \sqsubseteq)$, where (M, d) is a complete metric space and $\sqsubseteq \subseteq M^2$ is a closed binary relation. Such a structure is a model of MSE if it satisfies a strong metric form of extensionality and a weak approximation of comprehension. The strong form of extensionality, which we refer to as H-*extensionality*, requires that for any $a, b \in M$, $d(a,b) = d_{\mathrm{H}}(\{x : x \sqsubseteq a\}, \{x : x \sqsubseteq b\})$, where d_{H} is the Hausdorff metric on sets (Definition 2.2). We refer to H-extensional structures $(M, d, \sqsubseteq)$ as *metric set structures*.

The weak form of comprehension is the following principle: For any real-valued formula $\varphi(x, \bar{y})$, any real numbers $r < s$, and any tuple of parameters $\bar{a} \in M$, there is a set $b \in M$ such that for any c, if $\varphi(c, \bar{a}) \leq r$, then $c \sqsubseteq b$ and if $c \sqsubseteq b$, then $\varphi(c, \bar{a}) < s$ (Definition 2.7). Crucially, we make no guarantees about membership of those c's for which $\varphi(c, \bar{a})$ falls in the gap between r and s and so in this sense the principle is non-deterministic.

The word that we find most accurately captures this principle is *excision*, the idea being that we are only able to cut out a desired set somewhat crudely. From this we get the initialism MSE, for *Metric Sets with Excision*. Of course the nature of this principle depends entirely on what is meant by 'real-valued formula,' which is formalized in Section 2.2, but what matters is that these formulas are automatically uniformly continuous with regards to the metric. In particular, we have no direct access to the relation $\sqsubseteq$ as a $\{0,1\}$-valued predicate and instead can only use it in instances of bounded quantification, such as $\inf_{y \sqsubseteq z} \varphi(x, y)$.

After defining MSE and developing some techniques for constructing particular sets, we will establish that MSE is sufficiently strong and expressive by showing that it (with an axiom of infinity) interprets classical TSTI[1] (or, equivalently, full ωth-order arithmetic or the theory of a Boolean topos with a natural numbers object), which is well-known to be more than sufficient for the majority of everyday mathematics. In particular, we do this by considering *uniformly discrete sets*, which are better behaved than arbitrary sets in models of MSE. We then build canonical representatives of internal well-order types in models of MSE (which we call ordinals). We show that MSE has no ultrametric β-models (i.e., models that are correct about well-foundedness) by showing that the class of ordinals of any such model M admits an external map s to $(0,1]$ that is non-increasing and has dense image (implying that

[1] See [5, Section 1.1.2] for the definition of TSTI.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

the preimage of $(0, 1)$ under s has no least element). When we eventually construct models of MSE by using a non-standard modification of the ordinary construction of models of GPK^+, we show that they can have arbitrarily large standard ordinals (Theorem 7.15). This is of course similar to the situation with NF and NFU, which has no true β-models yet can have arbitrarily large well-founded parts,[2] but the mode of failure is more conceptually similar to the mechanism that prevents $\mathrm{C\L}_0$ from having ω-models in that it involves the difficulty of robustly formalizing induction for real-valued predicates. We also construct a pseudo-finite model of our theory (without infinity). This establishes that MSE without infinity has an incredibly low consistency strength, lower than Robinson arithmetic, in contrast to NFU and GPK^+ without infinity.[3] We also show that any complete metric space of diameter at most 1 can be embedded as an internal set of Quine atoms in a model of MSE, which in particular shows that not all models of MSE are ultrametric. Nevertheless, all models we are able to construct admit the map s as before, so we conjecture that this is in fact always the case.

Finally, we show how to formalize our theory in either continuous logic or Łukasiewicz predicate logic. In these contexts, we consider structures (without a given metric) of the form (M, e), where e is a binary $[0, 1]$-valued predicate on M. The intended interpretation of $e(x, y)$ is the quantity $\inf_{z \sqsubseteq y} d(x, z)$. Our strong form of extensionality ensures that the metric $d(x, y)$ can be recovered from $e(x, y)$ by the formula $d_e(x, y) := \inf_z |e(z, x) - e(z, y)|$. The H-extensionality axiom now takes the form

$$\sup_{xy} |e(x, y) - \inf_z \min(d_e(x, z) + 2e(z, y), 1)| = 0,$$

and the axiom scheme of excision consists of

$$\sup_{\bar{y}} \inf_z \sup_x \max(\min(e(x, z), -\varphi(x, \bar{y})), \min(\varepsilon_\varphi - e(x, z), \varphi(x, \bar{y}) - 1)) \leq 0$$

for each restricted $\mathcal{L}_e$-formula $\varphi(x, \bar{y})$, where ε_φ is a certain rational number directly computable from φ. We show that models of the above theory are precisely pre-models of MSE in the sense that the completion with regards to d_e yields a model of MSE (by taking $x \sqsubseteq y$ to be the relation $e(x, y) = 0$). Furthermore, all models of MSE arise in this way.

In Łukasiewicz logic, we use the predicate[4] $x \mathrel{\hat{\varepsilon}} y$ instead of $e(x, y)$, with the intended meaning being that $(x \mathrel{\hat{\varepsilon}} y) = 1 - e(x, y)$. The H-extensionality axiom is directly translated as

$$\forall x \forall y (x \mathrel{\hat{\varepsilon}} y \leftrightarrow \exists z (x =_e z \,\&\, z \mathrel{\hat{\varepsilon}} y \,\&\, z \mathrel{\hat{\varepsilon}} y)),$$

[2] This was shown by Jensen for NFU in [13] and the argument for NF is sketched in [12, Section 5.3].

[3] NFU's consistency strength without infinity is strictly between Robinson arithmetic and PA. GPK^+ is equiconsistent with full second-order arithmetic. Specker established in [20] that NF proves the axiom of infinity, so NF isn't relevant to this particular discussion.

[4] The hat in $\hat{\varepsilon}$ is merely to help visually distinguish it from the four other epsilon-like symbols in this article, $\in$, $\sqsubseteq$, e, and ε. We never use the symbol ε, and $\hat{\varepsilon}$ is only used in the last section of the article. $\in$ always refers to standard set-theoretic membership and ε is always a real number.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

where $x =_e y$ is the formula $\forall z(z\ \hat{\varepsilon}\ x \leftrightarrow z\ \hat{\varepsilon}\ y)$ (which is the same as $1 - d_e(x, y)$), and the axiom scheme of excision is shown to be equivalent to the scheme

$$\forall \bar{y}\exists z\forall x(x\ \hat{\varepsilon}\ z \vee (\neg\varphi\ \&\ \neg\varphi\ \&\ \neg\varphi)) \wedge ((\underbrace{\neg x\ \hat{\varepsilon}\ z\ \&\ \cdots\ \&\ \neg x\ \hat{\varepsilon}\ z}_{6\cdot\#\varphi\text{ times}}) \vee (\varphi\ \&\ \varphi\ \&\ \varphi))$$

for each Łukasiewicz formula $\varphi(x, \bar{y})$, where $\#\varphi$ is the number of instances of $\hat{\varepsilon}$ in φ.

§2. Specification of MSE.

2.1. H-extensionality and metric set structures. The structures we will be considering will be of the form $(M, d, \sqsubseteq)$, where (M, d) is a complete metric space, and $\sqsubseteq \subseteq M^2$ is a closed binary relation. As is suggested by the notation, $\sqsubseteq$ is meant to be interpreted as a set membership relation, and, as such, we would like for it to be extensional. Obviously we could just require extensionality of $\sqsubseteq$ as a binary relation in the standard sense, but for a few different reasons, we will opt to place a stronger condition on $\sqsubseteq$. To define this condition, recall one of many equivalent definitions of the *Hausdorff distance* between subsets of a metric space.

DEFINITION 2.1. The *Hausdorff distance between A and B*, written $d_{\mathrm{H}}(A, B)$, is the unique smallest element of $[0, \infty]$ such that for any $r > d_{\mathrm{H}}(A, B)$,

- for every $a \in A$, there is a $b \in B$ such that $d(a, b) < r$, and
- for every $b \in B$, there is an $a \in A$ such that $d(a, b) < r$.

On the full power set of a metric space M, d_{H} is an extended pseudo-metric, but on the collection of close subsets of M, it is an extended metric. We will be concerned exclusively with $[0, 1]$-valued metrics. In this context, it makes sense to modify the above definition to take d_{H} to be $[0, 1]$-valued as well. This only changes the distance between the empty set and non-empty sets. In particular, $d_{\mathrm{H}}(\varnothing, A) = 1$ for any non-empty A. This is the definition of the Hausdorff distance we will actually use.

The form of Definition 2.1 above makes it clear that d_{H} is a direct metric generalization of extensional equality of sets. $A = B$ if and only if for every $a \in A$, there is a $b \in B$ such that $a = b$ and for every $b \in B$, there is an $a \in A$ such that $a = b$. In this way, we take as our extensionality axiom a direct translation of the statement '$A = B$ if and only if A and B are coextensive.'

DEFINITION 2.2. Given a metric space (M, d) and a binary relation $\sqsubseteq \subseteq M^2$, we say that $\sqsubseteq$ is H-*extensional* if for any $a, b \in M$, $d(a, b) = d_{\mathrm{H}}(\{x : x \sqsubseteq a\}, \{x : x \sqsubseteq b\})$.

We say that $(M, d, \sqsubseteq)$ is a *metric set structure* if (M, d) is a complete metric space, d is $[0, 1]$-valued, and $\sqsubseteq \subseteq M^2$ is closed and H-extensional.

Metric set structures are a generalization of extensional digraphs (i.e., discrete models of the extensionality axiom). If δ is a $\{0, 1\}$-valued metric on V, then (V, δ, E) is a metric set structure if and only if (V, E) is an extensional digraph.

We should note that the definition of H-extensionality contains a somewhat arbitrary choice. After all, we could have just as easily required that $d(a, b) = (d_{\mathrm{H}}(\{x : x \sqsubseteq a\}, \{x : x \sqsubseteq b\}))^{1/2}$. That said, the choice we have made is reasonable

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

and seems to work well, so we have not investigated other possibilities in this article. Also, we should note that this notion of extensionality appears in a previous paper of the author [9, Definition 6.4].

A commonly cited benefit of extensionality is that it allows one to take $\in$ as the only primitive notion, with $x = y$ being defined as $\forall z(z \in x \leftrightarrow z \in y)$. It seems unlikely that we will be able to do something similar with $\sqsubseteq$, but we can do something similar with the natural $[0, 1]$-valued version of $x \sqsubseteq Y$, which is the distance from x to the elements of Y, commonly written $\operatorname{dist}(x, Y)$ or $d(x, Y)$. In the context of a set theory, $d(x, Y)$ is entirely unacceptable, being tantamount to writing $x = Y$ to mean $x \in Y$. $\operatorname{dist}(x, Y)$ is too long to use frequently, so we will use the following notation (which is used similarly in [3, 4]).

DEFINITION 2.3. We write $e(x, y)$ for $\inf\{d(x, z) : z \sqsubseteq y\}$.

Another useful characterization of the Hausdorff metric is this: $d_{\mathrm{H}}(A, B) = \sup_z |\operatorname{dist}(z, A) - \operatorname{dist}(z, B)|$. This means that if $(M, d, \sqsubseteq)$ is a metric set structure, we have that $d(x, y) = \sup_z |e(z, x) - e(z, y)|$. Since $x \sqsubseteq y$ if and only if $e(x, y) = 0$, it should be possible to take $e(x, y)$ as our only primitive notion. This is essentially the approach taken in the formalization of $\mathrm{C}\text{Ł}_0$ (see [8] and [17, Chapter 4.5]) and this is the approach we will take in the 'official' continuous logic formulation of MSE, which we will discuss in Section 6.

2.2. Formulas. Our formalism will be a small modification of first-order continuous logic, introduced in its modern form in [25]. Our only predicate symbol will be the metric, $d(x, y)$, but we will take bound quantifiers of the form $\sup_{x \sqsubseteq y}$ as a primitive notion. Note though that we cannot access $x \sqsubseteq y$ as a formula directly.

DEFINITION 2.4. Our set of *formulas*, written $\mathcal{L}_{\sqsubseteq}$, is the smallest non-empty set of expressions satisfying the following: For any $\varphi, \psi \in \mathcal{L}_{\sqsubseteq}$, variables x and y, and $r \in \mathbb{R}$, $\mathcal{L}_{\sqsubseteq}$ contains the expressions:

- 1,
- $d(x, y)$,
- $\varphi + \psi$,
- $\max(\varphi, \psi)$,
- $\min(\varphi, \psi)$,
- $r \cdot \varphi$,
- $\sup_x \varphi$,
- $\inf_x \varphi$,
- $\sup_{x \sqsubseteq y} \varphi$,
- $\inf_{x \sqsubseteq y} \varphi$.

When we need to be more specific, we will refer to elements of $\mathcal{L}_{\sqsubseteq}$ as $\mathcal{L}_{\sqsubseteq}$-formulas.

We refer to quantifiers of the form $\sup_{x \sqsubseteq y}$ or $\inf_{x \sqsubseteq y}$ as *bounded*. The *free variables* of a formula φ are defined in the obvious way. We write $\varphi(\bar{x})$ to indicate that the free variables of φ are included in $\bar{x}$.

For bookkeeping purposes, we will need the inductively defined quantity given by

- $v(1) = v(d(x, y)) = 1$,
- $v(\varphi + \psi) = v(\varphi) + v(\psi)$,
- $v(\max(\varphi, \psi)) = v(\min(\varphi, \psi)) = \max(v(\varphi), v(\psi))$,
- $v(r \cdot \varphi) = |r| v(\varphi)$,
- $v(\sup_x \varphi) = v(\inf_x \varphi) = v(\sup_{x \sqsubseteq y} \varphi) = v(\inf_{x \sqsubseteq y} \varphi) = v(\varphi)$.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

We are allowing ourselves arbitrary real numbers in Definition 2.4 because it will be convenient in several places. This convenience comes at a cost later in Section 6, however.

The intended interpretation of a given formula is clear, although we do need to specify the behavior of bounded quantifiers over empty sets. This is the first of two reasons why we defined the quantity $v(\varphi)$.

DEFINITION 2.5. Given a metric set structure $(M, d, \sqsubseteq)$ we define real-valued functions φ^M for $\varphi \in \mathcal{L}_{\sqsubseteq}$ inductively:

- $1^M = 1$.
- $(d(a,b))^M = d(a,b)$ for all $a, b \in M$.
- $(\varphi + \psi)^M = \varphi^M + \psi^M$. We define $(\max(\varphi, \psi))^M$, $(\min(\varphi, \psi))^M$, and $(r \cdot \varphi)^M$ similarly.
- $(\sup_x \varphi(x, \bar{a}))^M = \sup\{\varphi^M(b, \bar{a}) : b \in M\}$. We define $(\inf_x \varphi(x, \bar{a}))^M$ similarly.
- $(\sup_{x \sqsubseteq a} \varphi(x, a, \bar{b}))^M = \sup\{\varphi^M(c, a, \bar{b}) : c \sqsubseteq a\}$ if $c \sqsubseteq a$ for some $c \in M$. $(\inf_{x \sqsubseteq a} \varphi(x, a, \bar{b}))^M$ is defined similarly if $c \sqsubseteq a$ for some $c \in M$.
- $(\sup_{x \sqsubseteq a} \varphi(x, a, \bar{b}))^M = -v(\varphi)$ if $c \not\sqsubseteq a$ for all $c \in M$.
- $(\inf_{x \sqsubseteq a} \varphi(x, a, \bar{b}))^M = v(\varphi)$ if $c \not\sqsubseteq a$ for all $c \in M$.

We write expressions such as $M \models \varphi(\bar{a}) \leq \psi(\bar{b})$ to mean that $\varphi^M(\bar{a}) \leq \psi^M(\bar{b})$. We may also write expressions like $M \models \varphi(\bar{a}) = r$.

The conventions regarding suprema and infima of empty sets were chosen so that formulas would always be real-valued (rather than taking on values in $\mathbb{R} \cup \{\pm\infty\}$) and so that sup and inf are monotonic with regards to set inclusion, although we have to prove that this is actually the case.

We will often use commonsensical shorthand, such as $\varphi + \psi + \theta$ for $\varphi + (\psi + \theta)$, $\varphi - \psi$ for $\varphi + (-1) \cdot \psi$, and $|\varphi|$ for $\max(\varphi, -\varphi)$. We will abbreviate consecutive quantifiers with expressions such as $\sup_{xy}$. By an abuse of notation, we will also write $e(x, y)$ as shorthand for the formula $\inf_{z \sqsubseteq y} d(x, z)$.

In the context of a metric set structure M, we may often refer to formulas with parameters, such as $\varphi(\bar{x}, \bar{a})$ for some $\bar{a} \in M$, as formulas and write them with parameters suppressed.

Two important properties of formulas in continuous logic are that they only take on values in some bounded interval and that they are always uniformly continuous. The relevant interval and modulus of uniform continuity can be determined by the formula alone. We will need similar facts here.

LEMMA 2.6. *For any metric set structure $(M, d, \sqsubseteq)$ and formula $\varphi(\bar{x})$,*

(1) *$\varphi^M(\bar{a}) \in [-v(\varphi), v(\varphi)]$ for all $\bar{a} \in M$ and*
(2) *$\varphi^M : M^{|\bar{x}|} \to \mathbb{R}$ is $2v(\varphi)$-Lipschitz in the sense that for any $\bar{a}, \bar{b} \in M$, $|\varphi^M(\bar{a}) - \varphi^M(\bar{b})| \leq 2v(\varphi) d(\bar{a}, \bar{b})$ (where $d(\bar{a}, \bar{b})$ is the max metric on tuples).*

PROOF. 1 follows by an easy induction argument and the fact that d is $[0,1]$-valued. 2 follows similarly from the fact that $(x, y) \mapsto d(x, y)$ is 2-Lipschitz. ⊣

2.3. Excision. Our comprehension scheme is better defined in terms of its important consequence, rather than directly, as it takes a proof to establish that

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

this property is even axiomatizable. The principle can be informally justified like this:

Suppose that we run a chalk factory and we are contractually obligated to produce pieces of chalk that are no longer than 7.62 cm in length. (The chalk is boxed by another company and needs to fit in their boxes.) Of course, our machine, being cheap, actually produces pieces that are anywhere between roughly 7.4 cm and 7.8 cm. To deal with this, we add a second machine that measures length and rejects pieces that are too long. To maximize our output, we might say that we want it to reject a piece if and only if its length is strictly longer than 7.62 cm, but the realities of physical measurement mean that this is impossible to actually accomplish. Since the penalties for violating the contract are quite harsh, we need to give ourselves some leeway, but we also want to make sure we aren't throwing away too many acceptable pieces of chalk. So we configure the machine to accept chalk if it measures it to be no longer than 7.6 cm. We know that the error of the machine is no more than 0.01 cm, so we can guarantee that we will accept any piece of length at most 7.58 cm and reject any piece of length 7.62 cm or more, but we do not make any promises about the behavior of the machine in the gap between these bounds.[5]

This is the manner in which we will approximate comprehension. Given a formula $\varphi(x, \bar{y})$ (i.e., a 'measurable quantity'), bounds $r < s$, and parameters $\bar{a}$, we promise that we can deliver a set b such that for any c, if $\varphi(c, \bar{a}) \leq r$, then $c \sqsubseteq b$, and if $\varphi(c, \bar{a}) \geq s$, then $c \not\sqsubseteq b$, but we make no commitment about those c's for which $r < \varphi(c, \bar{a}) < s$.

DEFINITION 2.7. $(M, d, \sqsubseteq)$ satisfies *excision* if for any formula $\varphi(x, \bar{y})$, reals $r < s$, and $\bar{a} \in M$, there is a $b \in M$ such that for any $c \in M$, if $\varphi(c, \bar{a}) \leq r$, then $c \sqsubseteq b$, and if $c \sqsubseteq b$, then $\varphi(c, \bar{a}) < s$.

It is straightforward but worthwhile to see how this principle avoid Russell's paradox. We can consider a set a_r satisfying that if $1 - e(b, b) \leq r$, then $b \sqsubseteq a_r$ and if $b \sqsubseteq a_r$, then $1 - e(b, b) < 1$. As we pick r closer and closer to 1, we get better and better approximations of the Russell class, but for each r, we consistently have that $0 < e(a_r, a_r) < 1 - r$. So we see that while our theory is strictly speaking a $[0, 1]$-valued set theory like $\mathrm{C\L}_0$, there is something of a qualitative difference in its avoidance of Russell's paradox. While $\mathrm{C\L}_0$ is possibly[6] able to avoid Russell's paradox by Brouwer's fixed point theorem, our theory avoids it by virtue of the required gap between r and s (although these are not unrelated phenomena).

We are now finally able to define the class of models of our theory directly before defining the theory itself.

DEFINITION 2.8. We say that $(M, d, \sqsubseteq)$ is a model of MSE, written $(M, d, \sqsubseteq) \models \mathsf{MSE}$ or $M \models \mathsf{MSE}$, if it is a metric set structure that satisfies excision.

---

[5] Reality aside, a similar thing happens in the context of computable analysis: One can write a program that is able to discretely sort computable real numbers in the same manner as our chalk factory, but it is only able to do this if it is allowed to have non-deterministic behavior in some gap. Regardless, this is sufficient for certain purposes.

[6] Various fragments of this theory were shown to be consistent by a few authors in the 1950s and 1960s [3, 4, 19]. A full consistency proof was claimed by White in 1979 [24], but a seemingly fatal gap was discovered by Terui in 2010 [22] and consistency remains an open problem.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

MSE stands for *Metric Sets with Excision*.

The following notation will be useful.

DEFINITION 2.9. Given a metric set structure M, a formula $\varphi(x, \bar{y})$, tuple $\bar{a} \in M$, and reals $r < s$, we write

$$b = \left[x : \varphi(x, \bar{a}) \leq r \sim s\right]$$

to mean that for any c, if $\varphi^M(c, \bar{a}) \leq r$, then $c \sqsubset b$ and if $c \sqsubset b$, then $\varphi^M(c, \bar{a}) < s$.

Note of course that $\left[x : \varphi(x) \leq r \sim s\right]$ is not a uniquely specified object, but if M satisfies excision, it always exists.

It is immediate to show that models of MSE contain some of the familiar sets one expects to see in a set theory with a universal set.

PROPOSITION 2.10. *For any $M \models$ MSE, there are $a, b \in M$ such that for all $c \in M$, $c \not\sqsubset a$, and $c \sqsubset b$.*

PROOF. Let $a = \left[x : 1 \leq 0 \sim \frac{1}{2}\right]$ and $b = \left[x : 0 \leq \frac{1}{2} \sim 1\right]$. ⊣

Since these sets are unique by H-extensionality, we will write $\varnothing^M$ for $\left[x : 1 \leq 0 \sim \frac{1}{2}\right]$ and V^M for $\left[x : 0 \leq \frac{1}{2} \sim 1\right]$. We may drop the superscript M if no confusion will arise.

§3. Derived forms of comprehension. In this section we will show that models of MSE automatically satisfy certain instances of exact comprehension.

3.1. Relative excision. An important common construction in set theory is *separation*, i.e., comprehension relative to a given set. Ordinarily, separation is an easy consequence of comprehension—$\{x \in A : \varphi(x)\}$ is the same as the set $\{x : x \in A \wedge \varphi(x)\}$—but seeing that excision is merely an approximate form of comprehension, one might worry that we will only be able to find sets that are approximately subsets of other given sets. In other words, if B is a rough approximation of $\{x : x \in A \wedge \varphi(x)\}$, then it would only be the case that $x \sqsubset B \Rightarrow e(x, A) < r$ for some small but positive r. Fortunately, we are able to build exact subsets of a given set and thereby perform *relative excision*.

DEFINITION 3.1. For any $a, b \in M$, a metric set structure, we write $a \sqsubseteq b$ to mean that for all $c \in M$, if $c \sqsubset a$, then $c \sqsubset b$.

