EXISTENCE IN NSOP$_1$ THEORIES[†]

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Abstract. We show that Kim-forking satisfies existence in all NSOP$_1$ theories.

Recently NSOP$_1$ theories have been significantly studied mainly due to that in NSOP$_1$ theories, nice properties, such as symmetry, transitivity, and the Independence Theorem, hold over models with Kim-independence, as proved in [4, 5]. Later in [1, 3], the results are extended 'over arbitrary sets' in any NSOP$_1$ theory with existence. The class of NSOP$_1$ theories is considered as the proper superclass of that of simple theories, where parallel theoretic development to that on simple theories is expected to be made. In particular all simple theories have existence. Therefore it is a burning question to know whether every NSOP$_1$ theory has existence.[1] We investigate this problem in this article and achieve a partial success. Namely we show that Kim-forking satisfies existence (over all sets) in all NSOP$_1$ theories. More precisely, the failure of existence cannot be witnessed by Morley sequences, in all NSOP$_1$ theories. This implies that any complete type in NSOP$_1$ T has $\bigcirc^K$-Morley sequences in it. Hence in such a theory, it makes sense to define a new kind of 'dividing' of a formula witnessed by such a sequence (which is known to be equivalent to the usual 'Kim-dividing' when the theory has existence), although we do not at all intend to push forward its study in this note.

In Section 1, we collect basic definitions and facts necessary for the rest of the article. In Section 2, preliminary lemmas playing key roles in proving the main theorem are presented. In Section 3, we state and prove our main theorem that Kim-forking satisfies existence in all NSOP$_1$ T. We will discuss further related issues in Section 4.

§1. Introduction. In this section we summarize necessary definitions and facts (from [2–6], etc.) which will be used freely throughout the article. We work with a complete theory T in a language $\mathcal{L}$ with its large model $\mathcal{M}$. As usual, (small) sets $A, B, C, \ldots$ and (possibly infinite) tuples $a_i, b_i, \ldots$ are from $\mathcal{M}$ and models $M, N, \ldots$ are small elementary submodels of $\mathcal{M}$. Given a sequence $(A_i \mid i < \kappa)$ and

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[1]Shortly after the first draft of this article was presented, Mutchnik showed that there is an NSOP$_1$ theory that does not satisfy the existence axiom (see [9]).

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$j' < j < \kappa$, in addition to $A_{<j}$, $A_{\geq j}$, and so on, we use $A_{j'<<j}$, $A_{\neq k}$ to denote $(A_k \mid j' < k < j)$, $(A_i \mid i(\neq k) \in \kappa)$, respectively.

We begin to recall basic definitions related to dividing and forking. We also recall the notion of a *Morley sequence over* (A, B) originated from [10], which is useful in developing our arguments.

DEFINITION 1.1.

(1) We say a formula $\varphi(x, a_0)$ *divides* over A if there is an A-indiscernible sequence $\langle a_i \mid i < \omega \rangle$ such that $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent. A formula *forks* over A if the formula implies some finite disjunction of formulas, each of which divides over A. Then we say a type divides/forks over A if the type implies a formula which divides/forks over A.
(2) We write $A \mathop{\bigcup}_B C$ if $\mathrm{tp}(A/BC)$ does not fork over B. A sequence $I = \langle a_i \mid i < \omega \rangle$ is said to be $\bigcup$*-independent* (or just *independent*) over A (or *A-independent*) if $a_i \mathop{\bigcup}_A a_{<i}$ for each $i < \omega$; if additionally $a_i \equiv_A a_0$ then we say it is A-independent *in* $\mathrm{tp}(a_0/A)$. We say I is *independent over* (A, B) if $a_i \mathop{\bigcup}_A a_{<i} B$ for all $i \in \omega$.
(3) An A-independent sequence $I = \langle a_i \mid i < \omega \rangle$ is said to be *Morley* over A (or *A-Morley*) if it is A-indiscernible. When I is Morley over A, we also call it a Morley sequence *in* $\mathrm{tp}(a_0/A)$. I is said to be *Morley over* (A, B) if it is independent over (A, B) and I is AB-indiscernible.

We now point out the following handy fact which we will use frequently. A proof can be found, say in [6].

FACT 1.2. *Given κ, there is $\lambda = \lambda(\kappa)$ such that for any sequence $J = \langle a_i : i < \lambda \rangle$ of $\leq \kappa$-sized tuples and any set A of size $\leq \kappa$, there is an A-indiscernible sequence $\langle b_i : i < \omega \rangle$ which is finitely based on J over A, i.e., for any $b_{\leq n}$ there are $a_{i_{\leq n}}$ with $i_0 < \cdots < i_n \in \lambda$ such that $b_{\leq n} \equiv_A a_{i_{\leq n}}$.*

Straightforward consequences from the definitions of dividing and forking are summarized as follows.

FACT 1.3.

(1) (*Finite character*) $A \mathop{\bigcup}_B C$ *iff for any finite* $a \in A$, $c \in C$ *we have* $a \mathop{\bigcup}_B c$.
(2) (*Base monotonicity*) *If* $D \mathop{\bigcup}_A BC$ *holds then* $D \mathop{\bigcup}_{AB} C$ *follows. Hence if* I *is Morley over* (A, B) *then it is Morley over* AB *as well.*
(3) (*Left transitivity*) *If* $C \mathop{\bigcup}_A B$ *and* $D \mathop{\bigcup}_{CA} B$ *then we have* $DC \mathop{\bigcup}_A B$.
(4) (*Extension*) *For* $A \subseteq B \subseteq C$, *if* $D \mathop{\bigcup}_A B$ *then there is* $D' \equiv_B D$ *such that* $D' \mathop{\bigcup}_A C$.
(5) *Using extension and Fact 1.2, the equivalence of the following is obtained.*
   (a) *Any type over a set does not fork over the set.*
   (b) *For any* $p \in S(A)$ *there is a Morley sequence in* p.
   (c) *Any consistent formula* $\varphi(x, a)$ *does not fork over its finite parameter* a.
(6) (*Existence*) *We say* T *has existence if one of the equivalent conditions in* (5) *holds. Notice that* $a \mathop{\bigcup}_M M$ *holds for every model* M. *Hence given any complete type over a model, there is a Morley sequence in it.*

As is well-known a typical example where existence (over $\emptyset$) fails to hold is a circle with a ternary relation R on the circle such that $R(a, b, c)$ holds iff a, c are

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not antipodal and b is in the smaller arc connecting a and c. We will discuss this example again in Section 4.

DEFINITION 1.4.

(1) A formula $\varphi(x, a_0)$ *Kim-divides* over A if there is an A-Morley sequence $\langle a_i \mid i < \omega \rangle$ such that $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent. A formula *Kim-forks* over A if the formula implies some finite disjunction of formulas, each of which Kim-divides over A. A type Kim-forks over A if the type implies a formula which Kim-forks over A.

(2) We write $A \mathop{\smile\!\!\!\!\!\!|}^{K}_{B} C$ if $\mathrm{tp}(A/BC)$ does not Kim-fork over B. Obviously $A \mathop{\smile\!\!\!\!\!\!|}_{B} C$ implies $A \mathop{\smile\!\!\!\!\!\!|}^{K}_{B} C$.

A sequence $I = \langle a_i \mid i < \omega \rangle$ is said to be $\mathop{\smile\!\!\!\!\!\!|}^{K}$-*independent* over A if $a_i \mathop{\smile\!\!\!\!\!\!|}^{K}_{A} a_{<i}$ for each $i < \omega$; if additionally $a_i \equiv_A a_0$ then we say it is $\mathop{\smile\!\!\!\!\!\!|}^{K}$-independent (over A) *in* $\mathrm{tp}(a_0/A)$.