The following is a special case of relative excision, but we state it first because it is the only form of relative excision we will actually use and it is much easier to prove.

PROPOSITION 3.2 (Discrete separation). *Fix $M \models$ MSE and $\bar{a}$ and b in M. For any formula $\varphi(x, \bar{a})$ and $r < s$, if for all $c \sqsubset b$, $\varphi^M(c, \bar{a}) \leq r$ or $\varphi^M(c, \bar{a}) \geq s$, then there is an $f \sqsubseteq b$ such that $c \sqsubset f \Leftrightarrow \varphi^M(c, \bar{a}) \leq r$.*

PROOF. If $f_n = \left[x : \max\left(\varphi(x, \bar{a}) - r, e(x, a)\right) \leq 0 \sim 2^{-n}\right]$ for each $n \in \mathbb{N}$, then $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence that limits to the required set. ⊣

LEMMA 3.3. *Fix $M \models$ MSE. For any $a, b \in M$ and $\varepsilon > 0$, there is a $c \sqsubseteq a$ such that*

$$d(b, c) \leq \sup_{f \sqsubset b} \inf_{g \sqsubset a} d(f, g) + \varepsilon.$$

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

PROOF. Fix $\varepsilon > 0$. Let $c_0 = b$. For any n, let $t_n := \sup_{f \sqsubseteq c_n} \inf_{g \sqsubseteq a} d(f,g)$. At stage n, given c_n, let

$$c_{n+1} = \left[x : \max(e(x,a), e(x,b) - t_n) \leq 2^{-n-4}\varepsilon \sim 2^{-n-3}\varepsilon\right].$$

Note that for any $f \in M$, if $f \sqsubset a$ and $e(f,c_n) < t_n + 2^{-n-4}\varepsilon$, then $f \sqsubset c_{n+1}$, and if $f \sqsubset c_{n+1}$, then $e(f,a) < 2^{-n-4}\varepsilon$ and $e(f,c_n) < t_n + 2^{-n-3}\varepsilon$. In particular, $t_{n+1} \leq 2^{-n-3}\varepsilon$. Note also that by the definition of t_n, we have that for any $g \sqsubset c_n$, there is an $f \sqsubset a$ such that $d(f,g) < t_n + 2^{-n-4}\varepsilon$. Such a g must be an element of c_{n+1}. Since we can do this for any $f \sqsubset c_n$, we have that $d(c_n, c_{n+1}) \leq \max(t_n + 2^{-n-4}\varepsilon, t_n + 2^{-n-3}\varepsilon) = t_n + 2^{-n-3}\varepsilon$. Hence, for any $n > 0$, we have that $d(c_n, c_{n+1}) \leq 2^{-n-2}\varepsilon + 2^{-n-3}\varepsilon < 2^{-n-1}\varepsilon$. Therefore $(c_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $c = \lim_{n\to\infty} c_n$.

Since $t_n \to 0$ as $n \to \infty$, we have that $c \sqsubseteq a$. Now we just need to verify that $d(b,c) \leq t_0 + \varepsilon$. Our estimates give that

$$d(b,c) \leq d(c_0,c_1) + d(c_1,c) \leq t_0 + 2^{-4}\varepsilon + \sum_{n=1}^{\infty} 2^{-n-1}\varepsilon < t_0 + \varepsilon,$$

as required. ⊣

PROPOSITION 3.4 (Relative excision). *If $M \models \mathsf{MSE}$, then for any real-valued formula $\varphi(x, \bar{y})$, $\bar{a}, b \in M$, and reals $r < s$, there is a $c \in M$ with $c \sqsubseteq b$ such that for any $f \sqsubset b$, if $\varphi(f, \bar{a}) \leq r$, then $f \sqsubset c$, and if $f \sqsubset c$, then $\varphi(c, \bar{a}) < s$.*

PROOF. Fix $\varphi(x, \bar{y}) \in \mathcal{L}_{\sqsubset}$, $\bar{a}$ and b in M, and $\delta > 0$ with $\delta < \frac{1}{4v(\varphi)}$.

Let $c = \left[x : \max(\varphi(x, \bar{a}), e(x,b)) \leq 0 \sim \delta\right]$. Note that $\sup_{x \sqsubset c} \inf_{y \sqsubset b} d(x,y) \leq \frac{1}{2}\delta$. Apply Lemma 3.3 to c to get an $f \sqsubseteq b$ such that $d(c,f) < \delta$. We now have that for any g, if $g \sqsubset b$ and $\varphi(g, \bar{a}) \leq 0$, then $e(g,f) < \delta$ and if $e(g,f) \leq \frac{1}{6v(\varphi)}$, then $e(g,c) \leq \frac{1}{4v(\varphi)} + \delta < \frac{1}{2v(\varphi)}$ and so $\varphi(g, \bar{a}) < 2v(\varphi)\frac{1}{2v(\varphi)}$.

Since we can do this for any $\varphi(x, \bar{y})$ and $\bar{a} \in M$, we have that M satisfies Definition 6.5 relative to the set b. The proposition then follows by repeating the proofs of Lemma 6.6 and Proposition 6.7 relative to the set b. ⊣

In light of Propositions 3.2 and 3.4, we will write

$$c = \left[x \sqsubset b : \varphi(x, \bar{a}) \leq r \sim s\right]$$

to mean that for any $f \in M$, if $f \sqsubset b$ and $\varphi^M(f, \bar{a}) \leq r$, then $f \sqsubset c$ and if $f \sqsubset c$, then $f \sqsubset b$ and $\varphi^M(f, \bar{a}) < s$.

3.2. Comprehension for definable classes. Given Proposition 3.4, one might be tempted to ask whether we can just outright show that models of MSE satisfy a more conventional form of comprehension. Suppose we have a formula $\varphi(x)$ and we wish to form the set $\{x \in M : \varphi^M(x) = 0\}$. Could we not just form the sequence $a_n = \left[x : |\varphi(x)| \leq 0 \sim 2^{-n}\right]$ and take the limit? While we are perfectly able to form this sequence externally, the difficulty is that it will in general fail to be a Cauchy sequence.

Regardless, there are times when such a sequence of approximations does actually converge in the Hausdorff metric, giving us an instance of exact comprehension. This happens precisely when $\{x \in M : \varphi^M(x) = 0\}$ is a *definable set* in the sense of

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

continuous logic, although in the context of a set theory it would be more appropriate to refer to these as *definable classes*.

DEFINITION 3.5. A closed subset $D \subseteq M^n$ is a *definable class* if the function $\bar{x} \mapsto \inf_{\bar{a} \in D} d(\bar{x}, \bar{a})$ is a uniformly convergent limit of functions of the form $\varphi^M(\bar{x}, \bar{b})$ for $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_{\sqsubseteq}$ and $\bar{b} \in M$. D is definable *without parameters* if its definability is witnessed by formulas without parameters.

D is an *explicitly definable class*[7] if there is a $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_{\sqsubseteq}$ and a tuple $\bar{b}$ such that $\inf_{\bar{a} \in D} d(\bar{x}, \bar{a}) = \varphi^M(\bar{x}, \bar{b})$.

$\bar{x} \mapsto \inf_{\bar{a} \in D} d(\bar{x}, \bar{a})$ is called the *distance predicate of* D, which we may also write as $e(\bar{x}, D)$.

This definition is perhaps most strongly motivated by the fact that definable classes are precisely those that admit relative quantification.

LEMMA 3.6. *For any metric set structure M and $\bar{a} \in M$, if $\varphi(\bar{x}, \bar{a})$ is the distance predicate of a definable class $D \subseteq M^n$, then for any $\psi(\bar{x}, \bar{y}, \bar{z})$ and $\bar{c} \in M$,*

$$\left(\inf_{\bar{x}} \min(\psi(\bar{x}, \bar{c}, \bar{b}) + 2v(\psi)\varphi(\bar{x}, \bar{a}), v(\psi))\right)^M = \inf\{\psi^M(\bar{f}, \bar{c}, \bar{b}) : \bar{f} \in D\},$$

where $\inf \varnothing$ *is understood to be* $v(\psi)$.

PROOF. Let $r = \inf_{\bar{x}} \min(\psi(\bar{x}, \bar{c}, \bar{b}) + 2v(\psi)\varphi(\bar{x}, \bar{a}), v(\psi))$ and $s = \inf\{\psi^M(\bar{f}, \bar{c}, \bar{b}) : \bar{f} \in D\}$.

If D is empty, then $\varphi(\bar{x}, \bar{a}) = 1$ and the result holds.

If D is not empty, then we clearly have that $r \leq s$ since $\psi(\bar{x}, \bar{c}, \bar{b}) \in [-v(\psi), v(\psi)]$ by Lemma 2.6. For the other direction, fix $\bar{g} \in M$. For any $\varepsilon > 0$, there is an $\bar{f} \in D$ such that $d(\bar{f}, \bar{g}) < e(\bar{g}, D) + \varepsilon$. By Lemma 2.6, $\bar{x} \mapsto \psi(\bar{x}, \bar{c}, \bar{b})$ is $2v(\psi)$-Lipschitz, so $\psi(\bar{g}, \bar{c}, \bar{b}) + 2v(\psi)\varphi(\bar{g}, \bar{a}) \geq \psi(\bar{f}, \bar{c}, \bar{b})$ and therefore $\min(\psi(\bar{g}, \bar{c}, \bar{b}) + 2v(\psi)\varphi(\bar{g}, \bar{a}), v(\psi)) \geq \psi(\bar{f}, \bar{c}, \bar{b})$. Since we can do this for any $\bar{g} \in M$, we have that $r \geq s$ and we are done. ⊣

In continuous logic generally, definable classes can be characterized as those sets that admit relative quantification in the same sense as Lemma 3.6. In models of MSE moreover, definable classes of 1-tuples correspond precisely to sets. In particular, every definable class is explicitly definable by $e(x, a)$ for some a.

PROPOSITION 3.7. *Let $M \models \mathsf{MSE}$. A closed set $D \subseteq M$ is a definable class if and only if there is an $a \in M$ such that $D = \{b \in M : b \sqsubseteq a\}$.*

PROOF. The $\Leftarrow$ direction is obvious. To show the $\Rightarrow$ direction, find a formula $\varphi_n(x, \bar{a}_n)$ for each $n \in \mathbb{N}$ such that $\sup_x |\varphi_n(x, \bar{a}_n) - e(x, D)| < 2^{-n}$. Let $b_n = \left[x : \varphi_n(x, \bar{a}_n) \leq 2^{-n} \sim 2^{-n+1}\right]$. Note that if $c \in D$, then $c \sqsubseteq b_n$ and if $c \sqsubset b_n$, then $e(c, D) < 2^{-n+1} + 2^{-n} < 2^{-n+1}$. This implies that $d_{\mathrm{H}}(\{x \in M : x \sqsubseteq b_n\}, D) \leq 2^{-n+1}$, and so the sequence $(b_n)_{n \in \mathbb{N}}$ is a Cauchy sequence and $b = \lim_{n \to \infty} b_n$ has the property that $c \sqsubseteq b$ if and only if $c \in D$. ⊣

Proposition 3.7 allows us to answer some very basic questions that we haven't resolved yet.

[7] There is no standard term for explicit definability in continuous logic, as it's not a wholly natural concept.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

COROLLARY 3.8. *Fix* $M \models \mathsf{MSE}$.

(1) (Singletons) *For any* $a \in M$, *there is a* $b \in M$ *such that* $c \sqsubseteq b$ *if and only if* $c = a$.
(2) (Finite unions) *For any* $a, b \in M$, *there is a* $c \in M$ *such that* $f \sqsubseteq c$ *if and only if* $f \sqsubseteq a$ *or* $f \sqsubseteq b$.
(3) (Finite sets) *For any* $a_0, \dots, a_{n-1} \in M$, *there is a* $b \in M$ *such that* $c \sqsubseteq b$ *if and only if* $c = a_i$ *for some* $i < n$.
(4) (Closure-of-unions) *For any* $a \in M$, *there is a* $b \in M$ *such that* $c \sqsubseteq b$ *if and only if* b *is in the metric closure of* $\{f \in M : (\exists g \sqsubseteq a) f \sqsubseteq g\}$.

PROOF.

(1) This is witnessed by the formula $d(x, a)$.
(2) This is witnessed by the formula $\min(e(x, a), e(x, b))$.
(3) This follows from 1 and 2 by induction.
(4) This is witnessed by the formula $\inf_{y \sqsubseteq a} e(x, y)$. ⊣

Unfortunately, however, it is generally not the case that the distance to the intersection of two sets X and Y can be computed from the distances to X and Y. As such, *we cannot establish the existence of intersections in general*, and indeed as demonstrated in Proposition B.1 such intersections do not always exist. We do, however, get a kind of approximate intersection in the form of $\left[x \sqsubseteq a : e(x, b) \leq 0 \sim \varepsilon\right]$.

A minor corollary of Corollary 3.8 is that models of MSE satisfy $d(x, y) = \sup_z |e(x, z) - e(y, z)|$, as witnessed by the singleton $\{x\}$. In $[0, 1]$-valued set theories, the quantity $1 - \sup_z |e(x, z) - e(y, z)|$ is often referred to a *Leibniz equality*, as it represents the degree to which x and y cannot be discerned from each other.

In light of Corollary 3.8, we will write $\{a_0, a_1, \dots, a_{n-1}\}$ for the finite set containing $a_0, a_1, \dots, a_{n-1}$, $a \sqcup b$ for the union of a and b, and $\overline{\bigsqcup a}$ for the closure of the union of the elements of a.

Now that we have the ability to form finite sets by Corollary 3.8, we are free to code ordered pairs. While we certainly could use the standard Kuratowski ordered pair, Wiener's earlier definition is actually preferable to us for technical reasons.[8] As such, we will write $\langle a, b \rangle$ for $\{\{\{a\}, \varnothing\}, \{\{b\}\}\}$. Recall that $d(ab, cf) \coloneqq \max(d(a, c), d(b, f))$.

LEMMA 3.9. *Let* $M \models \mathsf{MSE}$. *For any* $a, b, c, f \in M$,

$$d(\langle a, b \rangle, \langle c, d \rangle) = d(ab, cf).$$

PROOF. Let $A = \{\{a\}, \varnothing\}$, $B = \{\{b\}\}$, $C = \{\{c\}, \varnothing\}$, and $F = \{\{f\}\}$. We have that

$$d(\{A, B\}, \{C, F\}) = \max(e(A, \{C, F\}), e(B, \{C, F\}), e(C, \{A, B\}), e(F, \{A, B\})).$$

[8] If $\langle a, b \rangle \coloneqq \{\{a\}, \{a, b\}\}$, then a straightforward but tedious calculation shows that $d(\langle a, b \rangle, \langle c, f \rangle) = \max(d(a, c), \min(d(b, f), \max(d(a, f), d(b, c))))$ and so $d(\langle a, b \rangle, \langle c, f \rangle) \leq d(ab, cf) \leq 3d(\langle a, b \rangle, \langle c, f \rangle)$. Setting $\langle a, b \rangle = \langle 2, 0 \rangle$ and $\langle c, f \rangle = \langle 1, 3 \rangle$ in $\mathbb{R}$ shows that this is sharp. If d is an ultrametric however, we do get $d(\langle a, b \rangle, \langle c, f \rangle) = d(ab, cf)$.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

For any x, $d(\{x\}, \varnothing) = 1$. This implies that $d(\{\{x\}, \varnothing\}, \{\{y\}\}) = 1$ for any x and y as well. This implies, for instance, that

$$e(A, \{C, F\}) = \min(d(A, C), d(A, F)) = \min(d(A, C), 1) = d(A, C).$$

This together with similar facts for the other three terms implies that

$$\begin{aligned} d(\{A, B\}, \{C, F\}) &= \max(d(A, C), d(B, F), d(C, A), d(F, B)) \\ &= \max(d(A, C), d(B, F)). \end{aligned}$$

Finally, $d(\{\{x\}\}, \{\{y\}\}) = d(x, y)$ and $d(\{\{x\}, \varnothing\}, \{\{y\}, \varnothing\}) = d(x, y)$ for any x and y, so we have that $d(\langle a, b\rangle, \langle c, f\rangle) = d(\{A, B\}, \{C, F\}) = d(ab, cf)$, as required. ⊣

Lemma 3.9 means that a sequence of ordered pairs can only converge to an ordered pair and that convergence of sequences of ordered pairs behaves in the expected way. In particular, the class of ordered pairs is closed. (Note that at the moment we don't know that the collection of ordered pairs is a set, but we will show this later in Corollary 3.15.)

With a little more work we can establish the existence of power sets.

PROPOSITION 3.10. (Power sets). *For any $a \in M$, there is a $b \in M$ such that $c \sqsubseteq b$ if and only if $c \sqsubseteq a$.*

PROOF. First note that by basic properties of the Hausdorff metric, the class $\mathcal{P}(a) \coloneqq \{c \in M : c \sqsubseteq a\}$ is necessarily closed. By Lemma 3.3, we know that for any $f \in M$,

$$\inf_{g \in \mathcal{P}(a)} d(f, g) \leq \sup_{y \sqsubseteq f} \inf_{z \sqsubseteq a} d(y, z).$$

On the other hand, for any $g \in \mathcal{P}(a)$, we must have that

$$\sup_{y \sqsubseteq f} \inf_{z \sqsubseteq a} d(y, z) \leq \sup_{y \sqsubseteq f} \inf_{z \sqsubseteq g} d(y, z) \leq d(f, g),$$

by monotonicity of inf. Therefore,

$$\sup_{y \sqsubseteq f} \inf_{z \sqsubseteq a} d(y, z) \leq \inf_{g \in \mathcal{P}(a)} d(f, g),$$

and so the two quantities are actually equal. Hence $\mathcal{P}(a)$ is a definable class and is coextensive with some element $b \in M$ by Proposition 3.7. ⊣

We will write $\mathcal{P}(a)$ for the power set of a.

3.3. Definable functions and replacement. Now that we are confident that ordered pairs exist, the next natural thing to consider is Cartesian products. In order to show that the class $a \times b \coloneqq \{\langle c, f\rangle : c \sqsubseteq a,\ f \sqsubseteq b\}$ is definable and therefore a set, what we would like to be able to do is write a real-valued formula like this:

$$\varphi(x) = \inf_{y \sqsubseteq a} \inf_{z \sqsubseteq b} d(x, \{\{\{y\}, \varnothing\}, \{\{z\}\}\}).$$

If we did have this formula, $\varphi(x)$ would of course be the point-set distance from x to the class $a \times b$. The issue is that the functions $x \mapsto \{x\}$ and $(x, y) \mapsto \{x, y\}$ and the constant $\varnothing$ are not formally part of our logic.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

Practically, however, it is commonly understood in the context of discrete logic that it is safe to pretend that certain functions—namely the *definable* functions—are formally part of the language in the following sense: Given a discrete structure M, a function $f : M^n \to M$ is *definable* if and only if for every formula $\varphi(\bar{x}, y, \bar{z})$, there is a formula $\psi(\bar{x}, \bar{z})$ such that for any $\bar{a}, \bar{b} \in M$, $M \models \varphi(\bar{a}, f(\bar{a}), \bar{b})$ if and only if $M \models \psi(\bar{a}, \bar{b})$. It is easy to show that this is equivalent to the graph of f being a definable subset of M^{n+1}.

In continuous logic, a similar thing can be done.

DEFINITION 3.11. Given a set $X \subseteq M^n$, a function $f : X \to M$ is *definable* if for every $\varepsilon > 0$, there is a $\varphi(\bar{x}, y, \bar{z}) \in \mathcal{L}_{\sqsubseteq}$ and a $\bar{c} \in M$ such that for any $\bar{a} \in X$ and $b \in M$, $|d(f(\bar{a}), b) - \varphi^M(\bar{a}, b, \bar{c})| < \varepsilon$.

f is *explicitly definable* if there is a $\varphi(\bar{x}, y, \bar{z}) \in \mathcal{L}_{\sqsubseteq}$ and a tuple $\bar{c}$ such that $d(f(\bar{a}), b) = \varphi^M(\bar{a}, b, \bar{c})$ for every $\bar{a} \in X$ and $b \in M^{n+1}$.

If $X = M^n$, we say that f is an *(explicitly) definable total function*. Otherwise it is an *(explicitly) definable partial function*.[9]

Given a function $g : M^n \to M$, we say that g is *(explicitly) definable on* X if $g \restriction X$ is (explicitly) definable.

When X is itself definable, it is not too hard to show that f is definable if and only if it is uniformly continuous and its graph is definable (in the sense of Definition 3.5 relative to the max metric on tuples).

While more general statements can be made (see [25, Section 9]), we really only need explicitly definable functions in this article.[10]

LEMMA 3.12. *For any metric set structure M, formula $\varphi(\bar{x}, y, \bar{z})$, and explicitly definable function $f(\bar{x})$ with domain $X \subseteq M^n$, there is a formula $\psi(\bar{x}, \bar{z})$ (possibly with parameters) such that for all $\bar{a} \in X$ and $\bar{b} \in M$, $\psi^M(\bar{a}, \bar{b}) = \varphi^M(\bar{a}, f(\bar{a}), \bar{b})$.*

PROOF. Let $f(\bar{x})$ be defined on X by $\chi(\bar{x}, y)$ (possibly with parameters). By the same argument as in the proof of Lemma 3.6, the formula $\inf_y \varphi(\bar{x}, y, \bar{z}) + 2v(\varphi)\chi(\bar{x}, y)$ is the required $\psi(\bar{x}, \bar{z})$. ⊣

A corollary of Lemma 3.12 is that compositions of explicitly definable functions are explicitly definable. (This is also true of definable functions, but we will not need it.)

It is fairly immediate that the operations we established in Section 3.2 are in fact definable.

PROPOSITION 3.13. *Let $M \models \mathsf{MSE}$. The following functions are explicitly definable*:

(1) $() \mapsto \varnothing^M$.
(2) $() \mapsto V^M$.
(3) $(x_0, x_1, \ldots, x_{n-1}) \mapsto \{x_0, x_1, \ldots, x_{n-1}\}$.
(4) $(x, y) \mapsto x \sqcup y$.
(5) $x \mapsto \overline{\bigsqcup x}$.

[9] Beware that, unlike in discrete first-order logic, definable partial functions do not always extend to definable total functions in continuous logic [11, Counterexample C.1.2].

[10] Note that with definable sets, in the special context of models of MSE, all definable sets are ultimately explicitly definable. It seems unlikely that this will also be true for definable functions.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

(6) $(x, y) \mapsto \langle x, y \rangle$.
(7) $x \mapsto \mathcal{P}(x)$.

PROOF. The definability of these functions other than (6) are witnessed by the following formulas:

(1) $d(y, \varnothing) = 1 - \inf_{z \sqsubseteq y} d(z, z)$.
(2) $d(y, V) = \sup_z e(z, y)$.
(3) $d(y, \{x_0, x_1, \ldots, x_{n-1}\}) = \sup_z |e(z, y) - \min(d(z, x_0), \ldots, d(z, x_{n-1}))|$.
(4) $d(z, x \sqcup y) = \sup_w |e(w, z) - \min(e(w, x), e(w, y))|$.
(5) $d\left(y, \overline{\bigsqcup x}\right) = \sup_z |e(z, y) - \inf_{w \sqsubseteq x} e(z, w)|$.
(7) $d(y, \mathcal{P}(x)) = \sup_z \left|e(z, y) - \sup_{u \sqsubseteq z} \inf_{v \sqsubseteq x} d(u, v)\right|$.

(6) follows from the fact that $(x, y) \mapsto \langle x, y \rangle$ is a composition of other explicitly definable functions. ⊣

Now finally we can return to the question of forming a Cartesian product of two sets. The relevant fact is this:

PROPOSITION 3.14. (Images of definable functions). *Let $M \models \mathsf{MSE}$. For any definable function $f : X \to M$ and any $a_0, \ldots, a_{n-1} \in M$, if $\bar{b} \in X$ for any tuple $\bar{b}$ satisfying $b_i \sqsubseteq a_i$ for each $i < n$, then the metric closure of $\{f(b_0, \ldots, b_{n-1}) : b_0 \sqsubseteq a_0, \ldots, b_{n-1} \sqsubseteq a_{n-1}\}$ is a definable class.*

PROOF. $\inf_{x_0 \sqsubseteq a_0} \ldots \inf_{x_{n-1} \sqsubseteq a_{n-1}} d(x, f(x_0, \ldots, x_{n-1}))$ is clearly the distance predicate of the class in question. By Lemma 3.12, this is equivalent to a formula. ⊣

COROLLARY 3.15. (Cartesian products). *Let $M \models \mathsf{MSE}$. For any $a, b \in M$, there is a $c \in M$ such that $f \sqsubseteq c$ if and only if $f = \langle g, h \rangle$ for some $g \sqsubseteq a$ and $h \sqsubseteq b$.*

PROOF. By Proposition 3.14, the metric closure of $a \times b \coloneqq \{\langle x, y \rangle : x \sqsubseteq a,\ y \sqsubseteq b\}$ is a definable class. By the discussion at the end of Section 3.2, $a \times b$ is already metrically closed, so we have that it is a definable class. By Proposition 3.7, we have that the required c exists. ⊣

We will write $a \times b$ for the set whose existence is established in Corollary 3.15.

One thing to note is that the proof of Corollary 3.15 actually establishes that the function $(x, y) \mapsto x \times y$ is explicitly definable as witnessed by the formula $\inf_{w \sqsubseteq x} \inf_{u \sqsubseteq y} d(z, \langle w, u \rangle)$.

It is occasionally useful to be able to project sets of ordered pairs onto their coordinates. Since the projection function $\pi_0(\langle x, y \rangle) = x$ is only partially defined, this is the first time we need the added generality of being able to talk about definable partial functions.

PROPOSITION 3.16. *Let $M \models \mathsf{MSE}$. Let π_0 and π_1 be the functions on the class of ordered pairs defined by $\pi_0(\langle x, y \rangle) = x$ and $\pi_1(\langle x, y \rangle) = y$. π_0 and π_1 are explicitly definable.*

PROOF. This is witnessed by the formulas $\inf_z d(\langle y, z \rangle, x)$ and $\inf_z d(\langle z, y \rangle, x)$. ⊣

The following facts will also be useful.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

LEMMA 3.17. *Fix closed sets $X \subseteq M^n$ and $Y \subseteq M^{n+1}$. Suppose that there is a formula $\varphi(\bar{x}, y)$ such that for every $\bar{a} \in X$ and $b \in M$, $\varphi^M(\bar{a}, b) = e(b, \{y : (\bar{a}, y) \in Y\})$. Then there is an explicitly definable function $f : X \to M$ such that for every $\bar{a} \in X$, $f(\bar{a})$ is coextensive with $\{y : (\bar{a}, y) \in Y\}$.*

PROOF. The formula $\psi(\bar{x}, z) = \inf_y |e(y, z) - \varphi(\bar{x}, y)|$ witnesses that the required function is explicitly definable. ⊣

LEMMA 3.18. *If $f : a \to M$ is an explicitly definable partial function on some set a, then the map $x \mapsto \overline{\{f(y) : y \sqsubset x\}}$ is an explicitly definable partial function on the set $\mathcal{P}(a)$.*

PROOF. Let $\varphi(y, z)$ be a formula (possibly with parameters) such that $\varphi^M(b, c) = d(b, f(c))$ for all $b \sqsubset a$ and $c \in M$. We now have that for any $g \sqsubseteq a$, $\inf_{z \sqsubset g} \varphi(y, z)$ is the distance predicate of $\overline{\{f(y) : y \sqsubset g\}}$. Therefore the required function is definable by Lemma 3.17. ⊣

3.4. Quotients by discrete equivalence relations. A commonly used operation is passing from an equivalence relation to its set of equivalence classes. We are able to do this for discrete equivalence relations.

DEFINITION 3.19. Fix a metric set structure M. Given a set $a \in M$, a formula $\varphi(x, y)$ is a *discrete equivalence relation on a* if for all $b, c \sqsubset a$, $\varphi^M(b, c)$ is either 0 or 1 and $\varphi^M(x, y) = 1$ is an equivalence relation on $\{b \in M : b \sqsubset a\}$.

Given a discrete equivalence relation $\varphi(x, y)$, we will refer to the equivalence classes of the equivalence relation $\varphi(x, y) = 1$ as *φ-equivalence classes*.

We need a small observation.

LEMMA 3.20. *Let $M \models \mathsf{MSE}$. If $\varphi(x, y)$ is a discrete equivalence relation on $a \in M$, then for any $b \sqsubset a$,*

$$\psi(x, b) \coloneqq \inf_{y \sqsubset a} \max(\varphi(y, b), d(x, y))$$

is the distance predicate of the φ-equivalence class of b.

PROPOSITION 3.21. *Let $M \models \mathsf{MSE}$. If $\varphi(x, y)$ is a discrete equivalence relation on $a \in M$, then there is a $b \in M$ containing precisely the φ-equivalence classes of a.*

Moreover, if c is a set such that $\varphi(x, y)$ is a discrete equivalence relation on every $a \sqsubset c$, then the map taking a to the set of φ-equivalence classes of a is an explicitly definable partial function on c.

PROOF. The formula in Lemma 3.20 defines a partial function on a that maps elements to their φ-equivalence classes. Therefore, by Proposition 3.14, the closure of the class of φ-equivalence classes of a is a set in M. By Lemma 3.20, the class of φ-equivalence classes is closed, so we are done.

The 'Moreover' statement follows from Lemma 3.17. ⊣

§4. Uniformly discrete sets and ordinary mathematics. A common feature of models of set theories that implement some kind of nearly unrestricted comprehension is that they have a class of tame sets in which unrestricted separation is consistent.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

In NF and NFU, the strongly Cantorian sets are well behaved in this way, and in GPK^+_∞, the closed sets of isolated points are likewise well behaved. In the context of MSE, the analogously well-behaved class seems to be that of the *uniformly discrete* sets.

DEFINITION 4.1. In a metric set structure M, an element $a \in M$ is *ε-discrete* if for any $b, c \sqsubset a$, either $b = c$ or $d(b,c) \geq \varepsilon$. a is *uniformly discrete* if it is ε-discrete for some $\varepsilon > 0$.

While it might make sense to simply use the term 'discrete' for sets that are ε-discrete for some $\varepsilon > 0$, 'discrete' has a pre-existing topological meaning (i.e., that each element is topologically isolated) which is weaker than what we are considering here.

DEFINITION 4.2. For any a and b, the *disjoint union of a and b*, written $a \boxplus b$, is $(a \times \{\varnothing\}) \sqcup (b \times \{\{\varnothing\}\})$.

Note that it is immediate that $(x, y) \mapsto x \boxplus y$ is an explicitly definable function, as it is a composition of explicitly definable functions.

LEMMA 4.3. *Let $M \models \mathsf{MSE}$. If $a, b \in M$ are ε-discrete, then $a \boxplus b$, $a \times b$, and $\mathcal{P}(a)$ are ε-discrete.*

PROOF. If a is empty, then the statements in the lemma are trivial, and if b is empty, then the statements for $a \boxplus b$ and $a \times b$ are trivial, so assume that a and b are both non-empty. Fix $c, f \sqsubset a$ and $g, f \sqsubset b$.

For the disjoint union, we have that $d(\langle c, \varnothing\rangle, \langle g, \{\varnothing\}\rangle) = d(\varnothing, \{\varnothing\}) = 1$. Furthermore, $d(\langle c, \varnothing\rangle, \langle f, \varnothing\rangle) = d(c, f)$ and $d(\langle g, \{\varnothing\}\rangle, \langle f, \{\varnothing\}\rangle) = d(g, f)$, so $a \boxplus b$ is ε-discrete.

For the Cartesian product, if $\langle c, g\rangle \neq \langle f, h\rangle$, then either $c \neq f$ or $g \neq h$. In either case we have that $d(\langle c, g\rangle, \langle f, h\rangle) \geq \varepsilon$.

For the power set, if $c \neq f$, then we may assume without loss of generality that there is a g such that $g \sqsubset c$ but $g \not\sqsubset f$. Since $g \neq h$ for all $h \sqsubset f$, we have that $e(g, h) \geq \varepsilon$ (since a is ε-discrete). Therefore $d(c, f) \geq \varepsilon$. ⊣

Note that Lemma 4.3 relies on our use of Wiener pairs over Kuratowski pairs. While seemingly a cosmetic issue, this matters in the treatment of uniformly discrete sets in the rest of this section and for much of the analysis of the global structure of models of MSE in Section 5.

Within uniformly discrete sets, we are generally able to reason in a familiar discrete manner. In order to do this, we will use a typed language for 'discrete formulas,' which we define next.

DEFINITION 4.4. Given a tuple $\bar{a}$ of formal type variables, we write $\bar{a}^*$ for the smallest collection of expressions containing $\{a_0, a_1, \dots\}$ and containing (the formal expressions) $b \times c$ and $\mathcal{P}(b)$ for any $b, c \in \bar{a}^*$.

We will assume that we have a sufficient collection of variable symbols assigned to each formal expression in $\bar{a}^*$, which are to be regarded as types.

DEFINITION 4.5. We write $\mathcal{L}_{\mathrm{dis}}(\bar{a})$ for the smallest collection of formulas satisfying the following (where b and c are elements of $\bar{a}^*$):

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

- For any x of type b and y of type c, $x = y$ is in $\mathcal{L}_{\text{dis}}(\bar{a})$.
- For any x of type b, y of type c, and z of type $b \times c$, $\langle x, y\rangle = z$ is in $\mathcal{L}_{\text{dis}}(\bar{a})$.
- For any x of type b and y of type $\mathcal{P}(b)$, $x \sqsubset y$ is in $\mathcal{L}_{\text{dis}}(\bar{a})$.
- For $\Phi, \Psi \in \mathcal{L}_{\text{dis}}(\bar{a})$, $\Phi \wedge \Psi$, $\Phi \vee \Psi$, $\Phi \to \Psi$, and $\neg\Phi$ are in $\mathcal{L}_{\text{dis}}(\bar{a})$.
- For $\Phi \in \mathcal{L}_{\text{dis}}(\bar{a})$ and x of type b, $(\exists x \sqsubset b)\Phi$ and $(\forall x \sqsubset b)\Phi$ are in $\mathcal{L}_{\text{dis}}(\bar{a})$.

If we wish to specify the variables and formal type variables of a formula $\Phi \in \mathcal{L}_{\text{dis}}(\bar{a})$, we will write $\Phi(\bar{x}; \bar{a})$ (where the free variables of Φ are among $\bar{x}$).

To distinguish discrete formulas from real-valued formulas, we will usually write discrete formulas with capital Greek letters.

The formal type variables in a formula of $\mathcal{L}_{\text{dis}}(\bar{a})$ are intended to be interpreted as uniformly discrete sets in a model of MSE, in which case a variable of type a is allowed to take on values in a. The interpretation of a formula is then clear.

Definition 4.6. Fix $M \models \mathsf{MSE}$. Given a formula $\Phi(\bar{x}; \bar{a}) \in \mathcal{L}_{\text{dis}}(\bar{a})$, a tuple $\bar{b} \in M$ of the same length as $\bar{a}$, and a tuple $\bar{c}$ of the same length as $\bar{x}$, we say that $\Phi(\bar{c}; \bar{b})$ is *well-typed* if for each $c_i \in \bar{c}$, if x_i is a variable of type $t(\bar{a})$ (where $t(\bar{a})$ is a formal type expression), then $c_i \sqsubset t(\bar{b})$ (where t is now interpreted as a literal expression involving the functions $\times$ and $\mathcal{P}$ in M).

If $\Phi(\bar{c}; \bar{b})$ is well-typed, we write $M \models \Phi(\bar{c}; \bar{b})$ to mean that $(M, \sqsubset)$ satisfies $\Phi(\bar{c}; \bar{b})$ as a discrete structure (where quantifiers, such as $\exists x \sqsubset t(\bar{b})$, are interpreted as $\exists x \sqsubset t(\bar{b})$).

Now, we will see that as long as the sets in $\bar{b}$ are ε-discrete, we can express these kinds of discrete formulas as real-valued formulas in a mostly uniform way.

Definition 4.7. Given a tuple $\bar{a}$ of formal type variables, a formula $\Phi(\bar{x}) \in \mathcal{L}_{\text{dis}}(\bar{a})$, and $\varepsilon > 0$, we write $[\![\Phi]\!]_\varepsilon(\bar{x}; \bar{a})$ for the real-valued formula defined by the following inductive procedure:

- $[\![x = y]\!]_\varepsilon = \max\left(1 - \frac{1}{\varepsilon} d(x, y), 0\right)$.
- $[\![\langle x, y\rangle = z]\!]_\varepsilon = \max\left(1 - \frac{1}{\varepsilon} d(\langle x, y\rangle, z), 0\right)$.
- $[\![x \sqsubset y]\!]_\varepsilon = \max\left(1 - \frac{1}{\varepsilon} e(x, y), 0\right)$.
- $[\![\Phi \wedge \Psi]\!]_\varepsilon = \min\left([\![\Phi]\!]_\varepsilon, [\![\Psi]\!]_\varepsilon\right)$.
- $[\![\neg\Phi]\!]_\varepsilon = 1 - [\![\Phi]\!]_\varepsilon$.
- $[\![(\exists x \sqsubset a)\Phi]\!]_\varepsilon = \sup_{x \sqsubset a} [\![\Phi]\!]_\varepsilon$.

The other Boolean connectives and the universal quantifier are defined from the above in the typical way.

Proposition 4.8. *Fix $M \models \mathsf{MSE}$. For any formal tuple $\bar{a}$ of type variable and any formula $\Phi(\bar{x}; \bar{a}) \in \mathcal{L}_{\text{dis}}(\bar{a})$, we have for any ε-discrete sets $\bar{b} \in M$ and any $\bar{c} \in M$ such that $\Phi(\bar{b}; \bar{c})$ is well-typed, $M \models \Phi(\bar{b}; \bar{c})$ if and only if $([\![\Phi]\!]_\varepsilon(\bar{b}; \bar{c}))^M = 1$ and $M \models \neg\Phi(\bar{b}; \bar{c})$ if and only if $([\![\Phi]\!]_\varepsilon(\bar{b}; \bar{c}))^M = 0$.*

Proof. This follows immediately by induction on the construction of formulas in $\mathcal{L}_{\text{dis}}(\bar{a})$. ⊣

Given Proposition 4.8, we can now confidently talk about familiar discrete concepts in the context of uniformly discrete sets. In particular, we can develop the notion of cardinalities.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

DEFINITION 4.9. Given a uniformly discrete sets $a, b \in M \models \mathsf{MSE}$, we write $M \models a \approx b$ to mean that there is a $c \sqsubseteq a \times b$ that is the graph of a bijection between a and b.

It is clear that there is an $\mathcal{L}_{\mathrm{dis}}(a,b)$-formula $\eta(x,y)$ with the property that if a and b are ε-discrete, $c \sqsubseteq a$, and $f \sqsubseteq b$, then $M \models [\![\eta(c,f)]\!]_\varepsilon$ if and only if $M \models c \approx f$. We will write $x \approx_{a,b} y$ for this formula. We will write $\approx_a$ for $\approx_{a,a}$. It is immediate that $[\![x \approx_a y]\!]_\varepsilon$ is an equivalence relation on $\mathcal{P}(a)$ whenever a is ε-discrete.

DEFINITION 4.10. Given an ε-discrete set a, a *cardinal of* a is a $[\![x \approx_a y]\!]_\varepsilon$-equivalence class of subsets of a. We write Card_a for the collection of cardinals of a.

Given $b \sqsubseteq a$, we write $|b|_a$ for the $[\![x \approx_a y]\!]_\varepsilon$-equivalence class of b.

It follows immediately from the above discussion and Proposition 3.21 that Card_a is a set for any uniformly discrete $a \in M \models \mathsf{MSE}$. Furthermore, $x \mapsto |x|_a$ is a definable function on $\mathcal{P}(a)$.

DEFINITION 4.11. Given a uniformly discrete set a and a set $b \sqsubseteq \mathcal{P}(a)$, we write $\mathrm{succ}_a(b)$ for the collection $\{c \sqsubset \mathcal{P}(a) : (\exists f \sqsubset b)[f \sqsubseteq c \wedge \exists!\, g(g \sqsubset f \wedge c \not\sqsubseteq f)]\}$.

It follows from Proposition 3.2 that $\mathrm{succ}_a(b)$ is a set for any uniformly discrete a and $b \sqsubseteq a$. Note that succ_a is an explicitly definable function by Lemma 3.17. Furthermore, if $b \in \mathrm{Card}_a$, then either $\mathrm{succ}_a(b) \in \mathrm{Card}_a$ or $\mathrm{succ}_a(b) = \varnothing$.

DEFINITION 4.12. We write 0 for the set $\{\varnothing\}$. (Note that 0 is always an element of Card_a.)

We write $\mathrm{ind}_a(x)$ for the $\mathcal{L}_{\mathrm{dis}}(a)$-formula $0 \sqsubset x \wedge (\forall y \sqsubset x)\, \mathrm{succ}_a(y) \in x \vee \mathrm{succ}_a(y) = \varnothing$. If $\mathrm{ind}_a(x)$ holds, we say that x is an *inductive* set.

We write $\mathbb{N}_a$ for the set $\{x \in \mathcal{P}^2(a) : (\forall y \in \mathcal{P}^3(a))\, \mathrm{ind}_a(y) \to x \sqsubset y\}$.

It is clear that $\mathrm{ind}_a(\mathrm{Card}_a)$ always holds and so $\mathbb{N}_a \sqsubseteq \mathrm{Card}_a$ for any uniformly discrete a.

Now we can finally state the local version of the axiom of infinity.

DEFINITION 4.13. We write $\mathsf{Inf}(a)$ for the $\mathcal{L}_{\mathrm{dis}}(a)$-sentence $|a|_a \notin \mathbb{N}_a$.

It is easy to show that $\mathsf{Inf}(a)$ holds if and only if $\mathbb{N}_a \neq \mathrm{Card}_a$.

Now provided that one can find a uniformly discrete $a \in M \models \mathsf{MSE}$ such that $M \models \mathsf{Inf}(a)$, we have that $(\mathbb{N}_a, \mathcal{P}(\mathbb{N}_a), \mathcal{P}^2(\mathbb{N}_a), \dots)$ is a model of full ωth-order arithmetic, which is sufficient to develop the majority of ordinary mathematics.

Given the local nature of our development here, one might worry that there could be uniformly discrete $a, b \in M \models \mathsf{MSE}$ for which $\mathsf{Inf}(a)$ and $\mathsf{Inf}(b)$ both hold but $\mathbb{N}_a$ and $\mathbb{N}_b$ are not internally isomorphic. Fortunately, since our theory is fully impredicative, this cannot happen.

PROPOSITION 4.14. *Fix $M \models \mathsf{MSE}$ and uniformly discrete $a \sqsubseteq b \in M$.*

(1) *The equivalence relation $\approx_a$ is the restriction of the equivalence relation $\approx_b$ to $\mathcal{P}(a) \times \mathcal{P}(a)$. Write ι for the induced map from Card_a to Card_b.*
(2) *If $M \models \mathsf{Inf}(a)$, then $M \models \mathsf{Inf}(b)$ and $\mathbb{N}_b$ is the image of $\mathbb{N}_a$ under ι.*
(3) *For any uniformly discrete c, if $M \models \mathsf{Inf}(a) \wedge \mathsf{Inf}(c)$, then $\mathbb{N}_a \approx \mathbb{N}_c$.*

PROOF. 1 follows from the fact that any bijection between subsets of a that exists as a subset of $b \times b$ is already a subset of $a \times a$. 2 and 3 follow from 1. ⊣

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

§5. Global structure of models of MSE. While Section 4 gives a satisfactory picture of the local structure of a model of MSE around some collection of uniformly discrete sets, the axiom of infinity is a global statement in that it says that there is some set, somewhere, that is infinite.

It is clear that we can take a more global view of cardinality for uniformly discrete sets in a model of MSE. $\approx$ is a perfectly well-defined equivalence relation externally and it would make sense to call its equivalence classes *cardinals*, but we cannot write it as a formula. While $0 = \{\varnothing\}$ and the class of all singletons, 1, are both sets, the class of doubletons in a model of MSE is never closed, which precludes it from being a set.[11] That said, the class of sets of size 1 or 2 (or more generally the class of non-empty sets of size at most n for any (externally) finite n) is a set by Propositions 3.7 and 3.14. Something similar to this observation is used in Definition 5.13 and Theorem 5.14 in order to construct canonical representatives of ordinals in MSE.

Suppose that there is a $\frac{1}{2}$-discrete set a such that $M \models \mathsf{Inf}(a)$. Does this necessarily imply that there is a 1-discrete set b such that $M \models \mathsf{Inf}(b)$? Can we somehow scale a set up in this way? We will see in Theorem 5.10 that the answer is no, and so in order to state the axiom of infinity in a global way, we will need to specify the scale at which infinity is to first appear.

5.1. Collecting ε-discrete sets and 'the' axiom of infinity. But before we can formalize that, we need to deal with another subtlety we have been ignoring up until now. How do we even know that we can find *any* uniformly discrete sets? Clearly $\varnothing$ is uniformly discrete, and likewise all hereditarily finite sets are, but it is not clear that the class of hereditarily finite sets is even a set.

What we would like to be able to do is make a formula $\varphi(x)$ that returns $\mathrm{dis}(x) := \sup\{r > 0 : x \text{ is } r\text{– discrete}\}$, but $\mathrm{dis}(x)$ is not a continuous function and so cannot possibly be a formula. If $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence limiting to a with $a_n \neq a$ for all $n \in \mathbb{N}$, then $(\{a_n, a\})_{n\in\mathbb{N}}$ will be $d(a_n, a)$-discrete but no better for every n, yet the limit, $\{a\}$, will be 1-discrete. (This is just the fact that the class of doubletons is not closed again.)

This makes it seem unlikely that we will be able to even approximately collect the r-discrete sets into a class. Nevertheless, we are able to do something nearly as good.

Fix $r > 0$ and consider the formula

$$\varphi_r(x) = \sup_{z,y \sqsubset x} \min(d(y,z), r - d(y,z)).$$

Note that $\varphi_r^M(a) \leq \varepsilon$ if and only if for every $b, c \sqsubset a$, either $d(y,z) \leq \varepsilon$ or $d(y,z) \geq r - \varepsilon$. If moreover $\varepsilon < \frac{1}{3}r$, this implies that the formula

$$E_r(x) = \max(\min(\tfrac{1}{r}(2r - 3d(x,y)), 1), 0)$$

defines a discrete equivalence relation on the elements of a. If a is already r-discrete, then this equivalence relation is equality. For any $\varepsilon > 0$ with $\varepsilon < \frac{1}{3}r$, let

$$X_r = \left\{x \in M : \varphi_r^M(x) < \tfrac{1}{3}r\right\}$$

[11] This phenomenon is also seen in models of GPK^+.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

and let $f_r : X_r \to M$ be the function that takes a to the set of E_r-equivalence classes of a. By Lemma 3.17 and Proposition 3.21, f_r is an explicitly definable partial function. Let $\psi_r(x, y)$ be a formula defining it (i.e., for any $a \in X_r$ and $b \in M$, $\psi_r^M(a, b) = d(f_r(a), b)$). Note that $\psi_r(x, y)$ does not need any parameters. Note moreover that for any $a \in X_r$, $f_r(a)$ is $\frac{1}{3}r$-discrete.

DEFINITION 5.1. For any $r, \varepsilon > 0$ with $\varepsilon < \frac{1}{3}r$, we let $\mathsf{Inf}_{r,\varepsilon}$ denote the condition

$$\sup_x \min\left(1 + \varepsilon - \varphi_e(x), [\![\mathsf{Inf}(f_r(x))]\!]_{\frac{1}{3}r}\right) \geq 1,$$

where $[\![\mathsf{Inf}(f_r(x))]\!]_{\frac{1}{3}r}$ means $\inf_z [\![\mathsf{Inf}(z)]\!]_{\frac{1}{3}r} + 2v([\![\mathsf{Inf}(-)]\!]_{\frac{1}{3}r})\psi_r(x, z)$.