(3) Given an A-indiscernible sequence $\langle a_i \mid i < \omega \rangle$ we say it is $\mathop{\smile\!\!\!\!\!\!|}^{K}$-*Morley* over A (or A-$\mathop{\smile\!\!\!\!\!\!|}^{K}$-*Morley*) if it is $\mathop{\smile\!\!\!\!\!\!|}^{K}$-independent over A; and we say it is a *total* $\mathop{\smile\!\!\!\!\!\!|}^{K}$-*Morley* sequence over A if $a_{\geq i} \mathop{\smile\!\!\!\!\!\!|}^{K}_{A} a_{<i}$ holds for each $i < \omega$. Due to base monotonicity and left transitivity, a Morley sequence over A is a total Morley sequence over A, so a total $\mathop{\smile\!\!\!\!\!\!|}^{K}$-Morley sequence over A as well.

In the definitions above if $A = \emptyset$ then we omit to state it. For example '$\varphi(x, a)$ (Kim-)divides/forks' means it does over $\emptyset$.

REMARK 1.5.

(1) Since we do not assume existence in Definition 1.4, given formulas $\varphi(x, a) \models \psi(x, b)$, it is unclear whether the following property holds: if $\psi(x, b)$ Kim-divides over A then so does $\varphi(x, a)$ over A. However this property holds with respect to Kim-forking (note that if $\varphi(x, a) \models \psi(x, b)$ and ψ implies a finite disjunction of Kim-dividing formulas, then φ also implies it). In particular, if $p(x) \models q(x)$ and $q(x)$ Kim-forks over A then so does $p(x)$ over A; Kim-forking of a type over A only depends on its solution set; and a type $p(x)$ Kim-forks over A iff $p(x)$ implies some finite disjunction of formulas, each of which Kim-divides over A.

(2) (Finite character for $\mathop{\smile\!\!\!\!\!\!|}^{K}$) $A \mathop{\smile\!\!\!\!\!\!|}^{K}_{B} C$ iff for any finite $a \in A$, $c \in C$ we have $a \mathop{\smile\!\!\!\!\!\!|}^{K}_{B} c$.

(3) (Extension for $\mathop{\smile\!\!\!\!\!\!|}^{K}$) By the same compactness argument we have the following: If a partial type p over C does not Kim-fork over a set then there is a completion of p over C which does not Kim-fork over the set. In particular, for $A \subseteq B \subseteq C$, if $a \mathop{\smile\!\!\!\!\!\!|}^{K}_{A} B$ then there is $a' \equiv_B a$ such that $a' \mathop{\smile\!\!\!\!\!\!|}^{K}_{A} C$.

(4) By the same techniques (using (3) and Fact 1.2) the equivalence of the following can be obtained.
   (a) Any type over a set does not Kim-fork over the set.
   (b) For any $p \in S(A)$ there is an $\mathop{\smile\!\!\!\!\!\!|}^{K}$-Morley sequence in p.
   We say Kim-forking has *existence* if one of the above equivalent conditions holds.

Now we recall a theory being (N)SOP$_1$, or simple.

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DEFINITION 1.6. We say a formula $\varphi(x, y) \in \mathcal{L}$ has SOP_1 if there is a tree of tuples $\{a_\eta \mid \eta \in 2^{<\omega}\}$ such that:

(1) for all $\xi \in 2^\omega$, $\{\varphi(x, a_{\xi\restriction\alpha}) \mid \alpha < \omega\}$ is consistent;
(2) for all $\eta \in 2^{<\omega}$, if $\nu \trianglerighteq \eta^\frown\langle 0\rangle$, then $\{\varphi(x; a_\nu), \varphi(x; a_{\eta^\frown\langle 1\rangle})\}$ is inconsistent,

where $\trianglelefteq$ denotes the tree partial order on $2^{<\omega}$. We say T has SOP_1 if some formula has SOP_1 in T, and say T is $NSOP_1$ if T *does not have* SOP_1.

FACT 1.7. [2, 4] *T has SOP_1 iff there is a formula $\varphi(x, y)$ and an indiscernible sequence $(b_i c_i \mid i < \omega)$ such that for each $i < \omega$, $b_i \equiv_{(bc)_{<i}} c_i$ (*); and $\{\varphi(x, b_i) \mid i \in \omega\}$ is consistent; while $\{\varphi(x, c_i) \mid i \in \omega\}$ is inconsistent. (Notice that by compactness the same holds even if we replace (*) by $b_i \equiv_{(bc)_{>i}} c_i$.)*

We say T is *simple* if $\mathop{\smile\!\!\!\!\!\!|}$ satisfies local character: For any set A and finite a, there is $A_0 \subseteq A$ with $|A_0| \leq |T|$ such that $a \mathop{\smile\!\!\!\!\!\!|}_{A_0} A$. As is well known, in simple T, existence holds; and dividing, forking, and Kim-forking are all equal over any set. Any simple T is NSOP_1. It is not known yet whether existence holds in all NSOP_1 theories. In this note, however, we show that Kim-forking satisfies existence in every NSOP_1 theory. We will only use the following two facts holding in any NSOP_1 theory.

FACT 1.8. [4] *Assume T is $NSOP_1$.*

(1) *A formula Kim-divides over a model M iff the formula Kim-forks over M. Moreover, $c \mathop{\smile\!\!\!\!\!\!|}^K_M M$ for any model M (since $c \mathop{\smile\!\!\!\!\!\!|}_M M$).*
(2) *(Kim's lemma for Kim-dividing over a model) A formula $\varphi(x, a_0)$ Kim-divides over a model M iff for any Morley sequence $\langle a_i \mid i < \omega\rangle$ over M, $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent.*

We state the following folklore which comes mainly from extension.

FACT 1.9. *Given a cardinal κ, let $I = \langle a_i \mid i < \kappa\rangle$ be $\mathop{\smile\!\!\!\!\!\!|}$-independent over (A, B).*

(1) *For any C there is $C' \equiv_{AB} C$ such that I is independent over (A, BC'). It follows that $I \mathop{\smile\!\!\!\!\!\!|}_A BC'$ and I is ABC'-independent.*
(2) *Suppose additionally κ is infinite and I is Morley over (A, B). Then again for any C, there is $C' \equiv_{AB} C$ such that I is Morley over (A, BC').*

PROOF. (1) It suffices to verify the first statement, since it implies the second one due to base monotonicity and left transitivity. Given $i < \kappa$, inductively assume that we have found $(b_j \mid j < i)$ such that $b_{<i} \equiv_{AB} a_{<i}$ and $(b_j \mid j < i)$ is independent over (A, BC) (*). It is enough (due to an AB-automorphism) to find b_i such that $b_{\leq i} \equiv_{AB} a_{\leq i}$ and $(b_j \mid j \leq i)$ is independent over (A, BC). Now by (*), there clearly is b'_i such that $b_{<i} b'_i \equiv_{AB} a_{<i} a_i$, so that $b'_i \mathop{\smile\!\!\!\!\!\!|}_A b_{<i}\, B$. Then by extension there is $b_i \equiv_{Ab_{<i}B} b'_i$ such that $b_i \mathop{\smile\!\!\!\!\!\!|}_A b_{<i}\, BC$. As wanted, $b_{\leq i} \equiv_{AB} b_{<i} b'_i \equiv_{AB} a_{\leq i}$ and $b_{\leq i}$ is independent over (A, BC).