We let Inf denote the collection of conditions $\{\mathsf{Inf}_{1,\varepsilon} : 0 < \varepsilon < \frac{1}{3}\}$.

PROPOSITION 5.2. *Fix $M \models \mathsf{MSE}$.*

(1) *For any $r, \varepsilon > 0$ with $\varepsilon < \frac{1}{3}r$, $M \models \mathsf{Inf}_{r,\varepsilon}$ if and only if for every $s \in (0, r - \varepsilon)$, there is an s-discrete set $a \in M$ such that $M \models \mathsf{Inf}(a)$.*
(2) *$M \models \mathsf{Inf}$ if and only if for every $r \in (0, 1)$, there is an r-discrete set $a \in M$ such that $M \models \mathsf{Inf}(a)$.*

PROOF. 2 follows immediately from 1. For the $\Rightarrow$ direction of 1, assume that $M \models \mathsf{Inf}_{r,\varepsilon}$. This implies that for any $\delta > 0$, there is a $b \in M$ such that $M \models 1 + \varepsilon - \varphi_r(b) > 1 - \delta$ and $M \models \inf_z [\![\mathsf{Inf}(z)]\!]_{\frac{1}{3}r} + 2v([\![\mathsf{Inf}(-)]\!]_{\frac{1}{3}r})\psi_r(b, z) > 1 - \delta$. The first condition implies that $\varphi_r^M(b) < \varepsilon + \delta$. For $\delta < \frac{1}{3}r - \varepsilon$, this implies that b is in X_r and so $f_r(b)$ is $(r - \varepsilon - \delta)$-discrete. Since $b \in X_r$, the second condition is equivalent to $M \models [\![\mathsf{Inf}(f_r(b))]\!]_{\frac{1}{3}r} \geq 1$, which is equivalent to $M \models \mathsf{Inf}(f_r(b))$ and we can take $f_r(b)$ to be the required a. Since we can do this for any $\delta > 0$ (and since an s-discrete set is t-discrete for any $t < s$), we have the required statement.

For the $\Leftarrow$ direction, fix $s \in (\frac{2}{3}r, r - \varepsilon)$ and let a be an s-discrete set such that $M \models \mathsf{Inf}(a)$. Now we clearly have that $\varphi_r(a) \leq r - s < \frac{1}{3}r$. Therefore $f_r(a)$ is defined (and equal to $\{\{x\} : x \sqsubset a\}$). Since $M \models \mathsf{Inf}(a)$, $M \models \mathsf{Inf}(f_r(a))$ as well. (There is an obviously definable bijection between a and $f_r(a)$. This exists as an element of $\mathcal{P}(a \times f_r(a))$.) Since we can do this for any sufficiently large $s < r - \varepsilon$, we're done. ⊣

By the discussion in Section 4, any of the axioms $\mathsf{Inf}_{r,\varepsilon}$ is a sufficient form of the axiom of infinity for the purposes of developing standard mathematics. Nevertheless, we propose the scheme Inf as a canonical choice for 'the axiom of infinity' in the context of MSE. One objection to this proposal might be that it is a scheme, rather than a single axiom, but as discussed in [10, Section 6.1], the concept of finite axiomatizability is murky in continuous logic.

For most of the models we construct in Section 7, there is a 1-discrete set a for which $\mathsf{Inf}(a)$ holds. This is obviously a more comfortable condition than $M \models \mathsf{Inf}$, but it is unclear whether it is actually axiomatizable. We could achieve it by adding a constant for some such a, but this is unsatisfying. Thus we have the following question.

QUESTION 5.3. *Is the class $\{M \models \mathsf{MSE} : (\exists a \in M)a$ is 1-discrete, $M \models \mathsf{Inf}(a)\}$ elementary in the sense of continuous logic*?

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

General pessimism leads us to believe that the answer to this is no, but we do not see an approach to resolving this question. One candidate might be the existence of the von Neumann ordinal ω, which is necessarily 1-discrete if it exists in an ω-standard model of MSE, but it is not clear that any model with a 1-discrete infinite set has ω as an element. It is also not clear whether it's possible to axiomatize the existence of ω in a satisfactory way.

5.2. Ordinals. Rather than develop the global structure of cardinals in models of MSE, we will focus on ordinals. We do this for a couple of reasons. Many of the technical details for cardinals and ordinals are similar but not quite similar enough to develop simultaneously in an expeditious way. Furthermore, more can be said about the structure of ordinals than of cardinals without assuming some form of the axiom of choice.

DEFINITION 5.4. Fix $M \models \mathsf{MSE}$. A *chain* in M is a set a such that for any $b, c \in a$, either $b \sqsubseteq c$ or $c \sqsubseteq b$.

Two uniformly discrete chains a and b are *order-isomorphic* if there is a bijection $f \in \mathcal{P}(a \times b)$ such that for any $g, h \sqsubseteq a$, it holds that $g \sqsubseteq h$ if and only if $f(g) \sqsubseteq f(h)$. We write $a \cong b$ to signify that a and b are order-isomorphic. The *order type of* a is the $\cong$-class of a, written $\mathrm{otp}(a)$.

A uniformly discrete chain a is *well-ordered* if for any non-empty $b \sqsubseteq a$, there is a $\sqsubseteq$-least element of b.

The $\cong$-classes of uniformly discrete well-ordered chains in M are referred to as the *ordinals* of M, and the collection of such is written Ord^M.

Note that we will typically use the term well-ordered to mean internally well-ordered. We will use the word 'externally' if we wish to emphasize that something is externally well-ordered.

Given a more general sort of linear order, namely a pair (a, b) with a uniformly discrete and $b \sqsubseteq a \times a$ the graph of a linear order, we can find a uniformly discrete chain c such that (a, b) and $(c, \sqsubseteq \restriction c \times c)$ are internally order-isomorphic. We just need to map each element f of a to the set $\{x \sqsubset a : \langle x, f \rangle \sqsubset c\}$ (i.e., the c-initial segment with largest element f). In this way we can see that uniformly discrete chains are sufficient to represent all uniformly discrete linear order types in models of MSE.

We denote order types of uniformly discrete well-ordered chains with lowercase Greek letters near the beginning of the alphabet, such as α and β. We write $\alpha \leq \beta$ to mean that for any a with $\mathrm{otp}(a) = \alpha$ and any b with $\mathrm{otp}(b) = \beta$, a is order-isomorphic to some initial segment of b. We write $\alpha < \beta$ to mean that $\alpha \leq \beta$ and $\alpha \neq \beta$. By a completely standard argument, we have that for any ordinals $\alpha, \beta \in \mathrm{Ord}^M$, either $\alpha < \beta$, $\beta < \alpha$, or $\alpha = \beta$.

It's important to emphasize that the objects in Ord^M are not internal sets in M, but rather externally defined equivalence classes of the equivalence relation $\cong$. To what extent can we get around this and approximate the class of well-ordered uniformly discrete chains with a set? As is typically the case in set theories with a universal set, something fishy needs to happen with regards to the class of ordinals, on pain of the Burali–Forti paradox. In particular, it is immediate that there cannot be a uniformly discrete set containing representatives of all ordinals of M.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

Using techniques similar to those in Section 5.1, we are able to collect representatives of all well-order types occurring below a certain scale. Just as there, we can't easily form sets that consist solely of r-discrete chains, only things that are in some sense 'approximate chains.' We can use a similar trick, however, to turn these into order-isomorphic chains.

DEFINITION 5.5. Let $\sigma(x, y) = \sup_{z \sqsubseteq x} e(z, y)$. Let $\mathrm{chn}(x) = \sup_{y,z \sqsubseteq x} \min(\sigma(y, z), \sigma(z, y))$.

Note that $\sigma(a, b) = 0$ if and only if $a \sqsubseteq b$. Note also that $d(a, b) = \max(\sigma(a, b), \sigma(b, a))$. Furthermore, $\mathrm{chn}(a) = 0$ if and only if a is a chain. Let φ_r, E_r, X_r, and f_r be defined as they were in Section 5.1.

LEMMA 5.6 (Trichotomy for approximate chains). *For any $r > 0$ and $a \in M \models \mathsf{MSE}$, if $\varphi_r(a) < \frac{1}{3}r$ and $\mathrm{chn}(a) < \frac{1}{3}r$, then for any $b, c \sqsubset a$, precisely one of the following holds: $d(a, b) < \frac{1}{3}r$, $\sigma(a, b) > \frac{2}{3}r$, or $\sigma(b, a) > \frac{2}{3}$.*

PROOF. Since $\varphi_r(a) < \frac{1}{3}r$, we have that for any $b, c \sqsubset a$, either $d(b, c) < \frac{1}{3}r$ or $d(b, c) > \frac{2}{3}r$. Since $\mathrm{chn}(a) < \frac{1}{3}r$, we have that for any $b, c \sqsubset a$, either $\sigma(b, c) < \frac{1}{3}r$ or $\sigma(c, b) < \frac{1}{3}r$. If $d(b, c) \not< \frac{1}{3}r$, then we must have either $\sigma(b, c) \geq \frac{1}{3}r$ or $\sigma(c, b) \geq \frac{1}{3}r$, whence either $\sigma(b, c) > \frac{2}{3}r$ and $\sigma(c, b) < \frac{1}{3}r$ or $\sigma(c, b) > \frac{2}{3}r$ and $\sigma(b, c) < \frac{1}{3}r$. ⊣

Define the formula

$$\sigma^\star(x, y) = \sup_{z \sqsubset x} \sup_{w \sqsubset y} \sigma(z, w).$$

While $\sigma(x, y)$ measures how close x is to being a subset of y, $\sigma^\star(x, y)$ measures how close every element of x is to being a subset of every element of y. Lemma 5.6 implies that if $\varphi_r(a) < \frac{1}{3}r$ and $\mathrm{chn}(a) < \frac{1}{3}$, then for any E_r-equivalence classes b and c of a, either $\sigma^\star(b, c) \leq \frac{1}{3}r$ or $\sigma^\star(b, c) \leq \frac{1}{3}r$. Furthermore, if both of these hold, then $b = c$.

Let C_r be the class $\{x \in X_r : \mathrm{chn}(x) < \frac{1}{3}r\}$. By the above observations, we have that the function $g_r : C_r \to M$ defined by

$$g_r(a) = \left\{\{x \sqsubset f_r(a) : \sigma^\star(x, y) \leq \tfrac{1}{3}r\} : y \in f_r(a)\right\}$$

is explicitly definable (without parameters). (Note that we do not need to take closures as $f_r(a)$ is $\frac{1}{3}r$-discrete.) Furthermore, it is immediate that $g_r(a)$ is a $\frac{1}{3}r$-discrete chain for any $a \in C_r$ and if $a \in C_r$ is a chain, then $g_r(a) = \{\{\{x\} : x \sqsubset a,\ x \sqsubseteq y\} : y \in a\}$ and moreover a and $g_r(a)$ are order-isomorphic as chains.

With this machinery, we are finally in a position to examine the global structure of ordinals in models of MSE.

DEFINITION 5.7. For any ordinal α of $M \models \mathsf{MSE}$, we write $s(\alpha)$ for the quantity

$$\sup\{r > 0 : (\exists \text{ well-ordered } r\text{-discrete chain } x \in M)\, \mathrm{otp}(x) = \alpha\}.$$

It is easy to see that if $\alpha \leq \beta$, then $s(\alpha) \geq s(\beta)$, so for any $M \models \mathsf{MSE}$, s is a non-increasing map from Ord^M to $(0, 1]$. Furthermore, it is always the case that $s(0) = 1$. By using Hartogs numbers, it is easy to show that for any ordinal $\alpha \in \mathrm{Ord}^M$, there is an ordinal $\beta \in \mathrm{Ord}^M$ of strictly larger cardinality, namely the Hartogs number of

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

$\mathcal{P}(a)$, where $\mathrm{otp}(a) = \alpha$. By an abuse of notation, we'll write this as $\aleph(\mathcal{P}(\alpha))$. Note that by Lemma 4.3 and the fact that the Hartogs number of X always embeds into $\mathcal{P}^3(X)$, we have that $s(\alpha) = s(\aleph(\mathcal{P}(\alpha)))$. This means that if $s(\beta) < s(\alpha)$, then β has much larger cardinality than α.

We'll write ω^M for the first limit ordinal in M, if it exists. The value of $s(\omega^M)$ is directly related to the axiom of infinity.

PROPOSITION 5.8. *Fix $M \models \mathsf{MSE}$. For any $r \in (0,1]$ and $\varepsilon \in (0, \frac{1}{3}r)$, $M \models \mathsf{Inf}_{r,\varepsilon}$ if and only if ω^M exists and $r \leq s(\omega^M)$. In particular, $M \models \mathsf{Inf}$ if and only if ω^M exists and $s(\omega^M) = 1$.*

PROOF. Let $\mathbb{N}^i_a$ be the set of initial segments of $\mathbb{N}_a$ for some uniformly discrete a. It is immediate that $\mathbb{N}^i_a$ is a well-ordered chain. If a is r-discrete, then we have by Lemma 4.3 that $\mathbb{N}^i_a$ is r-discrete as well. It is easy to show that $\mathrm{otp}(\mathbb{N}^i_a) = \omega^M$ if and only if $M \models \mathsf{Inf}(a)$. Conversely, if $\mathrm{otp}(b) = \omega^M$ for some well-ordered uniformly discrete chain b, then $M \models \mathsf{Inf}(b)$. The result now follows from Proposition 5.2. ⊣

LEMMA 5.9. *Let a and b be r-discrete chains in some $M \models \mathsf{MSE}$. If $d(a,b) < \frac{1}{2}r$, then $a \cong b$. Furthermore, if (M,d) is an ultrametric space, it is enough to assume that $d(a,b) < r$.*

PROOF. Fix s such that $d(a,b) < s < \frac{1}{2}r$. Since a and b are r-discrete, we have that the class $f = \{\langle x,y\rangle \sqsubset a \times b : d(x,y) \leq s\}$ is a set and is the graph of a bijection between a and b. Now we need to show that f is actually an order isomorphism between a and b. Suppose that we have $c, c' \sqsubset a$ with $c \sqsubseteq c'$ and $g, g' \sqsubset b$ with $d(c,g) < s$ and $d(c',g') < s$. If $c = c'$, then $g = g'$, so assume that $c \sqsubset c'$. Since $d(c,c') \geq r$, we can find an $h \sqsubset c'$ such that $e(h,c) > 2s$ (as $2s < r$). Since $d(c',g') < s$, we can find an $i \sqsubset g'$ such that $d(h,i) < s$. The triangle inequality implies that $e(i,g) \geq e(h,c) - d(h,i) - d(c,g) > 2s - s - s = 0$. Therefore $i \not\sqsubset g$ and it must be the case that $g \sqsubset g'$, as required.

The proof in the ultrametric case is essentially the same. ⊣

THEOREM 5.10. *Fix $M \models \mathsf{MSE}$ and $r \in \{s(\gamma) : \gamma \in \mathrm{Ord}^M\}$. Let $t = r$ if d is an ultrametric and let $t = \frac{1}{2}r$ otherwise.*

For any $s < t$, there is an ordinal $\alpha \in \mathrm{Ord}^M$ with $s \leq s(\alpha) < r$ such that for any $\beta \in \mathrm{Ord}^M$, if $s(\beta) \geq r$, then $\beta < \alpha$.

In particular, if d is an ultrametric, then $\{s(\alpha) : \alpha \in \mathrm{Ord}^M\}$ is dense in $(0,1]$ and $\{\alpha \in \mathrm{Ord}^M : s(\alpha) < 1\}$ has no least element.

PROOF. Fix positive $s < t$ and let $\delta = 1 - \frac{s}{t}$ (implying that $s = t(1-\delta)$). Assume without loss of generality that $t\delta < \frac{1}{3}r$. Let

$$a = \left[x : \max(\phi_r(x), \mathrm{chn}_r(x)) \leq 0 \sim t\delta\right].$$

Note that every element of a is an element of C_r. Let $b = \overline{\{g_r(x) : x \sqsubset a\}}$. Note that since the collection of chains in M is $\{x \in M : \mathrm{chn}^M(x) = 0\}$, it is metrically closed. Hence every element of b is a chain. It is also easy to see that every element of b is $r(1-\delta)$-discrete (regardless of whether d is an ultrametric).

Let $c = \{x \sqsubset b : x \text{ is well-ordered}\}$. Note that c is a set in M. Also note that for any $\beta \in \mathrm{Ord}^M$, if $s(\beta) \geq r$, then some element of c has order type β. The equivalence

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

relation $\cong$ is discretely definable on c, so we can form the set $f = \{\{y \sqsubset c : x \cong y\} : x \sqsubset c\}$. By Lemma 5.9, the set f is $t(1-\delta)$-discrete (regardless of whether d is an ultrametric) or, in other words, s-discrete. We can find a formula $\varphi(x, y)$ with the property that for any $x, y \sqsubset f$, $\varphi(x, y) \in \{0, 1\}$ and $\varphi(x, y) = 0$ if and only $(\forall z \sqsubset x)(\forall w \sqsubset y)\, \mathrm{otp}(z) \leq \mathrm{otp}(x)$. This formula defines a linear order on f which by a standard argument is a well-order. Let

$$\alpha = \mathrm{otp}(\{\{y : y \sqsubset f,\ \varphi(x, y) = 0\} : x \sqsubset f\}).$$

Since f contains representatives of all ordinals β with $s(\beta) \geq r$, we must have that α is larger than any such β. Therefore it cannot be the case that $s(\alpha) \geq r$. On the other hand, since f is s-discrete, it follows that $s(\alpha) \geq s$, as required.

The last statements in the theorem obviously follow from the rest of it. ⊣

COROLLARY 5.11. *If $M \models$ MSE has no infinite, uniformly discrete sets, then M has non-standard naturals.*

PROOF. By Corollary 3.8 and induction, any model of MSE contains hereditarily finite sets of every externally finite cardinality. Therefore for any standard natural n, $n \in \mathrm{Ord}^M$ and $s(n) = 1$. Theorem 5.10 implies that there are ordinals α in M (which must be internally finite) such that $s(\alpha) < 1$. ⊣

The behavior of ultrametric models of MSE in Theorem 5.10 is reminiscent of the behavior of ω in models of Cantor–Łukasiewicz set theory, as discovered by Hájek [17, Theorem 4.17]. In particular, they both exhibit a manifestation of the sorites paradox: an inability to formalize an induction principle of the form

$$\frac{\varphi(0) = 0 \qquad \forall\alpha[(\forall\beta < \alpha)(\varphi(\beta) = 0) \to \varphi(\alpha) = 0]}{\forall\alpha(\varphi(\alpha) = 0)}$$

for a real-valued predicate $\varphi(x)$ on some class of ordinals. For $\mathrm{C\L}_0$, this induction principle cannot hold even for ω, but, as we will see in Theorem 7.15, models of MSE can have arbitrarily large standard ordinals. Theorem 5.10 is also of course similar to the non-existence of β-models of NFU, although the mechanism by which models of NFU are ill-founded is different. One might idly wonder what could happen if we were to restrict excision to stratified formulas.

What is unclear at the moment is the status of non-ultrametric models of MSE. Theorem 5.10 does not preclude the possibility of β-models of MSE (i.e., models in which Ord^M is externally well-founded), but it seems unlikely that they exist. Every model of MSE we know how to produce contains a set that is an ultrametric model of MSE, whereby Theorem 5.10 applies. This leaves the following question.

QUESTION 5.12. *Does* MSE *have any β-models? Is it true that for any $M \models$ MSE, $\{s(\alpha) : \alpha \in \mathrm{Ord}^M\}$ is dense in $(0, 1]$?*

Given the behavior of the models constructed in Section 7, we conjecture that MSE has no β-models and $\{s(\alpha) : \alpha \in \mathrm{Ord}^M\}$ is always dense in $(0, 1]$.

Finally, although this is more or less a cosmetic nicety, we would like to show that we can build canonical 'tokens' representing well-order types, i.e., elements of M that

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

somehow canonically represent a well-order type α. In NFU, this is accomplished by taking $\{x : \mathrm{otp}(x) = \alpha\}$. In ZF and GPK^+_∞, this is accomplished by taking the von Neumann ordinal of that order type. Neither of these approaches will work for MSE, so we will have to do something new.

DEFINITION 5.13. Given a uniformly discrete set a, a set $b \sqsubseteq a \times V^M$, and an element $c \sqsubset a$, we write $b[c]$ for the class $\{f : \langle c, f\rangle \sqsubset b\}$. For any chain $a \in M \models \mathsf{MSE}$ and any set $b \sqsubseteq a \times V^M$, the *closed chain union of* b is

$$\chi(b) \coloneqq \overline{\left\{\overline{\bigsqcup\{b[f] : f \sqsubset a,\ f \sqsubseteq c\}} : c \sqsubset a\right\}}.$$

In other words, $\chi(b)$ is the (not necessarily uniformly discrete) chain constructed out of the image of b as a many-valued function on a.

Given a uniformly discrete chain $a \in M \models \mathsf{MSE}$, the *order token of* a is the class

$$\mathrm{otok}(a) \coloneqq \overline{\{\chi(b) : b \sqsubseteq a \times V^M\}}.$$

So $\mathrm{otok}(a)$ is roughly speaking the class (which we will shortly show is a set) of all chains that can be realized as a surjective order-preserving image of a. Thinking of $\mathrm{otok}(a)$ as a canonical token representing the ordinal of a is analogous to thinking of $\{\{a, b\} : a, b \in V\}$ (i.e., the set of sets of size 1 or 2) as a canonical token representing the cardinal 2, as discussed at the beginning of Section 5.

THEOREM 5.14. *Fix* $M \models \mathsf{MSE}$.

(1) $\mathrm{otok}(a)$ *is a set for any uniformly discrete chain* a.
(2) *For any* $r > 0$, *the map* $x \mapsto \mathrm{otok}(x)$ *is explicitly definable on the class of* r*-discrete chains.*
(3) *If* a *and* b *are well-ordered uniformly discrete chains, then* $\mathrm{otp}(a) \leq \mathrm{otp}(b)$ *if and only if* $\mathrm{otok}(a) \sqsubseteq \mathrm{otok}(b)$.

PROOF. 1 and 2 follow from Proposition 3.13 and Lemma 3.18.

For 3, assume that $\mathrm{otp}(a) \leq \mathrm{otp}(b)$. Let this be witnessed by an order isomorphism $f : a \to b$ to some initial segment of b. For any $c \sqsubseteq a \times V^M$, we can form the set $c = \{\langle f(x), y\rangle : \langle x, y\rangle \sqsubset c\}$ (because the map $\langle x, y\rangle \mapsto \langle f(x), y\rangle$ is definable) and we immediately have that $\chi(c) = \chi(c_f)$. Therefore $\mathrm{otok}(a) \sqsubseteq \mathrm{otok}(b)$.

Conversely, assume that $\mathrm{otok}(a) \sqsubseteq \mathrm{otok}(b)$. Let r be such that a and b are r-discrete. Find $c \sqsubseteq b \times V^M$ such that $d(a, \chi(c)) < \frac{1}{2}r$. By Lemma 5.9, we have that a and $\chi(c)$ are order-isomorphic as chains. Let this be witnessed by $f : a \to \chi(c)$. For any $x \sqsubset a$, let $g(x)$ be the smallest element of b such that $f(x) \sqsubseteq \overline{\bigcup\{c[z] : z \sqsubset b,\ z \sqsubseteq g(x)\}}$. (This is a set by Proposition 3.2 and the fact that a and b are r-discrete.) g is an injective order-preserving map from a to b, so by a standard argument, we have that $\mathrm{otp}(a) \leq \mathrm{otp}(b)$. ⊣

Now of course, given $\mathrm{otok}(a)$, we can build a canonical well-ordered chain with the same order type as a, namely

$$\mathrm{ord}(a) \coloneqq \{\mathrm{otok}(z) : z \text{ is a well-ordered uniformly discrete chain},\ \mathrm{otp}(z) < \mathrm{otp}(a)\}.$$

One can show that $x \mapsto \mathrm{ord}(x)$ is explicitly definable on the class of r-discrete well-ordered chains for any $r > 0$.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

Naturally, as discussed before, we could attempt to do something similar to Definition 5.13 with cardinalities, but without some form of the axiom of choice, we only seem to be able to build tokens representing equivalence classes of the $\approx^*$ relation (where $x \leq^* y$ if there is a surjection from some subset of y onto x and $x \approx^* y$ if $x \leq^* y$ and $y \leq^* x$). Specifically, if we define $\mathrm{ctok}^*(a) \coloneqq \overline{\{\overline{\{\pi_1(x) : x \sqsubseteq y\}} : y \sqsubseteq a \times V^M\}}$, we then have that $a \approx^* b$ if and only if $\mathrm{ctok}^*(a) = \mathrm{ctok}^*(b)$ for any uniformly discrete a and b. This raises an obvious question.

QUESTION 5.15. *Is there a function* $\mathrm{ctok}(x)$ *that is definable on the class of r-discrete sets for each* $r > 0$ *such that for any uniformly discrete a and b,* $a \approx b$ *if and only if* $\mathrm{ctok}(a) = \mathrm{ctok}(b)$?

§6. Formalizing MSE in continuous logic. Given a metric set structure $(M, d, \sqsubseteq)$, we can build a corresponding *general structure*[12] (M, e) by taking the function $e : M^2 \to [0, 1]$ to be the sole predicate. After doing so, the original structure can be recovered by taking $d(x, y) = \sup_z |e(z, x) - e(z, y)|$ and $\sqsubseteq = \{(x, y) \in M^2 : e(x, y) = 0\}$. Our goal in this section is to characterize the structures that arise in this way and show that they form an elementary class in the sense of continuous logic.

Let $\mathcal{L}_e$ be the language with a single $[0, 1]$-valued predicate symbol e. Given any $\mathcal{L}_e$-structure (M, e), we can define a pseudo-metric

$$d_e(x, y) \coloneqq \sup_z |e(z, x) - e(z, y)|.$$

Since this is a formula in the sense of continuous logic, d_e is a definable predicate on any $\mathcal{L}_e$-structure. Note that by construction, for any $\mathcal{L}_e$-structure (M, e) and $a \in M$, the function $y \mapsto e(a, y)$ is 1-Lipschitz with regards to d_e.

The first thing we need to do is write out an axiom that guarantees that $e(x, b)$ is a distance predicate with regards to d_e for any choice of b. This is done for non-empty definable sets in [25, Theorem 9.12], but we need to allow for the possibility of empty definable sets here (where we take the distance predicate of $\varnothing$ to be the constant function 1). This is possible to add in by hand, but we will give a single axiom that covers both cases in Appendix 1. Specifically, we show that a formula $\varphi(x)$ is a distance predicate (of a possibly empty set) if and only if $\sup_x |\varphi(x) - \inf_y \min(d(x, y) + 2\varphi(y), 1)| = 0$.

DEFINITION 6.1. The H-*extensionality axiom* is the $\mathcal{L}_e$-condition

$$\sup_{xy} |e(x, y) - \inf_z \min(d_e(x, z) + 2e(z, y), 1)| = 0.$$

We say that an $\mathcal{L}_e$-structure M is H-*extensional* if it satisfies the H-extensionality axiom.

Note that the H-extensionality axiom could more conventionally be written

$$\forall x \forall y (e(x, y) = \inf_z \min(d_e(x, z) + 2e(z, y), 1)).$$

[12] As defined in [14]. Such structure could also be described as 'metric structures without a metric.'

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

Given an $\mathcal{L}_e$-structure (M, e), we write M/e for the $d_e = 0$ quotient of M and we write $\overline{M/e}$ for the completion of this under d_e. Given $a \in M$, we write $[a]_e$ for the corresponding element of M/e, which we regard as a subset of $\overline{M/e}$.

LEMMA 6.2. *An $\mathcal{L}_e$-structure (M, e) satisfies the* H*-extensionality axiom if and only if for any $b \in M$, the function $x \mapsto e(x, b)$ is 1-Lipschitz with regards to d_e and if $f(x)$ is the extension of $e(x, b)$ to $\overline{M/e}$, then for any $a \in \overline{M/e}$, $f(a) = \inf\{d_e(a, c) : c \in \overline{M/e},\ f(c) = 0\}$, where* $\inf \varnothing = 1$.

PROOF. For the $\Rightarrow$ direction, suppose that M satisfies the H-extensionality axiom and fix $b \in M$. We have by the H-extensionality axiom that $e(x, b) = \inf_z \min(d_e(x, z) + 2e(z, b), 1)$ for all x. The function $x \mapsto \min(d_e(x, z) + 2e(z, b), 1)$ is 1-Lipschitz with regards to d_e for any z, therefore $e(x, b)$ is as well (since it is the infimum of a family of 1-Lipschitz functions). Let $f(x)$ be the extension of $e(x, b)$ to $\overline{M/e}$. It is immediate that $\overline{M/e}$ satisfies

$$\sup_x |f(x) - \inf_z \min(d_e(x, z) + 2f(z), 1)| = 0.$$

By Proposition A.1, we have that $f(x)$ is the distance predicate of its zero set, which is precisely the last condition in the statement of the lemma.

For the $\Leftarrow$ direction, suppose that $e(x, b)$ is 1-Lipschitz with regards to d_e for any $b \in M$ and that for any $a \in \overline{M/e}$, $f(a) = \inf\{d_e(a, x) : f(x) = 0\}$, where $f(x)$ is the unique continuous extension of $e(x, b)$ to $\overline{M/e}$. Fix $a \in M$ and let $r = e(a, b) = f([a]_e)$. Fix $\varepsilon > 0$. Find $c \in \overline{M/e}$ such that $f(c) = 0$ and $d([a]_e, c) < r + \frac{1}{4}\varepsilon$. Find $c' \in M$ such that $d(c, [c']_e) < \frac{1}{4}\varepsilon$. Since $f(x)$ is 1-Lipschitz, we have that $f([c']_e) = e(c', b) < \frac{1}{4}\varepsilon$. Note also that $d_e(a, c') < r + \frac{2}{4}\varepsilon$. We now have that

$$\begin{aligned}\inf_z \min(d_e(a, z) + 2e(z, b), 1) &\leq d_e(a, c') + 2e(c', b)\\ &\leq r + \frac{2}{4}\varepsilon + 2e(c', b)\\ &\leq r + \frac{2}{4}\varepsilon + 2\frac{1}{4}\varepsilon\\ &\leq r + \varepsilon.\end{aligned}$$

Since we can do this for any $\varepsilon > 0$, we have that

$$\inf_z \min(d_e(a, z) + 2e(z, b), 1) \leq \inf\{d_e(a, x) : f(x) = 0\} = e(a, b).$$

For the other direction of the inequality, let $s = \inf_z \min(d_e(a, z) + 2e(z, b), 1)$. If $s = 1$, then the above implies that $f(x) = 1$ for all $x \in \overline{M/e}$, so the H-extensionality axiom is satisfied. Otherwise assume that $s < 1$ and fix $\varepsilon > 0$ with $s + \varepsilon < 1$. Find $c \in M$ such that $\min(d_e(a, c) + 2e(c, b), 1) < s + \varepsilon$. We must have that $d_e(a, c) + 2e(c, b) < s + \varepsilon$. By assumption, $e(c, b) = f([c]_e) = \inf\{d_e([c]_e, x) : f(x) = 0\}$. Therefore

$$\begin{aligned}e(a, b) = \inf\{d_e([a]_e, x) : f(x) = 0\} &\leq d_e(a, c) + e(c, b)\\ &< d_e(a, c) + 2e(c, b)\\ &< s + \varepsilon.\end{aligned}$$

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

Since we can do this for any $\varepsilon > 0$, we have that $e(a,b) \leq s = \inf_z \min(d_e(a,z) + 2e(z,b), 1)$. Therefore $e(a,b) = \inf_z \min(d_e(a,z) + 2e(z,b)), 1$ for any $a, b \in M$ and the H-extensionality axiom holds. ⊣

Note that since $y \mapsto e(a,y)$ is automatically 1-Lipschitz with regards to d_e, the H-extensionality axiom implies that $(x,y) \mapsto e(x,y)$ is 2-Lipschitz with regards to d_e. This means that $e(x,y)$ extends to a unique continuous function on $\overline{M/e}$. By an abuse of notation we will also denote this as e. Note that in this case, $(\overline{M/e}, e)$ still satisfies the H-extensionality axiom (and is in fact elementarily equivalent to (M,e) as an $\mathcal{L}_e$-structure). In particular, by Lemma 6.2 applied to the structure $(\overline{M/e}, e)$, we have that $x \mapsto e(x,b)$ is a distance predicate for any $b \in \overline{M/e}$.

Given an $\mathcal{L}_e$-structure M for which $e(x,y)$ extends to $\overline{M/e}$, write $\sqsubseteq_e$ for the relation $\{(x,y) \in (\overline{M/e})^2 : e(x,y) = 0\}$. Now we will see the manner in which the H-extensionality axiom characterizes metric set structures.

PROPOSITION 6.3. *Fix an $\mathcal{L}_e$-structure (M,e). (M,e) satisfies the* H*-extensionality axiom if and only if $(\overline{M/e}, d_e, \sqsubseteq_e)$ exists and is a metric set structure.*

PROOF. This follows from Lemma 6.2 and the fact that $(d_e)_{\mathrm{H}}(\{x : x \sqsubseteq_e a\}, \{x : x \sqsubseteq_e b\}) = \sup_z |e(z,a) - e(z,b)| = d_e(a,b)$ for all $a, b \in \overline{M/e}$. ⊣

Axiomatizing excision will be more technical. For convenience, we'll take restricted $\mathcal{L}_e$-formulas to be defined as in [11, Section 1.3]: the only atomic formulas are those of the form $e(x,y)$ and we take as connectives $\varphi + \psi$, $\max(\varphi, \psi)$, $\min(\varphi, \psi)$, the constant 1, and $r \cdot \varphi$ for rational r. We should note though that the scheme described here would be sufficient with any definition of restricted formula, such as the one in [25, Section 3].

Given a restricted $\mathcal{L}_e$-formula φ, we can form a corresponding $\mathcal{L}_{\sqsubseteq}$-formula by replacing each instance of $e(x,y)$ with the $\mathcal{L}_{\sqsubseteq}$-formula $\inf_{z \sqsubseteq y} d(x,z)$, where z is taken to be any variable distinct from x and y. We write $\varphi_{\sqsubseteq}$ for the formula resulting from this translation. By an abuse of notation, we will write $v(\varphi)$ for $v(\varphi_{\sqsubseteq})$.

Later on, we will also need a way to translate $\mathcal{L}_{\sqsubseteq}$-formulas back to restricted $\mathcal{L}_e$-formulas. The difficulty here is that we allowed real coefficients in $\mathcal{L}_{\sqsubseteq}$-formulas but only rational coefficients in $\mathcal{L}_e$-formulas. With this issue in mind say that an $\mathcal{L}_{\sqsubseteq}$-formula is *rational* if all coefficients occurring in it are rational numbers. We define the $\mathcal{L}_e$-formula φ_e corresponding to a rational $\mathcal{L}_{\sqsubseteq}$-formula φ inductively as follows:

- $(d(x,y))_e = d_e(x,y)$,
- $(\inf_{x \sqsubseteq y} \varphi)_e = \inf_x \min(\varphi_e + 2v(\varphi)e(x,y), v(\varphi))$, and
- $(\sup_{x \sqsubseteq y} \varphi)_e = -(\inf_{x \sqsubseteq y} -\varphi)_e$,

with the other elements of the translation defined in the obvious way. The following facts are either standard results in continuous logic or easily verified.

FACT 6.4. *Fix an* H*-extensional $\mathcal{L}_e$-structure (M,e) with (M,d_e) complete.*

(1) *For any $\mathcal{L}_e$-formula $\varphi(\bar{x})$ and any $\bar{a} \in M$, $\varphi^{(M,e)}(\bar{a}) = \varphi_{\sqsubseteq}^{(M,d_e,\sqsubseteq_e)}(\bar{a})$.*
(2) *For any rational $\mathcal{L}_{\sqsubseteq}$-formula $\varphi(\bar{x})$ and any $\bar{a} \in M$, $\varphi^{(M,d_e,\sqsubseteq_e)}(\bar{a}) = \varphi_e^{(M,e)}(\bar{a})$.*
(3) *For any $\mathcal{L}_{\sqsubseteq}$-formula $\varphi(\bar{x})$ and $\varepsilon > 0$, there is a rational $\mathcal{L}_{\sqsubseteq}$-formula $\psi(\bar{x})$ such that $|\varphi^{(M,d_e,\sqsubseteq_e)}(\bar{a}) - \psi^{(M,d_e,\sqsubseteq_e)}(\bar{a})| < \varepsilon$ for all $\bar{a} \in M$.*

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

It follows from Lemma 2.6 and Fact 6.4 that if (M, e) is H-extensional with (M, d_e) complete, then for any restricted $\mathcal{L}_e$-formula $\varphi(\bar{x})$, the function $\bar{x} \mapsto \varphi^M(\bar{x})$ is $2v(\varphi)$-Lipschitz with regards to the max metric on tuples induced by d_e. By passing to the completion $(\overline{M/e}, d_e)$, this implies the same for any H-extensional (M, e).

For any formula $\varphi \in \mathcal{L}_e$, we define

$$\varepsilon_\varphi \coloneqq \frac{1}{\max(6v(\varphi), 3)}.$$

Note that for any H-extensional (M, e), if $|\varphi^M(\bar{a}) - \varphi^M(\bar{b})| \geq \frac{1}{2}$, then $d_e(\bar{a}, \bar{b}) > \varepsilon_\varphi$.

DEFINITION 6.5. The *axiom scheme of excision* is the collection of $\mathcal{L}_e$-conditions of the form

$$\sup_{\bar{y}} \inf_{z} \sup_{x} \max(\min(e(x, z), -\varphi(x, \bar{y})), \min(\varepsilon_\varphi - e(x, z), \varphi(x, \bar{y}) - 1)) \leq 0$$

for each restricted $\mathcal{L}_e$-formula $\varphi(x, \bar{y})$ (not containing z as a free variable).

Given an H-extensional $\mathcal{L}_e$-structure M, we say that M *satisfies* $\mathcal{L}_e$-excision to mean that M satisfies the axiom scheme of excision.

The axiom scheme of excision can be more conventionally stated like this: For all $\bar{y}, \delta > 0$, and $\varphi(x, \bar{y}) \in \mathcal{L}_e$, there is a z such that for all x,

- if $\varphi(x, \bar{y}) \leq -\delta$, then $e(x, z) < \delta$ and
- if $e(x, z) \leq \varepsilon_\varphi - \delta$, then $\varphi(x, \bar{y}) < 1 + \delta$.

It is also sufficient to assume merely that this holds for sufficiently small $\delta > 0$. This is clearly an approximation of a certain case of the excision principle in MSE, but we will now show that in H-extensional M with (M, d_e) complete, the axiom scheme of excision is enough to imply full excision.

LEMMA 6.6. *Fix an* H*-extensional* $\mathcal{L}_e$*-structure* M *with* (M, d_e) *complete. Suppose that* M *satisfies* $\mathcal{L}_e$*-excision. For any* $a \in M$ *and* $r, s \in [0, 1]$ *with* $r < s$*, there is a* $b \in M$ *such that for any* $c \in M$*, if* $e(c, a) \leq r$*, then* $c \sqsubseteq_e b$*, and if* $c \sqsubseteq_e b$*, then* $e(c, a) < s$*.*

PROOF. For readability, we will write d for d_e and $\sqsubseteq$ for $\sqsubseteq_e$.

Let $r_0 = \frac{2}{3}r + \frac{1}{3}s$, $s_0 = \frac{1}{3}r + \frac{2}{3}s$, and $b_0 = a$. For any n, let $\varphi_n(x, y) = \frac{e(x,y) - r_n}{s_n - r_n}$.

At stage n, suppose we are given b_n and rationals r_n and s_n with $0 < r_n < s_n$. Since M satisfies $\mathcal{L}_e$-excision, we have that for any $\gamma > 0$, there is an $f \in M$ such that

$$\forall x(e(x, f) < \gamma \vee \varphi_n(x, b_n) > -\gamma) \wedge (\varepsilon_{\varphi_n} - e(x, b_n) < \gamma \vee \varphi_n(x, b_n) - 1 < \gamma).$$

Let b_{n+1} be such an f with

$$\gamma = \delta_n \coloneqq \min\left(\frac{2^{-n-2}}{27}(s - r), \frac{1}{2}r_n, \varepsilon_{\varphi_n}\right).$$

We have that for any $c \in M$, if $\frac{e(c,b_n) - r_n}{s_n - r_n} \leq -\delta_n$ (i.e., if $e(c, b_n) \leq r_n - \delta_n(s_n - r_n)$), then $e(c, b_{n+1}) < \delta_n$. A fortiori, this implies that if $e(c, b_n) \leq r_n - \delta_n$, then $e(c, b_{n+1}) < \delta_n$.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

On the other hand, if $\varepsilon_{\varphi_n} - e(c, b_{n+1}) \geq \delta_n$ (i.e., if $e(c, b_{n+1}) \leq \varepsilon_{\varphi_n} - \delta_n$), then $\frac{e(c,b_n)-r_n}{s_n-r_n} - 1 < \delta_n$ and so $e(c, b_n) < r_n + (1 + \delta_n)(s_n - r_n)$. Since $\delta_n \leq \varepsilon_{\varphi_n}$, this implies that if $c \sqsubseteq b_{n+1}$, then $e(c, b_n) < r_n + 2s_n$.

Finally, pick r_{n+1} and s_{n+1} so that $2\delta_n < r_{n+1} < s_{n+1} < 3\delta_n$, and move to the next stage of the construction.

Claim. $(b_n)_{n<\omega}$ is a Cauchy sequence.

Proof of claim. For any $n > 0$ and $c \in M$, we have that if $c \sqsubseteq b_n$, then $e(c, b_{n+1}) < \delta_n \leq \frac{2^{-n-2}}{27}(s - r)$ and also that if $c \sqsubseteq b_{n+1}$, then $e(c, b_n) < r_n + 2s_n < 9\delta_{n-1} \leq \frac{2^{-n-1}}{3}(s - r)$. Therefore, by H-extensionality, $d(b_n, b_{n+1}) \leq \frac{2^{-n-1}}{3}(s - r)$. Since we can do this for any positive n, the claim follows. $\square_{\text{claim}}$

Let $b = \lim_{n\to\infty} b_n$. b is an element of M since (M, d) is complete.

Claim. For any $c \in M$, if $e(c, a) \leq r$, then $c \sqsubseteq b$.

Proof of claim. Since $e(c, a) \leq r$ and since $\delta_0 \leq \frac{2^{-2}}{27}(s - r) < \frac{1}{3}(s - r)$, we have that $e(c, b_0) = e(c, a) < r_0 - \delta_0$. Therefore, $r(c, b_1) < \delta_0$. For any n, suppose that we know that $e(c, b_{n+1}) < \delta_n$. We then have that

$$\begin{aligned} e(c, b_{n+1}) &< \delta_n \\ &< \tfrac{1}{2} r_{n+1} && \text{(by our choice of } r_{n+1}) \\ &\leq r_{n+1} - \delta_{n+1} && \text{(since } \delta_{n+1} \leq \tfrac{1}{2} r_{n+1}). \end{aligned}$$

Hence, $e(c, b_{n+2}) < \delta_{n+1} \leq \frac{2^{-n-1}}{27}(s - r)$.

Therefore $e(c, b) < \frac{2^{-n-1}}{27}(s - r) + d(b_{n+2}, b)$ for every n by induction. Since $b_n \to b$ we have that $e(c, b) = 0$, i.e., $c \sqsubseteq b$. $\square_{\text{claim}}$

Finally we just need to verify that if $c \sqsubseteq b$, then $e(c, a) < s$. By the above estimate, we know that $d(b_1, b) \leq \sum_{n=1}^{\infty} \frac{2^{-n-1}}{3}(s - r) = \frac{1}{6}(s - r)$. We have

$$\begin{aligned} d(a, b_1) = d(b_0, b_1) &\leq \max(\delta_0, r_0 + (1 + \delta_0)(s_0 - r_0)) \\ &\leq \max\left(\tfrac{1}{2} r_0, r_0 + \left(1 + \tfrac{1}{108}\right) \tfrac{1}{3}(s - r)\right) \\ &< \max\left(\tfrac{1}{2} r_0, r_0 + \tfrac{5}{4} \cdot \tfrac{1}{3}(s - r)\right) = \tfrac{1}{4} r + \tfrac{3}{4} s. \end{aligned}$$

Therefore,

$$d(a, b) \leq d(a, b_1) + d(b_1, b) < \tfrac{1}{4} r + \tfrac{3}{4} s + \tfrac{1}{6}(s - r) = \tfrac{1}{12} r + \tfrac{11}{12} s < s.$$

So if $c \sqsubseteq b$, then there is an $f \sqsubseteq a$ such that $d(c, f) < s$, implying that $e(c, a) < s$, as required. $\dashv$

PROPOSITION 6.7. *Let (M, e) be an* H*-extensional $\mathcal{L}_e$-structure with (M, d_e) complete. $(M, d_e, \sqsubseteq_e) \models$ MSE if and only if (M, e) satisfies $\mathcal{L}_e$-excision.*

Furthermore, all models of MSE *arise in this manner.*

PROOF. Let $\chi(x, \bar{y}, z) = \max(\min(e(x, z), -\varphi(x, \bar{y})), \min(\varepsilon_\varphi - e(x, z), \varphi(x, \bar{y})))$. Suppose that $(M, d_e, \sqsubseteq_e) \models$ MSE. Fix a restricted $\mathcal{L}_e$-formula $\varphi(x, \bar{y})$. Fix a tuple of parameters $\bar{a}$. Let $b = \left[x : \varphi_{\sqsubseteq}(x, \bar{a}) \leq 0 \sim \frac{1}{2}\right]$. Now for any c, we have that if $\varphi(c, \bar{a}) \leq 0$, then $e(c, b) = 0$. So $\min(e(c, b), -\varphi(c, \bar{a})) \leq 0$. Moreover, if $\varphi(c, \bar{a}) \geq 1$, then $d_e(c, f) > \varepsilon_\varphi$ for all $f \sqsubseteq_e b$ (since these all satisfy $\varphi^M(f, \bar{a}) < \frac{1}{2}$). Therefore $e(c, b) \geq \varepsilon_\varphi$. So $\min(\varepsilon_\varphi - e(c, b), \varphi(c, \bar{a})) \leq 0$. Since we can do this

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

for any $c \in M$, we have that $(M, e) \models \sup_x \chi(x, \bar{a}, b) \leq 0$, whereby $(M, e) \models \inf_z \sup_x \chi(x, \bar{a}, z) \leq 0$. Since we can do this for any $\bar{a} \in M$, we have that $(M, e) \models \sup_{\bar{y}} \inf_z \sup_x \chi(x, \bar{y}, z)$. Finally since this holds for any restricted $\mathcal{L}_e$-formula, we have that (M, e) satisfies $\mathcal{L}_e$-excision.