(2) Due to compactness, we can stretch out $I_\omega := \langle a_i \mid i < \omega\rangle (\subseteq I)$ to $I_\lambda = \langle a_i \mid i < \lambda\rangle$ for some sufficiently large λ (to apply Fact 1.2), which is still Morley over (A, B). Then by (1), there at least is $C_0 \equiv_{AB} C$ such that I_λ is independent over (A, BC_0). Now we apply Fact 1.2 to obtain ABC_0-indiscernible $J = (b_i \mid i < \omega)$ finitely based on I_λ over ABC_0, so that clearly J is Morley over (A, BC_0). Notice that $J \equiv_{AB} I_\omega$, since I_ω is AB-indiscernible. Hence there is C_1 such that $C_1 \equiv_{AB} C_0$ and I_ω is Morley over (A, BC_1). Again by compactness we can extend I_ω to I' of

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length κ such that I' is Morley over (A, BC_1). It also follows $I \equiv_{AB} I'$ and by an AB-automorphism we find a desired $C' \equiv_{AB} C_1$ such that the original I is Morley over (A, BC'). ⊣

Throughout this note we give the lexicographic order to $\mu \times \lambda$ so that for an array $(c_{ij} \mid (i, j) \in \mu \times \lambda)$, $c_{<ij}$ means $\{c_{i'j'} \mid i' < i$, or $(i' = i$ and $j' < j)\}$, and so on.

Definition 1.10. Assume an array $I = (c_{ij} \mid i, j \in \omega)$ is given with $I_i := (c_{ij} \mid j \in \omega)$.

(1) Recall that I is said to be *strongly indiscernible* over A if $(I_i \mid i < \omega)$ is A-indiscernible and each I_i is $AI_{\neq i}$-indiscernible.
(2) I is said to be a *Morley array* over A if it is strongly indiscernible over A, and $c_{ij} \mathop{\smile\!\!\!\!\!\!|}_A c_{<ij}$ holds for each $(i, j) \in \omega \times \omega$.
(3) I is a *weak* Morley array* over A if $c_{ij} \mathop{\smile\!\!\!\!\!\!|}_A c_{<ij}$ holds for each $(i, j) \in \omega \times \omega$, each I_i is A-indiscernible; and if additionally $(I_i \mid i \in \omega)$ is an A-Morley sequence then we call I a *weak Morley array* over A.

If I is a Morley array over A, then mainly by left transitivity and base monotonicity, I is a weak Morley array.

§2. Preliminaries. In this section we will present a couple of key lemmas critical in showing the main result Theorem 3.2. Sophisticated applications of array and tree indiscernibilities keeping $\mathop{\smile\!\!\!\!\!\!|}$-independence enable us to prove those lemmas.

Lemma 2.1. *Assume that $I = (c_{ij} \mid i, j \in \omega)$ is an A-Morley array. Then for any $(A \subseteq)B$, there is $I' \equiv_A I$ such that I' is a weak B-Morley array.*

Proof. We choose μ, κ to be the cardinals $\lambda(|B| + |I|)$ and $\lambda(\mu)$, respectively, as described in Fact 1.2. By compactness we can stretch out I to $(c_{ij} \mid i \in \mu,\ j \in \kappa)$ while keeping strong indiscernibility over A. Then it must be A-Morley as well (*). By induction on μ we will construct the following sequences. ⊣

Claim 2.1.1. *For each $i < \mu$ there is $J_i = (c'_{mj} \mid m \leq i;\ j \in \omega)$ such that*:

(i) *$J_i \equiv_A (c_{mj} \mid m \leq i;\ j \in \omega)$ and for $i' < i$, $J_{i'} \subseteq J_i$;*
(ii) *J_i is a weak* B-Morley array.*

Proof of Claim 2.1.1. Given $i < \mu$, inductively assume that for all $i' < i$, we have found $J_{i'}$ satisfying (i) and (ii). Now put $J_{<i} := \bigcup_{i'<i} J_{i'} = (c'_{mj} \mid m < i,\ j < \omega)$ (so if i is a successor then $J_{<i} = J_{i-1}$). Then due to (i), there is $J'_i = (d'_{ij} \mid j < \kappa)$ such that $J_{<i}J'_i \equiv_A (c_{mj} \mid m < i,\ j \in \omega)(c_{ij} \mid j \in \kappa)$.

In particular, due to (*) above, $d'_{ij} \mathop{\smile\!\!\!\!\!\!|}_A d'_{i,<j} J_{<i}$ holds for each $j < \kappa$, that is, J'_i is independent over $(A, J_{<i})$. Thus by Fact 1.9(1) with an $AJ_{<i}$-automorphism, there is $J^*_i = (d_{ij} \mid j < \kappa)$ such that $J^*_i \equiv_{J_{<i}A} J'_i$ (so that $J_{<i}J^*_i \equiv_A (c_{mj} \mid m \in i,\ j \in \omega)(c_{ij} \mid j \in \kappa)$ as well (**)), while $d_{ij} \mathop{\smile\!\!\!\!\!\!|}_B J_{<i} d_{i,<j}$ $(\dagger_j)$ holds for each $j < \kappa$.

Finally, due to our choice of κ, we can apply Fact 1.2 to produce a desired $L = (c'_{ij} \mid j < \omega)$, which is finitely based on J^*_i over $J_{<i}B$. Then due to (**) and strong indiscernibility over A, clearly $J_i := J_{<i}L$ satisfies (i). Moreover L is B-indiscernible, thus by $(\dagger_j)$ and the induction hypothesis for $J_{<i}$, (ii) for i holds too. ⊣

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We put $J := \bigcup_{i<\mu} J_i = ((c'_{ij} \mid j \in \omega) \mid i \in \mu)$ as in Claim 2.1.1. Owing to the choice of μ, we can use Fact 1.2 to find B-indiscernible $I' = ((c''_{ij} \mid j \in \omega) \mid i \in \omega)$ finitely based on J over B. As wanted, $I' \equiv_A I$ because of strong indiscernibility. Now by left transitivity and base monotonicity, I' is a weak B-Morley array.

We begin to recall the notions of trees and tree indiscernibility from [3, 4, 8]. In this article, by a *tree* we mean a set $\mathcal{T}$ with a partial order $\trianglelefteq$ such that for every $x \in \mathcal{T}$, the set $\{y \mid y \trianglelefteq x\} \subseteq \mathcal{T}$ is linearly ordered by $\trianglelefteq$. For any $x, y \in \mathcal{T}$, if there is a $\trianglelefteq$-greatest element $z \trianglelefteq x, y$, then we call z the *meet* of x and y, and write it as $x \wedge y$. Trees in this article also possess suitable notions of *level* and $<_{\text{lex}}$. For example for ordinals α, γ, we can see $\gamma^{\leq\alpha}$ as a tree in the language $L_{s,\alpha} := \{\trianglelefteq, \wedge, <_{\text{lex}}, (P_\beta)_{\beta\leq\alpha}\}$ with the usual interpretations of $\trianglelefteq, \wedge, <_{\text{lex}}$, and $P_\beta = \{\eta \in \gamma^{\leq\alpha} \mid \text{dom}(\eta) = \beta\}$, the set of elements with level β. In the rest, all the $L_{s,\alpha}$-structures we will consider are only trees.

Definition 2.2. Suppose that an $L_{s,\alpha}$-structure I is given.

(1) We say $(a_i : i \in I)$ from $\mathcal{M}$ is *s-indiscernible* over A if whenever $(s_0, \ldots, s_n)$, $(t_0, \ldots, t_n)$ are tuples from I with

$$\text{qftp}_{L_{s,\alpha}}(s_0, \ldots, s_n) = \text{qftp}_{L_{s,\alpha}}(t_0, \ldots, t_n),$$

then we have $a_{s_0} \ldots a_{s_n} \equiv_A a_{t_0} \ldots a_{t_n}$.

(2) We say $(b_i \mid i \in I)$ is *locally based* on $(a_i : i \in I)$ over A if given any finite set of formulas Δ from $\mathcal{L}(A)$ and a finite tuple $(t_0, \ldots, t_n)$ from I, there is a tuple $(s_0, \ldots, s_n)$ from I such that

$$\text{qftp}_{L_{s,\alpha}}(t_0, \ldots, t_n) = \text{qftp}_{L_{s,\alpha}}(s_0, \ldots, s_n)$$

and

$$\text{tp}_\Delta(b_{t_0} \ldots b_{t_n}) = \text{tp}_\Delta(a_{s_0} \ldots a_{s_n}).$$

(3) We say I-indexed trees have the *modeling property* if given any tree $(a_i : i \in I)$ indexed by I and any set A there is s-indiscernible $(b_i : i \in I)$ over A locally based on $(a_i : i \in I)$ over A.

Fact 2.3. [8, *Theorem* 4.3] *In any T, every $\omega^{<\omega}$-indexed tree has the modeling property, where $\omega^{<\omega}$ is considered as an $L_{s,\omega}$-structure.*

We now recall from [3, 4] an $L_{s,\alpha}$-tree $\mathcal{T}_\alpha$, while *making one change* as suggested in [7]. Namely we consider all $\beta \leq \alpha$ in the following definition (1) (so that every $\mathcal{T}_\alpha$ tree has a root), in order to make induction steps more palatable when we produce such trees of parameters possessing desired properties.

Definition 2.4. Suppose α is an ordinal. We define $\mathcal{T}_\alpha$ to be the set of functions η such that:

(1) $\text{dom}(\eta)$ is an end-segment of α of the form $[\beta, \alpha)$ for $\beta \leq \alpha$;
(2) $\text{ran}(\eta) \subseteq \omega$;
(3) (finite support) the set $\{\gamma \in \text{dom}(\eta) : \eta(\gamma) \neq 0\}$ is finite.

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We interpret $\mathcal{T}_\alpha$ as an $L_{s,\alpha}$-structure as follows:

- We put $\eta \trianglelefteq \xi$ if $\eta \subseteq \xi$. Write $\eta \perp \xi$ if $\neg(\eta \trianglelefteq \xi)$ and $\neg(\xi \trianglelefteq \eta)$.
- $\eta \wedge \xi := \eta{\restriction}[\beta, \alpha) = \xi{\restriction}[\beta, \alpha)$, where
$$\beta = \min\{\gamma \leq \alpha : \eta{\restriction}[\gamma, \alpha) = \xi{\restriction}[\gamma, \alpha)\}$$
(by finite support, β is not a limit ordinal).
- Put $\eta <_{\text{lex}} \xi$ if $\eta \lhd \xi$ or, $\eta \perp \xi$ with $\text{dom}(\eta \wedge \xi) = [\gamma + 1, \alpha)$ and $\eta(\gamma) < \xi(\gamma)$.
- For $\beta \leq \alpha$ and $\eta : [\beta, \alpha) \to \omega$ in $\mathcal{T}_\alpha$, we write $|\eta| := \beta$, the *level* of η. Then we put $P_\beta := \{\eta \in \mathcal{T}_\alpha : |\eta| = \beta\}$. Note that P_0 is the set of *top* level elements, and $P_\alpha = \{\emptyset\}$ is the set of the root.

DEFINITION 2.5. Suppose α, β are ordinals.

(1) (Concatenation) Suppose that $\eta : [\beta, \alpha) \to \omega$ in $\mathcal{T}_\alpha$ is given where $\beta \leq \alpha$. For $i < \omega$, we let $\eta^\frown\langle i\rangle$ denote the function $\eta \cup \{(\gamma, i)\} : [\gamma, \alpha) \to \omega$ in $\mathcal{T}_\alpha$, if $\beta = \gamma + 1$. In particular, $\langle i\rangle := \emptyset^\frown\langle i\rangle$ is in $\mathcal{T}_\alpha$ for *any successor* α.
On the other hand we let $\langle i\rangle^\frown\eta$ denote the function $\eta \cup \{(\alpha, i)\} : [\beta, \alpha + 1) \to \omega$, which is (due to our definition, always) in $\mathcal{T}_{\alpha+1}$.

(2) (Canonical inclusions) If $\beta \leq \alpha$, we define the map $\iota_{\beta\alpha} : \mathcal{T}_\beta \to \mathcal{T}_\alpha$ by $\iota_{\beta\alpha}(\eta) := \eta \cup \{(\gamma, 0) : \gamma \in [\beta, \alpha)\}$. Note that due to finite support, if α is a limit ordinal and $\xi(\neq \emptyset) \in \mathcal{T}_\alpha$, then there is $\beta < \alpha$ and $\eta \in \mathcal{T}_\beta$ such that $\iota_{\beta\alpha}(\eta) = \xi$, and hence $\mathcal{T}_\alpha = \bigcup_{\beta<\alpha} \iota_{\beta\alpha}(\mathcal{T}_\beta) \cup \{\emptyset\}$.

(3) (The all 0's path) Given $\beta \leq \alpha$, $\zeta_\beta \in \mathcal{T}_\alpha$ denotes the function with $\text{dom}(\zeta_\beta) = [\beta, \alpha)$ and $\zeta_\beta(\gamma) = 0$ for all $\gamma \in [\beta, \alpha)$.

We recall that due to compactness applied to Fact 2.3 we have the following modeling theorem for $\mathcal{T}_\alpha$ trees.

FACT 2.6. (T *is any theory.*) *For any ordinal* α, *all* $\mathcal{T}_\alpha$*-indexed trees have the modeling property.*

In the rest we assume trees of parameters are indexed by $L_{s,\alpha}$-trees having the modeling property (which we may simply say 'modeling' or 's-modeling').

DEFINITION 2.7. A sequence $(A_i \mid i < \kappa)$ of trees of parameters is called *mutually s-indiscernible* over B if for each $i < \kappa$, A_i is s-indiscernible over $BA_{\neq i}$.

We will freely use the following remark throughout the article.

REMARK 2.8.

(1) Let C be a tree of parameters. If C has the unique root $c_\emptyset$ then C is s-indiscernible over A if and only if it is s-indiscernible over $Ac_\emptyset$. Now for any set B, due to s-modeling there is C' which is s-indiscernible over B locally based on C over B. If already C is s-indiscernible over $B'(\subseteq B)$ then it easily follows that $C \equiv_{B'} C'$.

(2) If a sequence $(A_i \mid i < \omega)$ of trees is B-indiscernible and mutually s-indiscernible over B, then it is s-indiscernible over B.