Now assume that (M, e) satisfies $\mathcal{L}_e$-excision. Fix an $\mathcal{L}_{\sqsubseteq}$-formula $\varphi(x, \bar{y})$, $\bar{a} \in M$, and $r < s$. By passing to r' and s' with $r < r' < s' < s$ if necessary, we may assume that r and s are rational. By Fact 6.4, we can fix a rational $\mathcal{L}_{\sqsubseteq}$-formula $\psi(x, \bar{y})$ such that $\left|\psi(x, \bar{y}) - \frac{\varphi(x,\bar{y})-r}{s-r}\right| < \frac{1}{6}$ for all x and $\bar{y}$. Fix $\delta > 0$ with $\delta < \frac{1}{2}\varepsilon_{3\psi-1}$. Note that $\delta < \frac{1}{2}$. Apply $\mathcal{L}_e$-excision to the restricted $\mathcal{L}_e$-formula $3\psi_e(x, \bar{a}) - 1$ to get $b \in M$ such that for all $c \in M$, if $\psi_e(c, \bar{a}) \leq \frac{1}{3} - \frac{1}{3}\delta$, then $e(c, b) < \delta$ and if $e(c, b) < \varepsilon_{3\psi-1} - \delta$, then $\psi_e(c, \bar{a}) < \frac{2}{3} + \frac{1}{3}\delta$. Apply Lemma 6.6 to b to get a set f such that if $e(c, b) \leq \delta$, then $c \sqsubseteq f$ and if $c \sqsubseteq f$, then $e(c, b) \leq \frac{1}{2}\varepsilon_{3\psi-1} < \varepsilon_{3\psi-1} - \delta$.

For any $c \in M$, suppose that $\varphi(c, \bar{a}) \leq r$. We then have that $\psi(c, \bar{a}) = \psi_e(c, \bar{a}) < \frac{1}{6} < \frac{1}{3} - \frac{1}{3}\delta$. Therefore, $e(c, b) < \delta$ and so $c \sqsubseteq f$. On the other hand, suppose that $c \sqsubseteq f$. We then have that $e(c, b) < \varepsilon_{3\psi-1} - \delta$. Therefore $\psi_e(c, \bar{a}) < \frac{2}{3} + \frac{1}{3}\delta$, implying that $\frac{\varphi(c,\bar{a})-r}{s-r} < \frac{2}{3} + \frac{1}{3}\delta + \frac{1}{6} < 1$ and so $\varphi(c, \bar{a}) < s$.

Since we can do this for any $\varphi(x, \bar{a})$ and $r < s$, we have that $(M, d_e, \sqsubseteq_e) \models \mathsf{MSE}$.

The 'Furthermore' statement follows from the fact that if $(M, d, \sqsubseteq) \models \mathsf{MSE}$, then (M, e) (where e is defined from $\sqsubseteq$ and d) is H-extensional and satisfies $\mathcal{L}_e$-excision. ⊣

Given Proposition 6.7, we will also use MSE to denote the $\mathcal{L}_e$-theory consisting of the H-extensionality axiom and the axiom scheme of excision.

§7. Constructing models of MSE. In order to construct models of MSE, we need to borrow techniques from the construction of models of GPK. The construction also has something of the flavor of the construction of models of NFU in that it involves non-standard models of another set theory. In order to show that arbitrary metric spaces can be a set of Quine atoms[13] in a model of MSE, we will use a construction that combines elements of the tree structures in [23] and the construction presented at the end of [6, Section 2]. The construction we give here could be generalized to allow certain other metric set structures to be embedded in models of MSE, in the same vein as [6, Section 2], but we have not pursued this here. We work in the context of ZF (with a particular eye towards avoiding the axiom of choice in order to establish its independence from MSE).

In the following definition, Q is intended to be a set of Quine atoms in our resulting model, although the models we construct here always have precisely one additional Quine atom.

DEFINITION 7.1. Fix a set Q and a $[0, 1]$-valued metric d on Q. Assume that Q does not contain any ordinal-indexed sequences. For any ordinal α, we let $\mathcal{T}_\alpha(Q)$ be the set of all α-sequences x satisfying that:

[13] Recall that a *Quine atom*, sometimes called a *self-singleton*, is a set x satisfying $x = \{x\}$.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

- for every $\beta < \alpha$, $x(\beta) \subseteq Q \cup \mathcal{T}_\beta(Q)$ and
- for every $\beta < \gamma < \alpha$, $x(\beta) \cap Q = x(\gamma) \cap Q$ and $x(\beta) \setminus Q = \{y \restriction \beta : y \in x(\gamma) \setminus Q\}$.

Let $\sqsubseteq \subseteq \mathcal{T}_\alpha(Q)^2$ be a binary relation such that:

- for $x \in \mathcal{T}_\alpha(Q)$ and $y \in Q$, $x \sqsubseteq y$ holds if and only if $x = y$,
- for $x \in Q$ and $y \sqsubseteq \mathcal{T}_\alpha(Q) \setminus Q$, $x \sqsubseteq y$ holds if and only if $x \in y(\beta)$ for every $b < \alpha$, and
- for $x \in \mathcal{T}_\alpha(Q) \setminus Q$ and $y \in \mathcal{T}_\alpha(Q) \setminus Q$, $x \sqsubseteq y$ if and only if $x \restriction \beta \in y(\beta)$ for every $\beta < \alpha$.

Note that $\mathcal{T}_\alpha(Q)$ is well-defined, as $\mathcal{T}_0(Q) = Q \cup \{\varnothing\}$.

For any $x \in \mathcal{T}_\alpha(Q)$, we write $\mathrm{tc}(x)$ for the smallest subset of $\mathcal{T}_\alpha(Q)$ such that $\{y : y \sqsubseteq x\} \subseteq \mathrm{tc}(x)$ and if $z \sqsubseteq y \in \mathrm{tc}(x)$, then $z \in \mathrm{tc}(x)$. For any $x, y \in \mathcal{T}_\alpha(Q)$, we define

$$\begin{aligned}
\rho_0^{Q,\alpha}(x,y) &\coloneqq d_{\mathrm{H}}(\mathrm{tc}(x) \cap Q, \mathrm{tc}(y) \cap Q),\\
e_\beta^{Q,\alpha}(z,y) &\coloneqq \inf_{w \sqsubseteq y} \rho_\beta^{Q,\alpha}(z,w),\\
\rho_{\beta+1}^{Q,\alpha}(x,y) &\coloneqq \max\left(\sup_{z \sqsubseteq x} e_\beta^{Q,\alpha}(z,y), \sup_{w \sqsubseteq y} e_\beta^{Q,\alpha}(w,x)\right),\\
\rho_\lambda^{Q,\alpha}(x,y) &\coloneqq \sup_{\beta<\lambda} \rho_\beta^{Q,\alpha}(x,y),
\end{aligned}$$

for all β and limit λ, where $\sup \varnothing = 0$ and $\inf \varnothing = 1$.

We will often suppress the superscript $^{Q,\alpha}$. Since the supremum of a family of pseudo-metrics is always a pseudo-metric, an easy inductive argument shows that ρ_β is a pseudo-metric for every $\beta \in \mathrm{Ord} \cup \{\infty\}$. It is also immediate that for any $x \in Q$, $\mathrm{tc}(x) = \{x\}$, and so for $x, y \in Q$, $\rho_\beta(x,y) = d(x,y)$ for every β. Finally, it can be shown that if $x(\gamma) = y(\gamma)$ for all $\gamma \leq \beta$, then $\rho_\beta(x,y) = 0$.

Also, while we will not need it, we should note that $\mathcal{T}_\alpha(\varnothing)$ is precisely the tree structure of height α of [23] and in this case, $\rho_\beta(x,y)$ is 0 if and only if $x \restriction \beta = y \restriction \beta$ and is 1 otherwise. ρ_β is of course also closely related to the $\sim_\beta$ relation of [15].

LEMMA 7.2. *For any $a, b \in \mathcal{T}_\alpha$, $\beta \mapsto \rho_\beta(a,b)$ and $\beta \mapsto e_\beta(a,b)$ are both non-decreasing functions of β.*

PROOF. Proceed by induction on β. Limit stages are obvious, so assume that we know that $\gamma \mapsto \rho_\gamma(a,b)$ and $\gamma \mapsto e_\gamma(a,b)$ are increasing functions for any $a, b \in \mathcal{T}_\alpha$ on the interval $[0, \beta]$ and consider $\rho_{\beta+1}(x,y)$.

If $\beta = 0$, then we just need to argue that $\rho_1(a,b) \geq \rho_0(a,b) = d_{\mathrm{H}}(\mathrm{tc}(a) \cap Q, \mathrm{tc}(b) \cap Q)$. Suppose that $\rho_0(a,b) > r$. Without loss of generality, this implies that there is a $c \in \mathrm{tc}(a) \cap Q$ such that $\inf\{d(c,z) : z \in \mathrm{tc}(b) \cap Q\} > r$. Since $c \in \mathrm{tc}(a)$ and $c \sqsubseteq c$, there is an $f \sqsubseteq a$ such that $c \in \mathrm{tc}(f)$. Since $c \in \mathrm{tc}(f) \cap Q$ and since $\mathrm{tc}(g) \subseteq \mathrm{tc}(b)$ for any $g \sqsubseteq b$, we have that $\rho_0(f,g) > r$ for any $g \sqsubseteq b$. Therefore $\rho_1(a,b) \geq r$. Since we can do this for any r, we have that $\rho_1(a,b) \geq \rho_0(a,b)$.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

If $\beta > 0$, then for any $\gamma \leq \beta$, we have that $e_\gamma(u,v) \leq e_\beta(u,v)$ by the induction hypothesis, so

$$\max\left(\sup_{z \sqsubseteq a} e_\gamma(z,b), \sup_{w \sqsubseteq b} e_\gamma(w,a)\right) \leq \max\left(\sup_{z \sqsubseteq a} e_\beta(z,b), \sup_{w \sqsubseteq b} e_\beta(w,a)\right)$$

and therefore $\rho_\gamma(a,b) \leq \rho_{\gamma+1}(a,b) \leq \rho_{\beta+1}(a,b)$, as required. The fact that $e_{\beta+1}(a,b) \geq e_\beta(a,b)$ is immediate. here ⊣

LEMMA 7.3. *For any (Q,d) and ordinals $\alpha < \beta$, there is a unique $v_\alpha \in \mathcal{T}_\beta(Q)$ such that $(v_\alpha, \sqsubseteq)$ and $(V_\alpha, \in)$ are isomorphic and for any $\gamma \in (\alpha,\beta)$ and distinct $a, b \sqsubseteq v_\alpha$, $\rho_\gamma(a,b) = 1$.*

PROOF. Fix an ordinal β. We will prove this for all $\alpha < \beta$ by induction. For $V_0 = \varnothing$, the statement is witnessed by the sequence $v_0(\gamma) = \varnothing$ in $\mathcal{T}_\beta(Q)$.

Now assume that for some $\alpha < \beta$, the statement is known for all $\delta < \alpha$. If α is a successor and equal to $\gamma + 1$, let v_α be defined by $v_\alpha(0) = \{\varnothing\}$, $v_\alpha(\sigma+1) = \mathcal{P}(v_\gamma(\sigma))$, and $v_\alpha(\lambda) = \{x : x \text{ is a } \lambda \text{ sequence}, (\forall \sigma < \lambda) x{\restriction}\sigma \in v_\alpha(\sigma)\}$ for any limit ordinal λ. Since the statement holds for γ, we have that ρ_γ is $\{0,1\}$-valued on the $\sqsubseteq$-elements of v_γ, we get that $\rho_\alpha = \rho_{\gamma+1}$ is $\{0,1\}$-valued on the $\sqsubseteq$-elements of v_α. Furthermore, since $(v_\gamma, \sqsubseteq)$ is isomorphic to $(V_\gamma, \in)$, it follows immediately that $(v_\alpha, \sqsubseteq)$ is isomorphic to $(V_\alpha, \in)$.

If α is a limit, then let $v_\alpha(\sigma) = \bigcup_{\gamma<\alpha} v_\gamma(\sigma)$ for every $\sigma < \beta$. The required statements are obvious. ⊣

DEFINITION 7.4. For any Q and α, we let $\tau_{Q,\alpha}$ be the topology on $\mathcal{T}_\alpha(Q)$ generated by sets of the form $\{y \in \mathcal{T}_\alpha(Q) : \rho_\beta(x,y) < \varepsilon\}$ for $x \in \mathcal{T}_\alpha(Q)$, $\beta < \alpha$, and $\varepsilon > 0$.

It is immediate from basic topological facts that for any $X \subseteq \mathcal{T}_\alpha(Q)$, there is a unique smallest closed set $\overline{X}$ containing X. More importantly, we have the following.

PROPOSITION 7.5. *For any Q, limit α, and $\tau_{Q,\alpha}$-closed $F \subseteq \mathcal{T}_\alpha(Q)$, there is an $x \in \mathcal{T}_\alpha(Q)$ such that $F = \{y \in \mathcal{T}_\alpha(Q) : y \sqsubseteq x\}$.*

PROOF. For any $\beta < \alpha$, let $x(\beta) \coloneqq \{y \in \mathcal{T}_\beta(Q) : y \text{ extends to an element of } F\}$. x is clearly an element of $\mathcal{T}_\alpha(Q)$. Furthermore, we clearly have that if $y \in F$, then $y \sqsubseteq x$. So now we just need to show the converse.

Suppose that $y \sqsubseteq x$. We would like to show that y is in the closure of F and therefore is in F. In order to do this, it is sufficient to show that $\inf\{\rho_\beta(y,z) : z \in F\} = 0$ for each $\beta < F$. For each $\beta < \alpha$, find $z \in F$ such that $y(\beta+1)$ is an initial segment of z. We now have that $\rho_\beta(y,z) = 0$. Since we can do this for any $\beta < \alpha$, we have that y is in the closure of F. ⊣

What will ultimately be relevant to us is that the above facts are first-order properties of the structure $(V_{\alpha+\omega}, \alpha, Q, d, \sqsubseteq, \mathcal{T}_\alpha(Q))$ (assuming $\mathcal{T}_\alpha(Q)$ is an element of $V_{\alpha+\omega}$). This is part of the motivation for Definition 7.7.

We will also need the following.

LEMMA 7.6. *Fix a metric space (Q,d) and a limit ordinal α. Let $\overline{Q}$ be the $\tau_{Q,\alpha}$-closure of $Q \subset \mathcal{T}_\alpha(Q)$. For any $z \in \overline{Q}$, there is an $x \in Q$ such that $\rho_\beta(x,z) = 0$ for all $\beta < \alpha$.*

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

PROOF. First we need to show that if $x \in \overline{Q}$, then $|\operatorname{tc}(x) \cap Q| = 1$. Suppose that $\operatorname{tc}(x) \cap Q$ has more than one element. Let y and z be distinct elements of $\operatorname{tc}(x) \cap Q$. Suppose that $d(y,z) > r$. We now immediately have that $\rho_0(x,w) > \frac{1}{2}r$ for any $w \in Q$. Therefore $x \notin \overline{Q}$. On the other hand, suppose that $\operatorname{tc}(x) \cap Q = \varnothing$. Then likewise, $\rho_0(x,w) = 1$ for any $w \in Q$. Therefore $x \notin \overline{Q}$.

Now we need to argue that if $x \in \overline{Q}$, then for any $y \sqsubset x$, $y \in \overline{Q}$ as well. Suppose $y \sqsubset x$ and $y \notin \overline{Q}$. By definition, this implies that there is a $\beta < \alpha$ such that $\inf\{\rho_\beta(y,z) : z \sqsubset x\} = r > 0$, but this implies that $\rho_{\beta+1}(x,w) \geq r$ for all $w \in Q$ and so $x \notin \overline{Q}$.

For any $x \in \overline{Q}$, let $\pi(x)$ denote the unique element of Q that is in $\operatorname{tc}(x)$. We need to show that $\rho_\beta(x,\pi(x)) = 0$ for all $\beta < \alpha$. Clearly $\rho_0(y,\pi(x)) = 0$ for any $y \in \overline{Q}$ with $\pi(y) = \pi(x)$. Suppose that $\rho_\gamma(y,\pi(x)) = 0$ for all $\gamma < \beta$ and $y \in \overline{Q}$ with $\pi(y) = \pi(x)$. If β is a limit, then $\rho_\beta(x,\pi(x)) = 0$. Assume that $\beta = \delta + 1$ for some δ. Fix y with $\pi(y) = \pi(x)$. Fix $z \sqsubset y$. We clearly have that $\operatorname{tc}(z) \cap Q \subseteq \operatorname{tc}(y) \cap Q$. It also must be the case that $z \in \overline{Q}$. Therefore we must have that $\pi(z) = \pi(y) = \pi(x)$ as well, so by the induction hypothesis, we have that $\rho_\delta(z,\pi(x)) = 0$. Since we can do this for any $z \sqsubset y$, we have that $\rho_{\delta+1}(y,\pi(x)) = 0$, as required. ⊣

DEFINITION 7.7. Fix a tuple $(Q,d,\sqsubset)$ as in Definition 7.1 and an infinite ordinal α such that $\mathcal{T}_\alpha(Q)$ is an element of $V_{\alpha+\omega}$. We will assume that restricted $\mathcal{L}_e$-formulas are elements of $V_{\alpha+\omega}$.

Let $(M, \alpha^M, Q^M, d^M, \sqsubset^M, \mathcal{T}^M)$ be a (discrete) structure elementarily equivalent to $(V_{\alpha+\omega}, \alpha, Q, d, \sqsubset, \mathcal{T}_\alpha(Q))$.[14] We write $\rho_\beta^M(x,y)$ and $e_\beta^M(x,y)$ for the functions in M given by Definition 7.1 computed internally.

Given any $r \in \mathbb{R}^M$ satisfying $|r| \leq n$ for some standard natural n, the *standard part of* r, written $\operatorname{st}(r)$, is the unique standard real satisfying $r \geq t$ if and only if $\operatorname{st}(r) \geq t$ for all standard rationals t.

A *gauge on* M is a non-increasing function $s : \alpha^M \to [0,1]$ (where $[0,1]$ is the standard unit interval) with $s(0) = 1$. An *internal gauge on* M is a non-increasing function $s \in M$ from α^M to $[0,1]^M$ with $s(0) = 1$. An internal gauge on M is *ε-smooth* if:

- $s(0) = s(1)$,
- $s(\beta) = 0$ for all sufficiently large $\beta \in \alpha^M$,
- for every $\beta \in \alpha^M$, $s(\beta) < s(\beta+1) + \varepsilon$, and
- for any limit $\lambda \in \alpha^M$, there is a $\beta < \lambda$ such that $s(\beta) = s(\lambda)$.

Given an internal gauge s on M, the *standard part of* s, written s^{st}, is $\operatorname{st} \circ s$.

Given a gauge s on M, we define the functions

$$\rho_s(x,y) \coloneqq \sup_{\beta \in \alpha^M} \min(\rho_\beta^M(x,y), s(\beta))$$

and $e_s(x,y) = \inf_{w \sqsubset^M y} \rho_s(x,w)$. For any internal gauge s we write ρ_s and e_s for the corresponding quantities computed internally in M and we write ρ_s^{st} and e_s^{st} for their corresponding standard parts.

Given two gauges s_0 and s_1 on M, we write $\|s_0 - s_1\|$ for the quantity $\sup_{\beta \in \alpha^M} |s_0(\beta) - s_1(\beta)|$.

[14] Note that d^M is a function to $\mathbb{R}^M$, the M-internal reals. M is not a metric structure.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

Note that $\rho_s^{\mathrm{st}} = \rho_{s^{\mathrm{st}}}$ and $e_s^{\mathrm{st}} = e_{s^{\mathrm{st}}}$ for any M as in Definition 7.7. Since $\rho_s(x, y)$ is the supremum of a family of pseudo-metrics, it is itself a pseudo-metric. Finally, it is trivial that for any gauge s on M, $(\mathcal{T}^M, e_s)$ is an $\mathcal{L}_e$-structure.

In the statements of lemmas in the rest of this section, we will write '(M as in Definition 7.7.)' to mean that the structure $M = (\mathcal{T}^M, \alpha^M, \mathbb{R}^M)$ satisfies the conditions in Definition 7.7.

LEMMA 7.8 (M as in Definition 7.7). *Fix a restricted $\mathcal{L}_e$-formula $\varphi(\bar{x})$ and a tuple $\bar{a} \in \mathcal{T}^M$.*

(1) *For any gauges s and t on M,*

$$|\varphi^{(\mathcal{T}^M, e_s)}(\bar{a}) - \varphi^{(\mathcal{T}^M, e_t)}(\bar{a})| \leq v(\varphi)\|s - t\|.$$

(2) *For any internal gauge s on M,*

$$st((\varphi^{(\mathcal{T}^M, e_s)}(\bar{a}))^M) = \varphi^{(\mathcal{T}^M, e_s^{st})}(\bar{a}),$$

where $(\varphi^{(\mathcal{T}^M, e_s)}(\bar{a}))^M$ is the value of $\varphi^{(\mathcal{T}^M, e_s)}(\bar{a})$ computed internally in M.

PROOF. It is straightforward to show that for any $a, b \in \mathcal{T}^M$, $|\rho_s(a, b) - \rho_t(a, b)| \leq \|s - t\|$. This implies likewise that for any $a, b \in \mathcal{T}^M$, $|e_s(a, b) - e_t(a, b)| \leq \|s - t\|$. From this, 1 follows by an induction argument. 2 also follows from an easy induction argument. ⊣

LEMMA 7.9. (M as in Definition 7.7). *For any $\beta \in \alpha^M$,*

$$\rho_{\beta+1}^M(x, y) = \sup_{z \in \mathcal{T}^M} |e_\beta^M(z, x) - e_\beta^M(z, y)|.$$

PROOF. This follows immediately from the fact that ρ_β^M is a pseudo-metric on $\mathcal{T}^M$ and $\rho_{\beta+1}^M(x, y)$ is precisely the Hausdorff distance between $\{z : z \sqsubseteq^M x\}$ and $\{z : z \sqsubseteq^M y\}$ with respect to $\rho_{\beta+1}^M$. ⊣

LEMMA 7.10. (M as in Definition 7.7). *Fix $\varepsilon \in (0, 1]^M$ and an ε-smooth internal gauge s on M. Let $d_{e,s}(x, y) := \sup_z |e_s^M(z, x) - e_s^M(z, y)|$. The following statements hold internally in M:*

(1) *$\rho_s(a, b) \leq d_{e,s}(a, b) \leq \rho_s(a, b) + \varepsilon$ for all $a, b \in \mathcal{T}^M$.*
(2) *$|e_s(a, b) - \inf_z \min(d_{e,s}(a, z) + 2e_s(z, b), 1)| \leq \varepsilon$ for all $a, b \in \mathcal{T}^M$.*

PROOF. First note that by definition, $d_{e,s}(x, y)$ is the Hausdorff pseudo-metric induced by the pseudo-metric $\rho_s(x, y)$. So in particular we also have that

$$d_{e,s}(x, y) = \max\left(\sup_{z \sqsubseteq x} e_s(z, y), \sup_{w \sqsubseteq y} e_s(w, x)\right).$$

For 1, fix a and b in $\mathcal{T}$. If a and b are both $\sqsubseteq$-empty, then $d_{e,s}(a, b) = \rho_s(a, b) = 0$. If one of them is $\sqsubseteq$-empty and the other isn't, then $d_{e,s}(a, b) = \rho_s(a, b) = 1$. So assume that they are both non-$\sqsubseteq$-empty.

Suppose that $\rho_s(a, b) > r$. This implies that there is a $\beta \in \alpha^M$ such that $\min(\rho_\beta(a, b), s(\beta)) > r$, which implies that $\rho_\beta(a, b) > r$. If β is a limit ordinal, then $\rho_\beta(a, b) = \sup_{\gamma < \beta} \rho_\gamma(a, b)$, so, since s is non-increasing, we may assume that β is not a limit ordinal. Since $s(0) = s(1)$, we may assume that $\beta > 0$ by

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

Lemma 7.2. So let $\gamma + 1 = \beta$. We may now assume without loss of generality that there is a $c \sqsubseteq a$ such that $e_\gamma(c, b) > r$, implying that $\rho_\gamma(c, f) > r$ for all $f \sqsubseteq b$. Therefore we have that $\rho_s(c, f) \geq \min(\rho_s(c, f), s(\gamma)) > r$ for all $f \sqsubseteq b$, whence $e_s(c, b) \geq r$ and $d_{e,s}(a, b) \geq r$. Since we can do this for any $r < \rho_s(a, b)$, we have that $d_{e,s}(a, b) \geq \rho_s(a, b)$.