(3) For sets A, B and a tree of parameters $C = (c_\eta \mid \eta \in I)$, suppose that $A \mathop{\smile\!\!\!\!\!\!|}_B BC$ and C is s-indiscernible over B. If $C' = (c'_\eta \mid \eta \in I)$ is s-indiscernible over AB and locally based on C over AB, then still $A \mathop{\smile\!\!\!\!\!\!|}_B BC'$ holds: If not, then there

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is a forking formula $\varphi(x, b\bar{c}'_{\bar{\eta}})$ with finite $b \in B$ and a finite tuple $\bar{c}'_{\bar{\eta}} \in C'$, realized by finite $a \in A$. Owing to local basedness, there is $\bar{c}_{\bar{\nu}} \in C$ such that $\varphi(a, b\bar{c}_{\bar{\nu}})$ holds, where $\mathrm{qftp}(\bar{\eta}) \equiv_{L_{s,\alpha}} \mathrm{qftp}(\bar{\nu})$. Since C' is s-indiscernible over AB, we have $\bar{c}'_{\bar{\eta}} \equiv_{AB} \bar{c}'_{\bar{\nu}}$, in particular $\models \varphi(a, b\bar{c}'_{\bar{\nu}})$ as well, and $\varphi(x, b\bar{c}'_{\bar{\nu}})$ forks. Now due to (1), it follows $C' \equiv_B C$. Hence $\varphi(x, b\bar{c}_{\bar{\nu}})$ is also a forking formula realized by a, which contradicts $A \bigcup BC$.

The following lemma plays a key role in proving the main theorem. To show the lemma, we borrow techniques from [1] in order to produce s-indiscernible trees preserving suggested independence relations.

LEMMA 2.9. *Suppose that $C_0 = (a_\eta \mid \eta \in \mathcal{T}_\alpha)$ is s-indiscernible and $I = \langle b_i \mid i < \omega \rangle$ is Morley, where $b_0 = a_\emptyset \in C_0$. Then for any B, there is a sequence $L = ((a_{\eta,i} \mid \eta \in \mathcal{T}_\alpha) \mid i < \omega)$ and $B_0 \equiv B$ such that*:

(1) $C_0 = (a_{\eta,0} \mid \eta \in \mathcal{T}_\alpha)$;
(2) $\langle a_{\emptyset,i} \mid i < \omega \rangle \equiv I$ *and* $\langle a_{\emptyset,i} \mid 0 < i < \omega \rangle$ *is Morley over* $(\emptyset, C_0B_0)$;
(3) L *is* B_0*-indiscernible and mutually s-indiscernible over* B_0 (*so s-indiscernible over* B_0 *as well*).

PROOF. We first choose κ large enough with respect to $|B| + |C_0|$, and extend I to $I_\kappa = \langle b_i \mid i < \kappa \rangle$, which is still Morley. ⊣

CLAIM 2.9.1. *There are $B_0 \equiv B$ and $J = (d_i \mid i < \kappa) \equiv_{b_0} I_\kappa$ with $b_0 = d_0$ such that*:

(i) J *is Morley over* $(\emptyset, B_0)$;
(ii) C_0 *is s-indiscernible over* B_0J;
(iii) $J \setminus \{b_0\}$ *is independent over* $(\emptyset, C_0B_0)$.

PROOF OF CLAIM 2.9.1. By Fact 1.9(2), there is $B'_0 \equiv B$ such that I_κ is Morley over $(\emptyset, B'_0)$. Now by modeling, there is $(a_\emptyset \in)C'_0$ such that C'_0 is s-indiscernible over $a_\emptyset B'_0$ locally based on C_0 over $a_\emptyset B'_0$.

Notice now that $C_0 \setminus \{a_\emptyset\}$ is s-indiscernible over $a_\emptyset$. Thus by Remark 2.8(1) we have $C_0 \equiv_{a_\emptyset} C'_0$. Hence there is $J'B_0$ such that $J'B_0C_0 \equiv_{a_\emptyset} I_\kappa B'_0C'_0$. In particular C_0 is s-indiscernible over B_0 and J' is Morley over $(\emptyset, B_0)$ (*). In order to further satisfy (ii) and (iii), we will find a suitable $J \equiv_{B_0} J'$ as follows.

Write $J' := (d'_i \mid i < \kappa)$, where $d'_0 = b_0$. Using induction on κ, we will construct $J = (d_i \mid i < \kappa)$ $(d_0 = d'_0)$ such that for each $0 < i < \kappa$,

(a) $d_{\leq i} \equiv_{d_0B_0} d'_{\leq i}$;
(b) C_0 is s-indiscernible over $d_{\leq i}B_0$;
(c) $d_i \bigcup B_0C_0d_{<i}$.

Hence given $0 < i < \kappa$, assume that we have obtained d_j for all $(0 <)j < i$ (as said $d_0 = d'_0 = b_0$) satisfying (a)–(c). Now by extension with (a) and (*) above, there is d^*_i such that $d_{<i}d^*_i \equiv_{d_0B_0} d'_{\leq i}$ and $d^*_i \bigcup d_{<i}B_0C_0$ (**). Then by modeling there is C^*_0 s-indiscernible over $d_{<i}d^*_iB_0$ locally based on C_0 over $d_{<i}d^*_iB_0$ (***). By the induction hypothesis for (b) (even for $j = 0$) and Remark 2.8(1), we then have that $C^*_0 \equiv_{d_{<i}B_0} C_0$. Thus there is a wanted d_i such that $d_iC_0 \equiv_{d_{<i}B_0} d^*_iC^*_0$. Therefore (a) and (b) for i hold. Lastly, due to Remark 2.8(3) applied to (**) and (***) with the induction hypothesis for (b), we have $d^*_i \bigcup d_{<i}B_0C^*_0$. Hence (c) for i also holds.

In conclusion we have produced J; and due to (a)–(c), together with B_0C_0, Claim 2.9.1 is verified. ⊣

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We continue to work with $J = (d_i \mid i < \kappa)$ and B_0 produced in Claim 2.9.1. We further claim the following.

CLAIM 2.9.2. *For each $j < \kappa$, there is C_j such that*:

(i)* $d_{\geq j} C_j \equiv_{B_0} JC_0$;

(ii)* $\langle C_0, \dots, C_j \rangle$ *is mutually s-indiscernible over* $d_{>j} B_0$.

PROOF OF CLAIM 2.9.2. We suppose that for every $j' < j$ we have obtained $C_{j'}$ with the properties holding. If $j = 0$ then clearly C_0 (with Claim 2.9.1(ii)) serves it.

Now consider the case $0 < j$. Due to (ii)* for all $j' < j$ we have that $(C_{j'} \mid j' < j)$ is mutually s-indiscernible over $d_{\geq j} B_0$ $(\star)$.

Moreover there is D^*_j such that $d_{\geq j} D^*_j \equiv_{B_0} JC_0$, since J is B_0-indiscernible (Claim 2.9.1(i)). Hence d_j is the root of the tree D^*_j, and D^*_j is s-indiscernible over $d_{\geq j} B_0$, by Claim 2.9.1(ii). Then by modeling we take D'_j which is, over $d_{\geq j} C_{<j} B_0$, s-indiscernible and locally based on D^*_j (†). By Remark 2.8(1) we have $D'_j \equiv_{d_{\geq j} B_0} D^*_j$ as well (‡).

By now we choose $(D_{j'} \mid j' < j)$ as follows. If for all $k < j' (< j)$ we have selected D_k, then by modeling we choose $D_{j'}$ s-indiscernible over $D_{<j'} C_{j'<<j} D'_j d_{\geq j} B_0$ locally based on $C_{j'}$. We will show the following two properties:

(a)* The resulting $\{D_k \mid k < j\} \cup \{D'_j\}$ is mutually s-indiscernible over $d_{>j} B_0$.

(b)* $D_{<j} \equiv_{d_{\geq j} B_0} C_{<j}$.