Now suppose that $d_{e,s}(a, b) > r$ for some $r > 0$. We may assume without loss of generality that there is a $c \sqsubseteq a$ such that $e_s(c, b) > r$. So in particular, $\rho_s(c, f) > r$ for all $f \sqsubseteq b$. Therefore, for any such f, there is a $\beta_f \in \alpha^M$ such that $\min(\rho_{\beta_f}(c, f), s(\beta_f)) > r$. Since s is ε-smooth, there is a largest $\gamma \in \alpha^M$ such that $s(\gamma) > r$. Note that we must have $\gamma \geq \beta_f$ for all $f \sqsubseteq b$. So now we actually know that $\rho_\gamma(c, f) > r$ for all $f \sqsubseteq b$. Therefore $e_\gamma(c, b) \geq r$ and so $\rho_{\gamma+1}(a, b) \geq r$, whence $\rho_s(a, b) \geq \min(\rho_{\gamma+1}(a, b), s(\gamma + 1)) > r - \varepsilon$. Since we can this for any $r < d_{e,s}(a, b)$, we have that $\rho_s(a, b) \geq d_{e,s}(a, b) - \varepsilon$, as required.

For 2, it follows from the $\Leftarrow$ direction of the proof of Lemma 6.2 that $e_s(a, b) = \inf_z \min(\rho_s(a, z) + 2e_2(z, b), 1)$ for all $a, b \in \mathcal{T}^M$. It is immediate from part 1 that

$$|\inf_z \min(d_{e,s}(a, z) + 2e_2(z, b), 1) - \inf_z \min(\rho_s(a, z) + 2e_2(z, b), 1)| \leq \varepsilon$$

for all $a, b \in \mathcal{T}^M$, so the required result follows. ⊣

Lemma 7.11 (M as in Definition 7.7). *Fix $\varepsilon \in (0, 1]^M$ and let s be an ε-smooth internal gauge on M. For any*[15] *$\mathcal{L}_e$-formula $\varphi(\bar{a}, \bar{b})$ and any $\bar{a}, \bar{b} \in \mathcal{T}^M$, it holds internally in M that*

$$|\varphi^{(\mathcal{T}^M, e_s)}(\bar{a}) - \varphi^{(\mathcal{T}^M, e_s)}(\bar{b})| \leq 2v(\varphi) d_{e,s}(\bar{a}, \bar{b}),$$

where $d_{e,s}(\bar{a}, \bar{b}) = \max_{i<|\bar{a}|} d_{e,s}(a_i, b_i)$.

Proof. We prove this by induction on formulas. If φ is $e(x, y)$, then we have

$$\begin{aligned} |e(a_0, a_1) - e(b_0, b_1)| &\leq |e(a_0, a_1) - e(b_0, a_1)| + |e(b_0, a_1) - e(b_0, b_1)| \\ &\leq \rho_s(a_0, b_0) + d_{e,s}(a_1, b_1) \\ &\leq d_{e,s}(a_0, b_0) + d_{e,s}(a_1, b_1) \\ &\leq 2d_{e,s}(a_0 a_1, b_0 b_1) \end{aligned}$$

by Lemma 7.10. The argument for connectives and quantifiers the same as in Lemma 2.6. ⊣

Lemma 7.12. (M as in Definition 7.7). *Fix $\varepsilon \in (0, 1]^M$ and let s be an ε-smooth internal gauge on M. For any $\mathcal{L}_e$-formula $\varphi(x, \bar{y})$, it holds internally in M that*

$$\begin{aligned} (\mathcal{T}^M, e_s) \models \sup_{\bar{y}} \inf_z \sup_x \max(\min(e(x, z), -\varphi(x, \bar{y})), \\ \min(\varepsilon_\varphi - e(x, z), \varphi(x, \bar{y}) - 1)) \leq 2v(\varphi)\varepsilon. \end{aligned}$$

Proof. For any $\mathcal{L}_e$-formula $\varphi(x, \bar{y})$ and any $\bar{a} \in M$, let $B_0 = \{x \in \mathcal{T}^M : \varphi^{(\mathcal{T}^M, e_s)}(x, \bar{a}) \leq 0\}$. We need to argue that B_0 is closed in the topology on $\mathcal{T}^M$ given in Definition 7.4. Suppose that $c \notin B_0$. This means that $\varphi^{(\mathcal{T}^M, e_s)}(c, \bar{a}) > 0$.

[15] Possibly non-standard, although we do not need this.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

Since e_s is 2-Lipschitz with regards to ρ_s, this implies that there is a $\delta > 0$ such that for any $c' \in \mathcal{T}^M$ with $\rho_s(c, c') < \delta$, $\varphi^{(\mathcal{T}^M, e_s)}(c', \bar{a}) > 0$ as well. Since s is ε-smooth, there is a $\gamma \in \alpha^M$ such that $s(\gamma) = 0$. By Lemma 7.2, we know that if $\rho_\gamma(c, c') < \frac{1}{2}\delta$, then $\rho_s(c, c') < \delta$. Therefore we have that the set $\{x : \rho_\gamma(x, c) < \frac{1}{2}\delta\}$ is disjoint from B_0. Since we can do this for any $c \notin B_0$, we have that B_0 is closed.

Let b be the unique element of $\mathcal{T}^M$ coextensive with B_0 (which exists by Proposition 7.5). For any $c \in \mathcal{T}^M$, if $\varphi^{(\mathcal{T}^M, e_s)}(c, \bar{a}) \leq 0$, then $c \sqsubseteq b$ by our choice of c and so $e_s(c, b) = 0$. Therefore $\min(e_s(c, b), -\varphi(c, \bar{a})) \leq 0$ for all $c \in \mathcal{T}^M$. On the other hand, if $e_s(c, b) \leq \varepsilon_\varphi$, then there is an $f \sqsubseteq b$ such that $\rho_s(c, f) < \varepsilon_\varphi + \sigma$ for any $\sigma > 0$. Therefore, $\varphi(c, \bar{a}) < 2v(\varphi)d_{e,s}(c, f) < 2v(\varphi)(\varepsilon_\varphi + \sigma + \varepsilon)$ by Lemmas 7.10 and 7.11. Since $2v(\varphi)\varepsilon_\varphi < 1$ and since we can do this for any $\sigma > 0$, we have that $\varphi(c, \bar{a}) \leq 1 + 2v(\varphi)\varepsilon$. Therefore $\min(\varepsilon_\varphi - e(c, b), \varphi(c, \bar{a}) - 1) \leq 2v(\varphi)\varepsilon$ for any $c \in \mathcal{T}^M$. ⊣

LEMMA 7.13 (M as in Definition 7.7). *For any (external) gauge s on M with dense image in $[0, 1]$ and (standard) rational $\varepsilon \in (0, 1]$, there is an ε-smooth internal gauge t such that $\|s - t^{st}\| \leq \varepsilon$.*

PROOF. Find standard n large enough that $\frac{1}{n} < \frac{1}{2}\varepsilon$. For each $i < n$, find $\beta_i \in \alpha^M$ such that $\frac{i}{n} \leq s(\beta_i) < \frac{i+1}{n}$. Note that since the range of s is dense, none of the β_i's are 0 or 1. Also note that $(\beta_i)_{i<n}$ is a decreasing sequence of ordinals. For any $\gamma \in \alpha^M$, let

$$t(\gamma) = \begin{cases} 0 & \gamma \geq \beta_0 \\ \frac{i}{n} & 0 < i < n,\ \beta_{i-1} > \gamma \geq \beta_i \\ 1 & \beta_{n-1} > \gamma \end{cases}.$$

Clearly $t(\gamma) = 0$ for all sufficiently large γ. We also have that $s(\gamma) < s(\gamma + 1) + \frac{2}{n} < s(\gamma + 1) + \varepsilon$ for all $\gamma \in \alpha^M$. Finally, the limit ordinal condition in the definition of ε-smooth is clearly met, so t is ε-smooth.

Now for any γ, we have that if $t(\gamma) = 0$, then $\gamma \geq \beta_0$ and so $s(\gamma) \leq \frac{1}{n} < \varepsilon$. If $t(\gamma) \in (0, 1)$, then there is a positive $i < n$ such that $\beta_{i-1} > \gamma \geq \beta_i$, implying that $\frac{i-1}{n} \leq s(\gamma) \leq \frac{i+1}{n}$, so $|s(\gamma) - t(\gamma)| = |s(\gamma) - \frac{i}{n}| < \varepsilon$. And if $t(\gamma) = 1$, then $s(\gamma) \geq s(\beta_{n-1}) \geq \frac{n-1}{n}$, so $|s(\gamma) - t(\gamma)| \leq \frac{1}{n} < \varepsilon$. Therefore $\|s - t\| \leq \varepsilon$, as required. ⊣

In order to proceed we will need a fact from model theory. This is similar to the approach typically used to build partially standard models of NFU.

LEMMA 7.14. *For any ordinal σ, there is an $\alpha > \sigma$ and a structure $(M, \alpha^M) \equiv (V_{\alpha+\omega}, \alpha)$ such that:*

- *V_σ^M is isomorphic to V_σ and*
- *there is a set of M-ordinals less than α^M that is order-isomorphic to $\mathbb{Q}$.*

PROOF. Let $\kappa = |V_\sigma|$. Let $\alpha = \beth_{(2^\kappa)^+}$. Expand $(V_{\alpha+\omega}, \in)$ by Skolem functions. By [21, Lemma 7.2.12], we can find a V_σ-indiscernible sequence I with order type $\mathbb{Q}$ in some elementary extension N of $(V_{\alpha+\omega}, \in, \alpha, \text{Skolem functions})$ such that for any increasing sequence $a_0 < \cdots < a_{n-1}$ in I, there are ordinals $\delta_0, \ldots, \delta_{n-1} < \alpha$ with $\text{tp}(\bar{a}/V_\sigma) = \text{tp}(\bar{\delta}/V_\sigma)$. Let M be the Skolem hull of $V_\sigma \cup I$. For any element a of V_σ^M, there is a Skolem function f and tuples $\bar{b} \in V_\sigma$ and $\bar{c} \in I$ such that $a = f(\bar{b}, \bar{c})$. By

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

construction, there is a tuple $\bar{\delta} \in V_\alpha$ such that $\mathrm{tp}(\bar{c}/V_\sigma) = \mathrm{tp}(\bar{\delta}/V_\sigma)$. Since $f(\bar{b}, \bar{\delta}) \in V_\sigma$, we have that $f(\bar{b}, \bar{c}) = f(\bar{b}, \bar{\delta})$. Therefore $V_\sigma^M = V_\sigma$, as required. ⊣

THEOREM 7.15. *For any complete metric space (Q, d) with $[0,1]$-valued metric and any ordinal σ, there is a model N of* MSE *such that N contains*:

- *a set of Quine atoms isometric to (Q, d),*
- *all closed subsets of Q as elements, and*
- *a 1-discrete set v such that $(v, \sqsubseteq_e)$ is isomorphic to $(V_\sigma, \in)$ (implying that Ord^N has standard part of length at least σ and if σ is infinite, $N \models \mathsf{Inf}(v)$).*

In particular, MSE *is consistent.*

PROOF. Fix (Q, d) as in Definition 7.1. Fix some ordinal σ. We may assume that (Q, d) and the full power set of Q are in V_σ. We may also assume that σ is a limit ordinal. Apply Lemma 7.14 to get a structure M elementarily equivalent to $V_{\alpha+\omega}$ for some $\alpha > \sigma$ such that the standard part of M contains V_σ. Let J be a set of M-ordinals less than α^M order-isomorphic to $\mathbb{Q}$. Since $\mathcal{T}_\alpha(Q)$ is definable from Q and α, there is an element $\mathcal{T}^M$ in M realizing the same type over Q and α^M. In this way we can regard M as a structure satisfying the conditions of Definition 7.7.

J is also order-isomorphic to $(0,1) \cap \mathbb{Q}$. Let f be an order isomorphism witnessing this. Define $s : \mathrm{Ord}^M \to [0,1]$ by $s(\beta) = \inf\{f(\gamma) : \gamma \in J,\ \gamma \leq \beta\}$ with $\inf \varnothing = 1$. This is clearly a gauge on M. Let $N = (\mathcal{T}^M, e_s)$.

We need to show that for any axiom φ of MSE (i.e., those listed in Definitions 6.1 and 6.5) and any $\varepsilon > 0$, $N \models \varphi \leq \varepsilon$. If φ is the H-extensionality axiom, we can find with Lemma 7.13 a $\frac{1}{2}\varepsilon$-smooth internal gauge t such that $\|s - t^{st}\| \leq \frac{1}{2}\varepsilon$. By Lemmas 7.8 and 7.10, we have that $\varphi^N \leq \varphi^{(\mathcal{T}^M, e_t)} + \frac{1}{2}\varepsilon \leq \varepsilon$. If φ is the excision axiom for the formula ψ, then we can do the same with a $\frac{1}{2v(\psi)}\varepsilon$-smooth internal gauge t by Lemma 7.12. Since we can do this for any $\varphi \in$ MSE and $\varepsilon > 0$, we have that $N \models$ MSE.

Finally we just need to verify that the set of Quine atoms isomorphic to (Q, d), arbitrary closed subsets of Q, and the set isomorphic to V_σ exist in N. Let q be the element of $\mathcal{T}^M$ coextensive with the set Q^* defined in Lemma 7.6. We have that Q is a dense subset of q and furthermore ρ_β agrees with d on Q for all $\beta < \alpha$, therefore (q, ρ_s) is isometric to (Q, d), since Q is metrically complete. Since σ is a limit ordinal, the fact that arbitrary closed subsets of Q are coextensive with elements of N follows from Proposition 7.5.

Finally, let $v = v_\sigma$ as defined in the proof of Lemma 7.3. Since $\sigma(\sigma) = 1$, we have that v is 1-discrete. The relation $x \sqsubseteq_e y$ (i.e., $e(x, y) = 0$) agrees with $x \sqsubseteq y$ for $\sqsubseteq$-elements of v, so we have that $(v, \sqsubseteq_e)$ is isomorphic to $(V_\sigma, \in)$. ⊣

One thing to note with regards to Theorem 7.15 is that if M satisfies the axiom of choice, then the resulting structure N will satisfy the axiom of choice in all of its uniformly discrete sets. Conversely, if there is a set $x \in V_\alpha^M$ witnessing the failure of the axiom of choice, then the axiom of choice will fail for the corresponding 1-discrete set in N. Since we did not use the axiom of choice at any point in our construction, this establishes that choice for uniformly discrete sets is independent of MSE.

Recall that an $\mathcal{L}_e$-structure M is *pseudo-finite* if for every restricted $\mathcal{L}_e$-sentence φ and r, if $M \models \varphi < r$, then there is a finite $\mathcal{L}_e$-structure N such that $N \models \varphi < r$.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

THEOREM 7.16. *There is a pseudo-finite model of* MSE.

PROOF. Consider the structure $M = (V_{\omega+\omega}, \omega, \varnothing, \dots)$. For each $n \in \mathbb{N}$, let s_n be the scale on M defined by $s_n(i) = \min(\max(1 - \frac{i-1}{n}, 0), 1)$. This is easily seen to be $\frac{1}{n}$-smooth. The quotient of $\mathcal{T}_\omega(\varnothing)$ by the pseudo-metric ρ_{s_n} is finite, so any ultraproduct of the sequence $(\mathcal{T}_\omega(\varnothing)/\rho_{s_n}, e_{s_n})_{n\in\mathbb{N}}$ is a pseudo-finite model of MSE by Lemmas 7.10 and 7.12. ⊣

It is straightforward to show that no pseudo-finite model of MSE can satisfy Inf, as no pseudo-finite structure can interpret Robinson arithmetic.

§8. Formalizing MSE in Łukasiewicz logic. It was observed in [2] that there is a strong connection between continuous logic and Łukasiewicz–Pavelka predicate logic. This logic extends Łukasiewicz logic with 0-ary connectives for each rational $r \in [0, 1]$. Since every unary connective $f(x)$ in rational Pavelka logic is piecewise linear and has that $\frac{\mathrm{d}}{\mathrm{d}x} f(x) \in \mathbb{Z}$ for all but finitely many $x \in [0, 1]$, it is immediate that $\frac{x}{2}$ is not a connective that can be formed in it. One might think that this would prevent rational Pavelka logic from being logically complete in the sense of continuous logic, but as pointed out in [2, Proposition 1.18], the connective $x \mapsto \frac{1}{2}x$ is a uniform limit of connectives in rational Pavelka logic:

$$\lim_{n\to\infty} \max_{1\le i\le n} \min\left(\frac{i}{n}, x \dot{-} \frac{i}{n}\right) = \frac{1}{2}x$$

uniformly for all $x \in [0, 1]$. This implies that all $[0, 1]$-valued formulas in continuous logic are uniform limits of Łukasiewicz–Pavelka formulas.

Beyond this, it follows from results in [7] and was shown explicitly in [1, Theorem 6.3 and Corollary 6.4] via a Lindström's theorem argument that on the level of *conditions* rather than *formulas*, ordinary Łukasiewicz logic is already logically complete relative to continuous logic.[16] As a consequence of this, any continuous first-order theory or type (in a language with $[0, 1]$-valued predicates) can be axiomatized entirely in Łukasiewicz logic. We will give an elementary proof of this via a syntactic transformation here.

DEFINITION 8.1. A restricted formula φ is a *rational affine literal* if it is a rational affine combination of atomic formulas. A quantifier-free formula is in *maximal affine normal form* or *max ANF* if it is $\max_{n<N} \min_{m<M_n} \varphi_{nm}$, where each φ_{nm} is a rational affine literal. A formula φ is in *prenex max ANF* if it is a string of quantifiers followed by a max ANF formula.

It is not too hard to show (and is written out explicitly in [11, Proposition 1.4.12]) that every restricted formula is equivalent to a formula in prenex max ANF.

PROPOSITION 8.2. *Fix a continuous language $\mathcal{L}$ in which all predicates are* $[0, 1]$*-valued. For every restricted $\mathcal{L}$-formula $\varphi(\bar{x})$ (in either the sense of* [25, *Section* 3] *or the more permissive sense of* [11, *Section* 1.3]*), there is a formula $\psi(\bar{x})$ using only*

[16] Note however that not all continuous functions form $[0, 1]^n \to [0, 1]$ are uniform limits of expressions in propositional Łukasiewicz logic, as any such expression maps $\{0, 1\}^n$ to $\{0, 1\}$, so Łukasiewicz–Pavelka logic is stronger in at least one sense that matters to continuous logic.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

the connectives 1 *and* $x \dot{-} y$ *such that for any metric structure* M *and tuple* $\bar{a} \in M$, $\varphi^M(\bar{a}) \leq 0$ *if and only if* $\psi^M(\bar{a}) \leq 0$.

PROOF. By McNaughton's theorem [16, Theorem 1], for any integers a and $b_0, \ldots, b_{n-1}$, the function $M(\bar{x}; a, \bar{b}) = \min(\max(a + b_0 x_0 + b_1 x_1 + \cdots + b_{n-1} x_{n-1}, 0), 1)$ can be expressed using the connectives of Łukasiewicz logic.

Let $\varphi \leq r$ be a restricted closed condition. We may assume without loss of generality that $r = 0$. By the above discussion we can rewrite φ as an equivalent prenex max ANF formula

$$\psi(\bar{x}) = \operatorname*{qnt}_{x_0} \operatorname*{qnt}_{x_1} \operatorname*{qnt}_{x_2} \ldots \max_{n<N} \min_{m<M_n} \left(a_{nm} + \sum_{k<K_{nm}} b_{nmk} \chi_{nmk} \right),$$

where each qnt is either inf or sup and each χ_{nmk} is an atomic formula. Let ℓ some number larger than the denominators of the coefficients in ψ. Consider now the formula $\psi^\dagger(\bar{x})$ defined as

$$\operatorname*{qnt}_{x_0} \operatorname*{qnt}_{x_1} \operatorname*{qnt}_{x_2} \ldots \max_{n<N} \min_{m<M_n} M(\chi_{n,m,0}, \ldots, \chi_{n,m,K_{nm}-1}; \ell! \cdot a_{nm}, \ell! \cdot b_{n,m,0}, \ldots, \ell! \cdot b_{n,m,K_{nm}-1}).$$

Note that $\psi^\dagger$ is equivalent to $\min(\max(\ell! \cdot \psi, 0), 1)$. We clearly have that in any structure M, $\psi(\bar{a}) \leq 0$ if and only if $\psi^\dagger(\bar{a}) \leq 0$, but this latter condition can be expressed in Łukasiewicz logic by McNaughton's theorem. In particular, if we interpret $M(-; \ell! \cdot a_{nm}, \ell! \cdot b_{n,m,0}, \ldots, \ell! \cdot b_{n,m,K_{nm}-1})$ as an expression in Łukasiewicz logic, then we have that $M \models \psi(\bar{a}) \geq 0$ if and only if M satisfies

$$\neg Q x_0 Q x_1 Q x_3 \cdots \bigvee_{n<N} \bigwedge_{n<M_n} M(\chi_{n,m,0}, \ldots, \chi_{n,m,K_{nm}-1}; \ell! \cdot a_{nm}, \ell! \cdot b_{n,m,0}, \ldots, \ell! \cdot b_{n,m,K_{nm}-1}),$$

where $Q x_i$ is $\exists x_i$ if qnt_{x_i} is $\sup_{x_i}$ and $\forall x_i$ if qnt_{x_i} is $\inf_{x_i}$. ⊣

There are some minor differences in the treatment of equality (i.e., the metric) in continuous logic and Łukasiewicz logic and, relatedly, the intended semantics of continuous logic is more specific than that of Łukasiewicz logic, but for structures without equality (i.e., general structures such as our $\mathcal{L}_e$-structures) there is no difference in expressive power.

While the above discussion is sufficient to prove that it exists, we will now give an explicit axiomatization of MSE in Łukasiewicz logic. For the sake of compatibility with the existing Łukasiewicz logic literature, we will switch to the convention of regarding 1 as true. As such, we will write $x \mathrel{\hat{\varepsilon}} y$ for $1 - e(x, y)$. We will write $A \to B$ for connective $1 - (A \dot{-} B) = \min(1 - A + B, 1)$ and $\bot$ for 0. Formulas are formed from $x \mathrel{\hat{\varepsilon}} y$ using the connectives $\to$ and $\bot$ and the quantifiers $\exists$ and $\forall$. We'll write $\text{Ł}_{\hat{\varepsilon}}$ for this set of formulas, which we will regard as a subset of the set of restricted $\mathcal{L}_e$-formulas (where we interpret $\exists x$ as $\sup_x$ and $\forall x$ as $\inf_x$). For an $\mathcal{L}_e$-structure M, a tuple $\bar{a} \in M$, and a formula $\varphi(\bar{x}) \in \text{Ł}_{\hat{\varepsilon}}$, we say that M *satisfies* $\varphi(\bar{a})$ if $M \models \varphi(\bar{a}) = 1$.

It is a basic fact that the connectives $\to$ and $\bot$ can be used to define the following: $A \leftrightarrow B := 1 - |x - y|$, $\neg A := 1 - A$, $A \wedge B := \min(A, B)$, $A \vee B := \max(A, B)$, and $A \,\&\, B := \max(A + B - 1, 0)$.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

First we need to define extensional equality: We will write $x =_e y$ as shorthand for the formula $\forall z(z \mathrel{\hat{\varepsilon}} x \leftrightarrow z \mathrel{\hat{\varepsilon}} y)$. (This is the same thing as $1 - d_e(x, y)$.) With this we can now write the H-extensionality axiom as

$$\forall x \forall y(x \mathrel{\hat{\varepsilon}} y \leftrightarrow \exists z(x =_e z \,\&\, z \mathrel{\hat{\varepsilon}} y \,\&\, z \mathrel{\hat{\varepsilon}} y)). \tag{1}$$

It is easy to verify that this is a literal transcription of Definition 6.1.