For (a)*, we firstly show that given $k < j$, D_k is s-indiscernible over $D_{\neq k} D'_j d_{>j} B_0$. To lead a contradiction let us deny this. Hence there are

$$k_0 < \cdots < k_i = k < \cdots < k_n < j \ (i \leq n);$$

$\varphi(\bar{x}_0, \dots, \bar{x}_n, \bar{y}) \in \mathcal{L}(d_{>j} B_0)$; finite tuples of indices $\bar{\eta}_0, \dots, \bar{\eta}_n, \bar{\mu}_i, \bar{\nu} \in \mathcal{T}_\alpha$ with

$$\mathrm{qftp}(\bar{\eta}_i) \equiv_{L_{s,\alpha}} \mathrm{qftp}(\bar{\mu}_i)$$

and corresponding tuples $\bar{u}^0_{\bar{\eta}_0} \in D_{k_0}, \dots, \bar{u}^n_{\bar{\eta}_n} \in D_{k_n}, \bar{u}^i_{\bar{\mu}_i} \in D_{k_i}, \bar{w}_{\bar{\nu}} \in D'_j$ such that

$$\models \varphi(\bar{u}^0_{\bar{\eta}_0}, \dots, \bar{u}^i_{\bar{\eta}_i}, \dots, \bar{u}^n_{\bar{\eta}_n}, \bar{w}_{\bar{\nu}}) \wedge \neg\varphi(\bar{u}^0_{\bar{\eta}_0}, \dots; \bar{u}^i_{\bar{\mu}_i}; \dots, \bar{u}^n_{\bar{\eta}_n}, \bar{w}_{\bar{\nu}}).$$

By the construction of $D_{k_{i+1}}, \dots, D_{k_n}$ and local basedness, we can consecutively find $\bar{v}^n_{\bar{\nu}_n} \in C_{k_n}, \dots, \bar{v}^{i+1}_{\bar{\nu}_{i+1}} \in C_{k_{i+1}}$ (where of course $\mathrm{qftp}(\bar{\eta}_n) \equiv_{L_{s,\alpha}} \mathrm{qftp}(\bar{\nu}_n)$, and so on) such that

$$\varphi(\bar{u}^0_{\bar{\eta}_0}, \dots, \bar{u}^i_{\bar{\eta}_i}; \bar{v}^{i+1}_{\bar{\nu}_{i+1}}, \dots, \bar{v}^n_{\bar{\nu}_n}, \bar{w}_{\bar{\nu}}) \wedge \neg\varphi(\bar{u}^0_{\bar{\eta}_0}, \dots; \bar{u}^i_{\bar{\mu}_i}; \bar{v}^{i+1}_{\bar{\nu}_{i+1}}, \dots, \bar{v}^n_{\bar{\nu}_n}, \bar{w}_{\bar{\nu}})$$

holds. But then due to the s-indiscernibility of $D_k = D_{k_i}$ over the rest of the parameters, this leads to a contradiction.

Secondly, by the similar arguments with (†) above, it also easily follows that D'_j is s-indiscernible over $D_{<j} d_{>j} B_0$. We have shown (a)*.

Now (b)* can be shown straightforwardly, by iterated applications of Remark 2.8(1) using $(\star)$ above and the way of choosing $D_{<j}$.

We can now return our attention to finding wanted C_j. Due to (b)* above, there is C_j such that

$$C_{<j} C_j \equiv_{d_{\geq j} B_0} D_{<j} D'_j.$$

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Then due to above (‡) and (a)*, it follows that C_j and $C_{\leq j}$ satisfy (i)* and (ii)* for j. We have shown Claim 2.9.2. ⊣

We are about to finish our proof of Lemma 2.9. We work again with the selected $J = (d_i \mid i < \kappa)$ and B_0 from Claim 2.9.1, and $\langle C_i \mid i < \kappa \rangle$ from Claim 2.9.2. In particular, $d_{\geq i} C_i \equiv_{B_0} JC_0$ and $(C_i \mid i < \kappa)$ is mutually s-indiscernible over B_0 (♯). Now since κ is sufficiently large, by Fact 1.2, there is B_0-indiscernible $(C_i' \mid i < \omega)$ finitely based on $(C_i \mid i < \kappa)$ over B_0. By an automorphism we can assume $C_0' = C_0$. Then we put a wanted L as $(C_i' \mid i < \omega)$. Clearly (1) holds. Since $J \equiv I_\kappa$ (Claim 2.9.1), the first condition of (2) is satisfied. The second condition of (2) holds as well, due to that $(C_i' \mid 0 < i < \omega)$ is $B_0 C_0$-indiscernible, and Claim 2.9.1(iii) with (♯). Moreover (3) comes from (♯). The proof of Lemma 2.9 is completed.

§3. Existence and the main theorem. In this section we consider the case where Kim-forking fails to have existence. Hence after naming parameters, we have consistent formulas $\psi(x) \in \mathcal{L}$ and $\varphi_i(x, b_i)$ $(i \leq n)$ such that $\models \psi(x) \to \varphi_0(x, b_0) \vee \cdots \vee \varphi_n(x, b_n)$, and each $\varphi_i(x, b_i)$ Kim-divides. Then replacing $\varphi_i(x, b_i)$ by $\psi(x) \wedge \varphi_i(x, b_i)$ (and renaming it as $\varphi_i(x, b_i)$) allow us the following.

In this section we assume

$$\models \psi(x) \leftrightarrow \varphi_0(x, b_0) \vee \cdots \vee \varphi_n(x, b_n)$$

holds (all are consistent formulas) and each $\varphi_i(x, b_i)$ $(i \leq n)$ Kim-divides witnessed by a Morley sequence $I_i = (b_i^j \mid j < \omega)$, where $b_i^0 = b_i$. We now put $D := b_0 \dots b_n$. Notice that for any $b_i' \equiv b_i$, $\varphi_i(x, b_i') \models \psi(x)$, and for any $b_0' \dots b_n' \equiv D$, $\psi(x)$ is equivalent to $\varphi_0(x, b_0') \vee \cdots \vee \varphi_n(x, b_n')$ as well.

LEMMA 3.1. *Given a cardinal κ and each ordinal $\alpha < \kappa$, there is s-indiscernible $C_\alpha = (a_\eta \mid \eta \in \mathcal{T}_\alpha)$ and $D_\alpha = b_0^\alpha \dots b_n^\alpha \equiv D$, and there is $f : \kappa \to n+1$, which satisfy the following*:

(1) *Suppose that $\alpha = \gamma + 1$. Then*
   (a) $(a_{\langle i \rangle} \mid i \in \omega) \equiv I_{f(\gamma)}$ *(where $\langle i \rangle \in \mathcal{T}_\alpha$)*;
   (b) *for each $0 < i < \omega$, we have that*

$$a_{\langle i \rangle} \downarrow a_{\trianglerighteq \langle 0 \rangle} (a_{\langle j \rangle} \mid j < i) D_\alpha.$$

(2) *C_α is s-indiscernible over D_α, and $a_\emptyset = b^\alpha_{f(\alpha)} \in D_\alpha$.*
(3) *For every $\eta \in P_0$, $\{\varphi_{f(|\nu|)}(x, a_\nu) \mid \nu \trianglelefteq \eta\}$ is consistent.*
(4) *For $\alpha' < \alpha$, $C_{\alpha'} \subseteq C_\alpha$ via the canonical inclusion.*

PROOF. Suppose that we have determined C_β, D_β, and $f(\beta)$ for all $\beta < \alpha$ satisfying the properties. We want to choose C_α, D_α, and $f(\alpha)$.

If $\alpha = 0$ then choose C_0 to be any $b_i \in D_0 := D$ and $f(0) = i$. This choice clearly works.

Now if α is a limit ordinal, notice that $C_{<\alpha}$, the union of C_β for all $\beta < \alpha$ is also s-indiscernible, due to the canonical inclusion. Then by modeling we choose $C'_{<\alpha}$, which is s-indiscernible over D and locally based on $C_{<\alpha}$ over D. Then since $C_{<\alpha}$ is s-indiscernible, we have $C_{<\alpha} \equiv C'_{<\alpha}$. Hence there is a wanted $D_\alpha = b_0^\alpha \dots b_n^\alpha$ such that $C_{<\alpha} D_\alpha \equiv C'_{<\alpha} D$, so $C_{<\alpha}$ is s-indiscernible over D_α (*). Notice that by the induction

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hypothesis with finite support of $\mathcal{T}_{<\alpha}$, given $\eta \in P_0$ (of $\mathcal{T}_{<\alpha}$), $\{\varphi_{f(|\nu|)}(x, a_\nu) \mid \nu \trianglelefteq \eta\}$ is also consistent. Moreover due to s-indiscernibility, for $\nu \in P_\beta$, we have $a_\nu \equiv b_{f(\beta)}$. Thus obviously

$$\{\varphi_{f(|\nu|)}(x, a_\nu) \mid \nu \trianglelefteq \eta\} \models \psi(x).$$

Since $\models \psi(x) \leftrightarrow \varphi_0(x, b_0^\alpha) \vee \cdots \vee \varphi_n(x, b_n^\alpha)$ as well, there is some $i_0 \leq n$ such that

$$\{\varphi_{f(|\nu|)}(x, a_\nu) \mid \nu \trianglelefteq \eta\} \cup \{\varphi_{i_0}(x, b_{i_0}^\alpha)\}$$

is consistent (**). Then we put $f(\alpha) := i_0$ and $a_\emptyset := b_{i_0}^\alpha$ (in C_α indexed by $\mathcal{T}_\alpha$). Since $a_\emptyset \in D_\alpha$, C_α is s-indiscernible over D_α, due to (*) above. The rest of (2) and (4) for α are already satisfied. Moreover since C_α is s-indiscernible over its root $b_{i_0}^\alpha$, (3) for C_α also holds by (**).

It remains to check when $\alpha = \gamma + 1$. By the induction hypothesis, C_γ is s-indiscernible. Then by applying Lemma 2.9 to C_γ and Morley $I_{f(\gamma)}$, we find a sequence $L = ((a_{\eta,i} \mid \eta \in \mathcal{T}_\alpha) \mid i < \omega)$ with $D_\alpha \equiv D$ such that:

(i) $C_\gamma = (a_{\eta,0} \mid \eta \in \mathcal{T}_\alpha)$;
(ii) $\langle a_{\emptyset,i} \mid i < \omega\rangle \equiv I_{f(\gamma)}$ and $\langle a_{\emptyset,i} \mid 0 < i < \omega\rangle$ is Morley over $(\emptyset, C_\gamma D_\alpha)$;
(iii) L is D_α-indiscernible and mutually s-indiscernible over D_α (so s-indiscernible over D_α as well).

In order to find a wanted tree $C_\alpha = (a_\eta \mid \eta \in \mathcal{T}_\alpha)$, we canonically put $a_{\triangleright\emptyset} := L$. Then (1) for α is fulfilled due to (ii) above. Now using the same argument in the previous paragraph, we can find $j_0 \leq n$ so that we put $f(\alpha) := j_0$; $a_\emptyset := b_{j_0}^\alpha \in D_\alpha$; and (2)–(4) for α again follow due to (iii). ⊣

We are ready to show the main theorem.

THEOREM 3.2. *Assume T is NSOP$_1$. Then Kim-forking satisfies existence. Equivalently any complete type has an $\bigcup^K$-Morley sequence in it.*

PROOF. We keep our starting assumption in this section that Kim-forking fails to have existence, and work with chosen $b_0 \ldots b_n$ and I_i. We will deduce a contradiction under T being NSOP$_1$ (which is preserved regardless of naming parameters).

We also work with Lemma 3.1 and selected C_α. There we take κ sufficiently large (to apply Fact 1.2) and in the tree $C_{<\kappa} = \bigcup_{\alpha<\kappa} C_\alpha = (a_\eta \mid \eta \in \mathcal{T}_{<\kappa})$, we consider the sequence $\langle L_\alpha \mid \alpha < \kappa\rangle$, where $L_\alpha = (a_{\zeta_{\alpha+1}{}^\frown\langle i\rangle} \mid i \in \omega)$. By Fact 1.2, there is an indiscernible sequence

$$J_{<\omega} := \langle J_i = (c_{i,j} \mid j < \omega) \mid i < \omega\rangle$$

finitely based on $\langle L_\alpha \mid \alpha < \kappa\rangle$. Notice that due to Lemma 3.1, we have that all c_{ij} and some b_i, say b_0 have the same type; and $\{\varphi_0(x, c_{i,0}) \mid i < \omega\}$ is consistent. Moreover due to s-indiscernibility, each J_i is $J_{>i}$-indiscernible. Then since T is NSOP$_1$, it follows that $\{\varphi_0(x, c_{i,1}) \mid i < \omega\}$ is consistent as well, by Fact 1.7. Again due to s-indiscernibility from Lemma 3.1, we more have that

$$J'_{<\omega} := \langle J'_i = (c_{i,j} \mid 1 \leq j < \omega) \mid i < \omega\rangle$$

is a Morley array, in particular strongly indiscernible.

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Now there is a model M such that $J'_{<\omega}$ is weak Morley over M, due to Lemma 2.1. Hence both $(c_{i,1} \mid i < \omega)$ and $(c_{0,j} \mid 1 \leq j < \omega)$ are Morley in $\mathrm{tp}(c_{0,1}/M)$. However since $(c_{0,j} \mid 1 \leq j < \omega) \equiv I_0$ as in Lemma 3.1, $\{\varphi_0(x, c_{0,j}) \mid 1 \leq j < \omega\}$ is inconsistent. This contradicts Kim's lemma over a model in $\mathrm{NSOP}_1 T$. We have proved the theorem. ⊣

We give two remarks before finishing this section.

Remark 3.3.

(1) If an arbitrary theory T satisfies existence, then existence for $\mathop{\smile\!\!\!\!\!\!|}^K$ follows trivially since $\mathop{\smile\!\!\!\!\!\!|}$ implies $\mathop{\smile\!\!\!\!\!\!|}^K$. On the other hand, if in T there is no Morley sequence over $\emptyset$ at all, then existence for $\mathop{\smile\!\!\!\!\!\!|}^K$ over $\emptyset$ again follows trivially since there is no Kim-dividing formula over $\emptyset$.
However even if existence fails to hold in an $\mathrm{NSOP}_1 T$ so that say $p(x) \in S(\emptyset)$ forks over $\emptyset$, there may exist Morely sequences in other complete types over $\emptyset$. Thus even in the case, the proof of Theorem 3.2 is not trivialized. For example, as mentioned in the beginning, Mutchnik in [9] gives a two-sorted example of NSOP_1 theory without existence. In one sort, Morley sequences (over $\emptyset$) do not exist, while they exist in the other sort.

(2) Originally, in [4, Definition 3.13] Kaplan and Ramsey define $\mathop{\smile\!\!\!\!\!\!|}^K$ using invariant Morley sequences, which, as proved in [4, Theorem 7.7], coincide with Definition 1.4(1) and (2) using Morley sequences when the theory is NSOP_1 and the base is a model. In any theory, existence for $\mathop{\smile\!\!\!\!\!\!|}^K$ (as defined in Definition 1.4(1) and (2)) implies existsnce for $\mathop{\smile\!\!\!\!\!\!|}^K$ defined using invariant Morley sequences, since every invariant Morley sequence is a Morley sequence. Recently, in an unpublished note, James Hanson shows $\mathop{\smile\!\!\!\!\!\!|}^K$ with respect to invariant Morley sequences satisfies existence in any theory.