Given any formula $\varphi \in Ł_{\hat{\varepsilon}}$, let $\#\varphi$ be the number of instances of $\hat{\varepsilon}$ in φ.[17] For the axiom scheme of excision, we have

$$\forall \bar{y} \exists z \forall x(x \mathrel{\hat{\varepsilon}} z \vee (\neg\varphi \,\&\, \neg\varphi \,\&\, \neg\varphi)) \wedge ((\underbrace{\neg x \mathrel{\hat{\varepsilon}} z \,\&\, \cdots \,\&\, \neg x \mathrel{\hat{\varepsilon}} z}_{6\cdot\#\varphi \text{ times}}) \vee (\varphi \,\&\, \varphi \,\&\, \varphi)) \tag{2_φ}$$

for every formula $\varphi(x, \bar{y}) \in Ł_{\hat{\varepsilon}}$ that does not contain z as a free variable. As this is not a literal transcription of the axiom scheme of excision given in Definition 6.5, we need to prove that it is equivalent.

PROPOSITION 8.3. *An $\mathcal{L}_e$-structure M is* H*-extensional and satisfies $\mathcal{L}_e$-excision if and only if it satisfies* (1) *and* (2_φ) *for all $\varphi \in Ł_{\hat{\varepsilon}}$.*

PROOF. We clearly have that M is H-extensional if and only if it satisfies (1). Therefore we may assume without loss of generality that (M, d_e) is a complete metric space.

An easy inductive argument shows that for any $\varphi \in Ł_{\hat{\varepsilon}}$, $v(\varphi) = \#\varphi$. In particular, any such φ is $(2 \cdot \#\varphi)$-Lipschitz relative to d_e.

For the $\Rightarrow$ direction, by Proposition 6.7, $(M, d_e, \sqsubseteq_e) \models \mathsf{MSE}$. Fix a formula $\varphi(x, \bar{y}) \in Ł_{\hat{\varepsilon}}$ and a tuple $\bar{a} \in M$. If $\#\varphi = 0$, then $\varphi(x, \bar{a})$ is a constant that does not depend on $\bar{a}$, so (2_φ) is witnessed either by $\varnothing^M$ or by V^M. If $\#\varphi > 0$, consider the set $b = \left[x : 1 - \varphi(x, \bar{a}) \leq \frac{1}{3} \sim \frac{2}{3}\right]$. For any $c \in M$, we have that if $\varphi^M(c, \bar{a}) \geq \frac{2}{3}$, then $(c \mathrel{\hat{\varepsilon}} b)^M = 1$ and if $(c \mathrel{\hat{\varepsilon}} b)^M = 1$, then $\varphi^M(c, \bar{a}) > \frac{1}{3}$. In particular, this implies that if $\varphi^M(f, \bar{a}) \leq \frac{1}{3}$, then $(f \mathrel{\hat{\varepsilon}} b)^M \leq 1 - \frac{1}{6\cdot\#\varphi}$ (which implies that M satisfies $\neg f \mathrel{\hat{\varepsilon}} b \,\&\, \cdots \,\&\, \neg f \mathrel{\hat{\varepsilon}} b$ with $6 \cdot \#\varphi$ instances of $\neg f \mathrel{\hat{\varepsilon}} b$). Furthermore M satisfies

$$\forall x(x \mathrel{\hat{\varepsilon}} b \vee (\neg\varphi(x, \bar{a}) \,\&\, \neg\varphi(x, \bar{a}) \,\&\, \neg\varphi(x, \bar{a})))$$

and

$$\forall x(\underbrace{\neg x \mathrel{\hat{\varepsilon}} b \,\&\, \cdots \,\&\, \neg x \mathrel{\hat{\varepsilon}} b}_{6\cdot\#\varphi \text{ times}}) \vee (\varphi(x, \bar{a}) \,\&\, \varphi(x, \bar{a}) \,\&\, \varphi(x, \bar{a})).$$

Since we can do this for any $\varphi \in Ł_{\hat{\varepsilon}}$ and any $\bar{a} \in M$, we have that M satisfies (2_φ) for all $\varphi \in Ł_{\hat{\varepsilon}}$.

For the $\Leftarrow$ direction, assume that M satisfies (1) and (2_φ) for all $\varphi \in Ł_{\hat{\varepsilon}}$. Fix a restricted $\mathcal{L}_e$-formula $\varphi(x, \bar{y})$. Assume without loss of generality that $\varphi(x, \bar{y})$ contains an instance of the predicate e and that $\varphi(x, \bar{y})$ is in prenex max ANF. Pick a sufficiently large $\ell > 1$ and let $\varphi^\dagger(x, \bar{y})$ be defined as above. In particular, we may

[17] That is to say, the number of instances of $e(x, y)$ in the corresponding restricted $\mathcal{L}_e$-formula.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

think of $\varphi^\dagger(x,\bar{y})$ as a formula in $\text{Ł}_{\hat{\varepsilon}}$ which has the property that for any M and $a,\bar{b}\in M$, $(\varphi^\dagger)^M(a,\bar{b})=\min(\max(\varphi^M(a,\bar{b}),0),1)$.

Fix some $\bar{a}\in M$ and $\delta>0$ with $\delta<1$ and apply $(2_{\neg\varphi^\dagger})$ to $\bar{a}$ to get a $b\in M$ such that for every $x\in M$,

- either $(x\ \hat{\varepsilon}\ b)^M>1-\delta$ or $(\neg\neg\varphi^\dagger)^M(x,\bar{a})>\frac{1-\delta}{3}$ and
- either $(\neg x\ \hat{\varepsilon}\ b)^M>\frac{1-\delta}{6\cdot\#\varphi^\dagger}$ or $(\neg\varphi^\dagger)^M(x,\bar{a})>\frac{1-\delta}{3}$.

This means that for every $x\in M$,

- if $\varphi(x,\bar{a})\leq\frac{1-\delta}{3\cdot\ell!}$, then $e(x,b)<\delta$ and
- if $e(x,b)=0$, then $\varphi(x,\bar{a})<\frac{2+\delta}{3\cdot\ell!}$.

The first of these clearly implies that if $\varphi(x,\bar{a})\leq-\delta$, then $e(x,z)<\delta$. Now suppose that for some $c\in M$, $e(c,b)\leq\varepsilon_\varphi-\delta$. We then have that there is an $f\in M$ with $(f\ \hat{\varepsilon}\ b)^M=1$ such that $d(c,f)<\varepsilon_\varphi$. Since $(f\ \hat{\varepsilon}\ b)^M=1$, we have that $e(f,b)=0$, so $\varphi(f,\bar{a})<\frac{2+\delta}{3\cdot\ell!}$ and therefore $\varphi(c,\bar{a})<\varphi(f,\bar{a})+2v(\varphi)d(c,f)<\frac{2+\delta}{3\cdot\ell!}+2v(\varphi)\varepsilon_\varphi<\frac{1}{\ell!}+\frac{2v(\varphi)}{6v(\varphi)}<\frac{1}{2}+\frac{1}{3}<1<1+\delta$.Since we can do this for any sufficiently small $\delta>0$, we have that M satisfies the excision axiom for $\varphi(x,\bar{y})$. So since we can do this for any $\varphi(x,\bar{y})\in\mathcal{L}_e$, we have that M satisfies $\mathcal{L}_e$-excision. ⊣

It is of course also possible to translate our axioms of infinity and other sentences described in this article to Łukasiewicz logic, but doing so is more involved.

8.1. Comparison to CŁ$_0$. While we have been primarily conceptualizing MSE as a theory that describes closed subsets of a metric space, it is also of course a $[0,1]$-valued set theory in the same vein as CŁ$_0$. As such it makes sense to compare them. In set theory without a primitive equality notion, extensionality is often formalized as

$$\forall x\forall y(x=_l y\leftrightarrow x=_e y),\tag{Ext}$$

where $x=_l y:=\forall z(x\ \hat{\varepsilon}\ z\leftrightarrow y\ \hat{\varepsilon}\ z)$ is *Leibniz equality* (see [8] and [17, Chapter 4.5]). (Ext) is inconsistent with CŁ$_0$. It is also easy to see semantically that our H-extensionality axiom is strictly stronger than it. When rendered in continuous logic, (Ext) says that $\sup_z|e(x,z)-e(y,z)|=\sup_z|e(z,x)-e(z,y)|$ for any x and y. The H-extensionality axiom is equivalent to (Ext) plus the additional requirement that for any a, $x\mapsto e(x,a)$ is a distance predicate relative to the metric defined by $d(x,y):=\sup_z|e(x,z)-e(y,z)|=\sup_z|e(z,x)-e(z,y)|$. (Although, as evidenced by the structure of this article, we find the presentation of it in Definition 2.2 preferable.)

Uniform continuity is ultimately responsible for the fact that CŁ$_0$ is inconsistent with (Ext). Comprehension says that for any formula $\varphi(x)$, we can find a set a such that $e(x,a)=\varphi(x)$ for any x. Coincidence implies that $x\mapsto e(x,a)$ is always a 1-Lipschitz function with regards to the metric, but $\varphi(x)$ easily fails to be 1-Lipschitz. The results of this article, and in particular Proposition 3.7, have demonstrated that when restricted to a certain natural class of 1-Lipschitz formulas (namely distance predicates of definable classes), comprehension is consistent with (Ext) (in that it is consistent with the strong H-extensionality axiom). This prompted an anonymous referee to suggest the following question.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

QUESTION 8.4. *For which classes of* 1*-Lipschitz formulas is* (*Ext*) *consistent with the full comprehension scheme* $\exists y \forall x (x \mathrel{\hat{\varepsilon}} y \leftrightarrow \varphi(x))$ (*i.e.*, $\inf_y \sup_x |e(x,y) - \varphi(x)|$)?

To the best of our knowledge, there are no limitative results in this area. It's possible that (Ext) is consistent with comprehension for arbitrary 1-Lipschitz formulas. That said, it seems unlikely that the techniques of this article could easily be adapted to prove this, as they're fairly specialized to the case of $x \mapsto e(x,a)$ being a distance predicate.

Appendix A. A single axiom for distance predicates in continuous logic. Here we will give a single condition axiomatizing distance predicates of possibly non-empty sets in continuous logic. It is a slight modification of the condition E_2 in [25, Chapter 9] that obviates the need for E_1 and covers the case of empty definable sets (where the distance predicate of $\varnothing$ in a $[0,1]$-valued metric is the constant function 1).

It is possible to prove the following proposition using [25, Theorem 9.12], but doing so seems to roughly as hard as proving the equivalence directly, so we will do this for the sake of making this article more self-contained.

PROPOSITION A.1. *In any metric structure* M *with a* $[0,1]$*-valued metric, a formula* $\varphi(x)$ *is the distance predicate of a* (*possibly empty*) *definable set if and only if*

$$M \models \sup_x |\varphi(x) - \inf_y \min(d(x,y) + 2\varphi(y), 1)| = 0.$$

PROOF. First suppose that $\varphi(x)$ is the distance predicate of a definable set D in the structure M. If D is empty, then $\varphi(x) = r$ for all x and so $\min(d(x,y) + 2\varphi(y), 1) = r$ for any x and y, implying that $|\varphi(x) - \inf_y \min(d(x,y) + 2\varphi(y), 1)| = 0$ for every x.

If D is not empty, fix $x \in M$. Let $t = \inf_y \min(d(x,y) + 2\varphi(y), 1)$. We need to show that $t = \varphi(x)$. Fix an $\varepsilon > 0$ and $z \in D$ such that $d(x,z) < \varphi(x) + \varepsilon = \inf_{w \in D} d(x,w) + \varepsilon$. Clearly

$$t \leq \min(d(x,z) + 2\varphi(z), 1) = \min(d(x,z), 1) = d(x,z) < \varphi(x) + \varepsilon.$$

Since this is true for any $\varepsilon > 0$, we have that $t \leq \varphi(x)$.

On the other hand, fix some $y \in M$ and consider the quantity $\min(d(x,y) + 2\varphi(y), 1)$. Find a $w \in D$ such that $d(y,w) < \varphi(y) + \varepsilon$. We now have that

$$\varphi(x) \leq d(x,w) \leq d(x,y) + d(y,w) < d(x,y) + \varphi(y) + \varepsilon \leq d(x,y) + 2\varphi(y) + \varepsilon.$$

Since this is true for any $\varepsilon > 0$, we have that $\varphi(x) \leq d(x,y) + 2\varphi(y)$ for any $y \in M$. Therefore $\varphi(x) \leq t$, whereby $t = \varphi(x)$.

Now suppose that M satisfies the condition in the statement of the proposition. If $\varphi(x) = 1$ for all x, then $\varphi(x)$ is the distance predicate of the empty set. So assume that there is an $a \in M$ such that $\varphi(a) < 1$.

Note that for any x, we have that

$$\varphi(x) = \inf_y \min(d(x,y) + 2\varphi(y), 1) \leq \min(d(x,x) + 2\varphi(x), 1) = \min(2\varphi(x), 1).$$

Therefore it cannot be the case that $\varphi(x) < 0$, whereby we must have that $\varphi(x) \geq 0$ for all x.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

Let D be the zero set of $\varphi(x)$. For any $x \in M$ and $y \in D$, we have that $\varphi(x) \leq \min(d(x,y) + 2\varphi(y), 1) = d(x,y)$. Therefore $\varphi(x)$ is upper bounded by the distance to the set D. We now just need to show that for any $x \in M$ with $\varphi(x) < 1$ and any $\varepsilon > 0$, there is a $y \in D$ such that $d(x,y) < \varphi(x) + \varepsilon$.

Fix some $x \in M$ with $\varphi(x) < 1$ and some $\varepsilon > 0$. Find $\delta > 0$ such that $\varphi(x) + \delta < 1$ and $\delta < \varepsilon$. Find y_0 such that $\varphi(x) \leq \min(d(x,y_0) + 2\varphi(y_0), 1) < \varphi(x) + \delta$. Since $\varphi(x) + \delta < 1$, we actually have that

$$\varphi(x) \leq d(x,y_0) + 2\varphi(y_0) < \varphi(x) + \delta.$$

We will construct a sequence $(y_n)_{n\in\mathbb{N}}$ inductively. Our induction hypothesis will be that

$$d(x,y_0) + \sum_{i<n}(d(y_i,y_{i+1}) + \varphi(y_i)) + 2\varphi(y_n) < \varphi(x) + \delta.$$

This holds in the $n = 0$ case.

Given y_n satisfying the induction hypothesis, find a $\gamma > 0$ small enough that $\gamma < 2^{-n}$ and $\varphi(y_n) + \gamma < 1$ and

$$d(x,y_0) + \sum_{i<n}(d(y_i,y_{i+1}) + \varphi(y_i)) + 2\varphi(y_n) + \gamma < \varphi(x) + \delta.$$

Find a y_{n+1} such that $\varphi(y_n) \leq \min(d(y_n,y_{n+1}) + 2\varphi(y_{n+1}), 1) < \varphi(y_n) + \gamma$. Since $\varphi(y_n) + \gamma < 1$, we actually have that

$$\varphi(y_n) \leq d(y_n,y_{n+1}) + 2\varphi(y_{n+1}) < \varphi(y_n) + \gamma.$$

Clearly we get

$$\begin{aligned}
&d(x,y_0) + \sum_{i<n+1}(d(y_i,y_{i+1}) + \varphi(y_i)) + 2\varphi(y_{n+1}) \\
&= d(x,y_0) + \sum_{i<n}(d(y_i,y_{i+1}) + \varphi(y_i)) + \varphi(y_n) + d(y_n,y_{n+1}) + 2\varphi(y_{n+1}) \\
&< d(x,y_0) + \sum_{i<n}(d(y_i,y_{i+1}) + \varphi(y_i)) + \varphi(y_n) + \varphi(y_n) + \gamma \\
&< \varphi(x) + \delta,
\end{aligned}$$

and so the induction hypothesis holds for $n+1$. Note also that $0 \leq \varphi(y_{n+1}) < \frac{1}{2}\varphi(y_n)$.

Since $d(y_n,y_{n+1}) \leq \varphi(y_n) + \gamma < \varphi(y_n) + 2^{-n}$ and since $\varphi(y_{n+1}) < \frac{1}{2}\varphi(y_n)$ for each n, $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let y be its limit. Since $\lim_{n\to\infty}\varphi(y_n) = 0$, we must have that $\varphi(y) = 0$ or in other words that $y \in D$. By construction, we have that

$$d(x,y) \leq d(x,y_0) + \sum_{n\in\mathbb{N}} d(y_n,y_{n+1}) \leq \varphi(x) + \delta < \varphi(x) + \varepsilon.$$

Since we can do this for any $\varepsilon > 0$, we are done. ⊣

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

For $[0, r]$-valued metrics more generally, it is immediate that $\sup_x |\varphi(x) - \inf_y \min(d(x, y) + 2\varphi(y), r)| = 0$ is equivalent to $\varphi(x)$ being a distance predicate (where in this case the distance predicate of $\varnothing$ is the constant function r).

Appendix B. A model of MSE in which the intersection of two sets is not a set. Here we will give an example of a model MSE in which the intersection of two sets is not always a set. This construction is an easy example in continuous logic of two definable sets with undefinable intersection.

PROPOSITION B.1. *There is a model $M \models \mathsf{MSE}$ with $A, B \in M$ such that no element of M is coextensive with the intersection of A and B.*

PROOF. Let (Q, d) be a metric space consisting of an infinite sequence $(a_n, b_n)_{n\in\mathbb{N}}$ of pairs with $d(a_n, b_n) = 2^{-n}$ and $d(a_n, b_m) = 1$ for $n \neq m$. Let $A = \{a_n : n \in \mathbb{N}\}$ and $B = \{b_n : n \in \mathbb{N}\}$. By Theorem 7.15, we can build a model $N \models \mathsf{MSE}$ containing Q as a set of Quine atoms and containing A and B as sets. Let M be an $\aleph_0$-saturated (in the sense of $\mathcal{L}_e$-structures) elementary extension of N.

First we need to show that the intersection of $\{x \in M : M \models x \sqsubset A\}$ and $\{y \in M : M \models y \sqsubset B\}$ is non-empty in M. The partial type $\Sigma(x, y)$ generated by the conditions $x \sqsubset A$, $y \sqsubset B$, and $d(x, y) \leq 2^{-n}$ for each $n \in \mathbb{N}$ is finitely satisfiable in N and is therefore realized in M. Therefore, by $\aleph_0$-saturation, there exists an $f \in M$ such that $M \models f \sqsubset A$ and $M \models f \sqsubset B$. Now let $C \in M$ be an element satisfying that for any $f \in M$, if $M \models f \sqsubset C$, then $M \models f \sqsubset A$ and $M \models f \sqsubset B$. Clearly it must be the case that for any $n \in \mathbb{N}$, $M \models e(a_n, C) = 1$ and $M \models e(b_n, C) = 1$. Therefore the partial type $\Pi(x, y)$ generated by the condition $x \sqsubset A$, $y \sqsubset B$, $e(x, C) = 1$, $e(y, C) = 1$, and $d(x, y) \leq 2^{-n}$ for each $n \in \mathbb{N}$ is finitely satisfiable in M. By $\aleph_0$-saturation, it is realized in M, whereby there is a $g \in M$ such that $M \models g \sqsubset A$ and $M \models g \sqsubset B$ but $M \not\models g \sqsubset C$.

Since we can do this for any such C in M, the class $\{x \in M : M \models x \sqsubset A,\ M \models x \sqsubset B\}$ is not a set in M. ⊣

What's unclear at the moment is whether this is possible in an ω-model.

QUESTION B.2. *Is there an ω-model of* MSE *containing two sets with no intersection*?

REFERENCES

[1] X. CAICEDO, *Maximality of continuous logic*, ***Beyond First Order Model Theory***, vol. 1 (J. Iovino, editor), Chapman & Hall/CRC, Philadelphia, 2020.

[2] X. CAICEDO and J. N. IOVINO, *Omitting uncountable types and the strength of* [0, 1]*-valued logics*. ***Annals of Pure and Applied Logic***, vol. 165 (2014), no. 6, pp. 1169–1200.

[3] C. C. CHANG, *The axiom of comprehension in infinite valued logic*. ***Mathematica Scandinavica***, vol. 13 (1963), pp. 9–30.

[4] J. E. FENSTAD, *On the consistency of the axiom of comprehension in the Łukasiewicz infinite valued logic*. ***Mathematica Scandinavica***, vol. 14 (1964), no. 1, pp. 65–74.

[5] T. E. FORSTER, ***Set Theory with a Universal Set: Exploring an Untyped Universe***, Clarendon Press, Oxford, 1992.

[6] M. FORTI and F. HONSELL, *A general construction of hyperuniverses*. ***Theoretical Computer Science***, vol. 156 (1996), nos. 1–2, pp. 203–215.

[7] P. HÁJEK, J. PARIS, and J. SHEPHERDSON, *Rational Pavelka predicate logic is a conservative extension of Łukasiewicz predicate logic*. ***Journal of Symbolic Logic***, vol. 65 (2000), no. 2, pp. 669–682.

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press

[8] P. Hájek, *On arithmetic in the Cantor–Łukasiewicz fuzzy set theory*. ***Archive for Mathematical Logic***, vol. 44 (2005), pp. 763–782.

[9] J. Hanson, *Analog reducibility*. ***Journal of Logic and Computation***, vol. 31 (2021), pp. 1561–1597.

[10] ———, *Metric spaces are universal for bi-interpretation with metric structures*. ***Annals of Pure and Applied Logic***, vol. 174 (2023), no. 2, p. page 103204.

[11] ———, ***Definability and categoricity in continuous logic***, Ph.D. thesis, University of Wisconsin–Madison, 2020.

[12] M. R. Holmes and S. Wilshaw, *NF is consistent*, preprint, 2025.

[13] R. B. Jensen, *On the consistency of a slight (?) modification of quine's new foundations*. ***Synthese***, vol. 19 (1968), pp. 278–291.

[14] H. Jerome Keisler, *Model theory for real-valued structures*, preprint, 2020, arXiv:2005.11851.

[15] R. J. Malitz, ***Set theory in which the axiom of foundation fails***, Ph.D. thesis, University of California, 1976.

[16] R. McNaughton, *A theorem about infinite-valued sentential logic*. ***Journal of Symbolic Logic***, vol. 16 (1951), no. 1, pp. 1–13.

[17] F. Montagna, editor, ***Petr Hájek on Mathematical Fuzzy Logic***, Springer International Publishing, Switzerland, 2015.

[18] M. Randall Holmes, *Alternative axiomatic set theories*, ***The Stanford Encyclopedia of Philosophy*** (E. N. Zalta, editor), Metaphysics Research Lab, Stanford University, Stanford, California, 2021, edition, 2021.

[19] T. Skolem, *Bemerkungen zum komprehensionsaxiom. Dem andenken an heinrich scholz gewidmet*. ***Zeitschrift fur mathematische Logik und Grundlagen der Mathematik***, vol. 3 (1957), nos. 1–5, pp. 1–17.

[20] E. P. Specker, *The axiom of choice in quine's new foundations for mathematical logic*. ***Proceedings of the National Academy of Sciences***, vol. 39 (1953), no. 9, pp. 972–975.

[21] K. Tent and M. Ziegler, ***A Course in Model Theory***, Lecture Notes in Logic, Cambridge University Press, Cambridge, 2012.

[22] K. Terui, *A flaw in R.B. White's article "The consistency of the axiom of comprehension in the infinite-valued predicate logic of Łukasiewicz"*. 2014, Unpublished.

[23] E. Weydert, ***How to approximate the naive comprehension scheme inside of classical logic***, Ph.D. thesis, University of Bonn, 1989.

[24] R. B. White, *The consistency of the axiom of comprehension in the infinite-valued predicate logic of Łukasiewicz*. ***Journal of Philosophical Logic***, vol. 8 (1979), no. 1, pp. 509–534.

[25] I. B. Yaacov, C. Alexander Berenstein, W. Henson, and A. Usvyatsov, ***Model Theory for Metric Structures***, volume 2 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 2008, pp. 315–427.

DEPARTMENT OF MATHEMATICS
IOWA STATE UNIVERSITY
AMES, IOWA
USA
E-mail: jameseh@iastate.edu

https://doi.org/10.1017/jsl.2025.10147 Published online by Cambridge University Press