§4. Discussion. In this section we will discuss a couple of issues. First of all it is natural to ask whether there is a theory T where Kim-forking does not have existence. As proved in Section 3, if exists then it should not be NSOP_1.

As pointed out in Section 1, after Fact 1.3, the typical circle example with the ternary R does not have existence (over $\emptyset$). It is because there are three distinct points a_0, a_1, a_2 on the circle such that $\models x = x \leftrightarrow R(a_0, x, a_1) \vee R(a_1, x, a_2) \vee R(a_2, x, a_0)$, and each $R(a_i, x, a_{i+1})$ (mod 3) divides over $\emptyset$. But the dividing of each formula is witnessed *not* by a Morley sequence but only by an $\mathop{\smile\!\!\!\!\!\!|}^d$-Morley sequence. Thus this (although not NSOP_1) does not serve as an example where Kim-forking fails to have existence.

With respect to this question, at least we observe the following.

Remark 4.1.

(1) Suppose that a consistent formula $\psi(x) \in \mathcal{L}$ forks, witnessed by two consistent formulas $\varphi(x, a_0)$ and $\phi(x, b_0)$. Namely $\models \psi(x) \leftrightarrow \varphi(x, a_0) \vee \phi(x, b_0)$ and dividing of $\varphi(x, a_0)$ and $\phi(x, b_0)$ is witnessed by indiscernibles $I = (a_i \mid i < \omega)$ and $J = (b_i \mid i < \omega)$, respectively. We claim that I (hence J as well) can neither be $\mathop{\smile\!\!\!\!\!\!|}^d$-Morley, nor Morley. (In general, the reversed sequence of a non ($\mathop{\smile\!\!\!\!\!\!|}^d$-)Morley sequence can be ($\mathop{\smile\!\!\!\!\!\!|}^d$-)Morley. But the

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proof below says I is not $\bigcirc^d$-Morley in either direction. Moreover the circle example above says the argument below does not work if three (or more) formulas witness forking of $\psi(x)$.)

Note that there are $k(\geq 1)$ and $k' \geq 2$ such that $\varphi(x, a_0)$ $(k+1)$-divides, but not k-divides, via I; and $\phi(x, b_0)$ k'-divides, via J (*). Consider an $\mathcal{L}$-formula

$$\chi(x_k, \dots, x_0) := \exists z(\psi(z) \wedge \varphi(z, x_k) \wedge \cdots \wedge \varphi(z, x_1)) \wedge \neg \exists z'(\psi(z') \wedge \varphi(z', x_k) \wedge \cdots \wedge \varphi(z', x_0)).$$

Clearly $(a_k, \dots, a_0)$ realizes χ. It suffices, by left transitivity of $\bigcirc^d$, to show that $\chi(x_k, \dots, x_1; a_0)$ divides.

By compactness there clearly is an indiscernible $(a_0^j b_0^j \mid j < \omega)$ such that $a_0^0 b_0^0 = a_0 b_0$, and $(b_0^j \mid j \in \omega) \equiv J$. Then $\chi(x_k, \dots, x_1; a_0)$ k'-divides via $(a_0^j \mid j < \omega)$, namely, $\chi(x_k, \dots, x_1; a_0^0) \wedge \cdots \wedge \chi(x_k, \dots, x_1; a_0^{k'-1})$ is inconsistent. Why? Suppose not, say realized by $c_k \dots c_1$. Note that for each $j < k'$,

$$\psi(z) \wedge \varphi(z, c_k) \wedge \cdots \wedge \varphi(z, c_1) \models \neg\varphi(z, a_0^j) \wedge \psi(z) \models \phi(z, b_0^j).$$

Since $\psi(z) \wedge \varphi(z, c_k) \wedge \cdots \wedge \varphi(z, c_1)$ is consistent, $\{\phi(x, b_0^j) \mid j < k'\}$ is then consistent, contradicting (*) above.

(2) Consider a 2-sorted structure $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \in)$, where $\in$ is the usual membership relation. One can find sequences $(B_i)_{i<\omega}$ and $(B'_i)_{i<\omega}$ in $\mathcal{P}(\mathbb{N})$ satisfying $B'_0 = \mathbb{N} \setminus B_0$, $B_i \cap B_j = \emptyset$, $B'_i \cap B'_j = \emptyset$ for each $i \neq j$, and $|B_i| = |\mathbb{N} \setminus B_i| = |B'_i| = |\mathbb{N} \setminus B'_i| = \aleph_0$ for each i. Since any permutation of $\mathbb{N}$ induces an automorphism of the structure, it easily follows that $\mathrm{tp}(B_i) = \mathrm{tp}(B'_j)$ for all i, j. Put $B = B_0, B' = B'_0$. Now $\models x = x \leftrightarrow x \in B \vee x \notin B$. Moreover each of the formulas $x \in B$, and $x \notin B$ (equivalent to $x \in B'$) 2-divides over $\emptyset$, witnessed by the sequences $(B_i)_{i<\omega}$ and $(B'_i)_{i<\omega}$, respectively. This is an example of (1), which is a variation of an example failing existence mentioned in [11].

Now we list the following questions.

QUESTION 4.2.

(1) *Is there a theory where Kim-forking does not have existence? As mentioned in Remark 3.3(2), if Kim-forking is defined using invariant Morley sequences, then every theory has Kim-forking existence.*
(2) *In NSOP$_1$ T, can one obtain the same Theorem 3.2 even if each I_i in Section 3 is weakened to be an $\bigcirc^d$-Morley sequence?*

In addition to the questions above, in all NSOP$_1$ theories we are now able to define a new kind of dividing witnessed by $\bigcirc^K$-Morley sequences, due to the main result Theorem 3.2. As mentioned before Section 1, it is intriguing to know how far a theoretical development can be made using the new dividing for NSOP$_1$ theories, although the new one is known to be equal to Kim-dividing and Kim-forking in NSOP$_1$ theories with existence.

By the way, a quick consequence of Theorem 3.2 is as follows.

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REMARK 4.3. Assume T is NSOP_1. Define $A \mathop{\smile\!\!\!\!\!\!|}^{*}_{B} C$ if there is a model $(B \subseteq) M$ such that $M \mathop{\smile\!\!\!\!\!\!|}^{K}_{B} AC$ and $A \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} C$.

- (Symmetry of $\mathop{\smile\!\!\!\!\!\!|}^{*}$) Due to symmetry of $\mathop{\smile\!\!\!\!\!\!|}^{K}$ over a model, $A \mathop{\smile\!\!\!\!\!\!|}^{*}_{B} C$ iff $C \mathop{\smile\!\!\!\!\!\!|}^{*}_{B} A$.
- (Existence of $\mathop{\smile\!\!\!\!\!\!|}^{*}$) Given A, B, by Theorem 3.2, there is $(B \subseteq) M$ such that $M \mathop{\smile\!\!\!\!\!\!|}^{K}_{B} A$. Then since $B \subseteq M$, we have $A \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} B$. In other words, $A \mathop{\smile\!\!\!\!\!\!|}^{*}_{B} B$ holds.
- If T has existence then $A \mathop{\smile\!\!\!\!\!\!|}^{*}_{B} C$ iff $A \mathop{\smile\!\!\!\!\!\!|}^{K}_{B} C$ [1, Proposition 3.5].

However it is unclear whether in T, $\mathop{\smile\!\!\!\!\!\!|}^{*}$ has other properties, such as extension, transitivity, and so on.

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