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Numerical Analysis for a Nonlocal Parabolic Problem

M. Mbehou[1,2,*], R. Maritz[1] and P.M.D. Tchepmo[1]

[1] *Department of Mathematical Sciences, University of South Africa, Pretoria 0003, South Africa.*
[2] *Department of Mathematics, University of Yaounde I, Cameroon.*

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Abstract. This article is devoted to the study of the finite element approximation for a nonlocal nonlinear parabolic problem. Using a linearised Crank-Nicolson Galerkin finite element method for a nonlinear reaction-diffusion equation, we establish the convergence and error bound for the fully discrete scheme. Moreover, important results on exponential decay and vanishing of the solutions in finite time are presented. Finally, some numerical simulations are presented to illustrate our theoretical analysis.

AMS subject classifications: 65N30

Key words: Convergence, numerical simulation, Crank-Nicolson schemes, Galerkin finite element method, nonlinear parabolic equation, nonlocal diffusion term.

1. Introduction

Partial differential equations often arise in mathematical models describing physical, chemical, biological and ecological systems. Chipot & Lovat [1] studied the following nonlocal problem:

$$\begin{cases} u_t - a(l(u))\Delta u = f(x,t) \ \text{ in } \Omega \times (0,T)\,, \\ u(x,t) = 0 \ \text{ on } \partial\Omega \times (0,T)\,, \\ u(x,0) = u_0(x) \ \text{ in } \Omega\,, \end{cases} \tag{1.1}$$

where the domain Ω is a subset of $\mathbb{R}^d$, $d \geq 1$ with a smooth boundary $\partial\Omega$, T is an arbitrary time, $a(l(u))$ is some function from $\mathbb{R}$ to $(0,\infty)$, and Δ denotes the Laplacian operator. For instance, this problem describes the density u of a population (e.g. bacteria) subject to spreading, where the diffusion coefficient a may depend upon the entire population in the domain rather than the local density. The model is said to be nonlocal if the evolution is guided by the global state, as it is in that case [2]. Further applications of this model are discussed in Refs. [3,4] and references therein.

*Corresponding author. *Email address:* mohamm@unisa.ac.za (M. Mbehou)

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Chipot & Lovat [1] proved the existence and uniqueness of the solution to the problem (1.1), and Correa *et al.* [5] extended their analysis on assuming that $a(l(u))$ and $f(x,u)$ are continuous functions. However, there are few studies on the numerical solution of nonlocal problems, and they are restricted to nonlocal reaction terms or nonlocal boundary conditions. Yin & Xu [7] applied the finite volume method to obtain approximate solutions for a nonlocal problem on reactive flows in porous media, and derived the optimal convergence order in the L^2 norm. Sidi Ammi & Torres [8] applied the finite element method to the space variables and the Euler or Crank-Nicolson method to the time, in fully discretising a nonlocal thermistor problem. They also proved optimal rates of convergence in the L^2 norm. Almeida *et al.* [9] presented convergence analysis for a fully discretised approximation to a nonlocal problem involving a parabolic equation with moving boundaries, with the finite element method applied for the space variables and the Crank–Nicolson method for the time.

We consider the following parabolic problem with nonlocal nonlinearity:

$$\begin{cases} u_t - a(l(u))\Delta u + \alpha|u|^{p-2}u = f(u) \quad \text{in } \Omega \times (0,T], \\ u(x,t) = 0 \ \text{ on } \partial\Omega \times (0,T], \\ u(x,0) = u_0(x) \ \text{ in } \Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 1$ is again a domain with a smooth boundary $\partial\Omega$, a and f are functions to be defined in the next section, and $l : L^2(\Omega) \to \mathbb{R}$ is a continuous linear form. Simsen & Ferreira [6] have discussed not only the existence and uniqueness of solutions for this problem, but also continuity with respect to initial values, the exponential stability of weak solutions, and important results on the existence of a global attractor. Our goal here for this nonlocal reaction–diffusion problem (1.2) is to check the asymptotic behaviour of the solution for large time t numerically, and prove the convergence of a fully discrete approximation using the Crank-Nicolson-Galerkin finite element method.

In Section 2, we discuss hypotheses on the data and the weak variational formulation of the problem, followed by consideration of the asymptotic behaviour of the solution for large time t. Section 3 is concerned with the space-discrete problem and its convergence, and the convergence of the discrete solution is proven in Section 4. Finally, numerical results are presented in Section 5 to illustrate our theoretical analysis.

2. Preliminaries, Weak Formulation and Asymptotic Solution Behaviour

Throughout this article, $H^k(\Omega)$ denotes the usual Sobolev space of order $k \in \mathbb{N}$ with norm $\|\cdot\|_k$ and $H_0^k(\Omega)$ the closure of $\mathscr{C}_0^\infty(\Omega)$ in $H^k(\Omega)$. The Lebesgue space is denoted by $L^r(\Omega)$, $1 < r \leq \infty$ with norm $\|\cdot\|_{L^r}$ that is simplified to $\|\cdot\|$ in $L^2(\Omega)$, and $\mathscr{D}'(\Omega)$ is the space of distributions on Ω. We also employ the standard notation of Bochner spaces, such as $L^q(0,T,X)$ with norm $\|\cdot\|_{L^q(X)}$ where X is an Hilbert space, and denote the potential energy associated with the problem (1.2) by

$$E(t) = \frac{1}{2}\|u(t)\|^2 .$$

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Let us now suppose that α is a nonnegative constant and $p > 1$. Simsen & Ferreira [6] proved the existence and uniqueness of global solution under the following hypotheses.

Hyp 1: $u_0 \in L^2(\Omega)$.

Hyp 2: $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz–continuous function — i.e. there exists $\gamma > 0$ such that $|f(s) - f(t)| \leq \gamma |s - t|$ for all s, $t \in \mathbb{R}$ and $f(0) = 0$.

Hyp 3: $a : \mathbb{R} \to \mathbb{R}$ is bounded with $0 < m \leq a(s) \leq M$ for all $s \in \mathbb{R}$ and $\lambda_1 > \gamma/m$, where λ_1 is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$.

Hyp 4: $a : \mathbb{R} \to \mathbb{R}$ is Lipschitz–continuous, $|a(s_1) - a(s_2)| \leq A|s_1 - s_2|$ for all $s_1, s_2 \in \mathbb{R}$.

Hyp 5: $l : L^2(\Omega) \to \mathbb{R}$ is a continuous linear form — i.e. there exists $g \in L^2(\Omega)$ such that $l(u) = l_g(u) = \int_\Omega g(x)u(x)\,dx$ for all $u \in L^2(\Omega)$.

Theorem 2.1 (Existence and uniqueness of solution — cf. Ref [6])**.** *If the hypotheses* (**Hyp 1**)–(**Hyp 5**) *hold, then problem* (1.2) *possesses a unique solution — i.e. there exists a unique function u such that*

$$u \in L^2(0, T, H_0^1(\Omega) \cap L^p(\Omega)) \cap \mathscr{C}([0, T]; L^2(\Omega))\,, \tag{2.1}$$

$$u_t \in L^2(0, T, H^{-1}(\Omega))\,, \tag{2.2}$$

$$u(x, 0) = u_0(x) \ \text{in}\ \Omega\,, \tag{2.3}$$

$$\int_\Omega u_t v\,dx + a(l(u))\int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v\,dx + \alpha\int_\Omega |u|^{p-2}uv\,dx = \int_\Omega f(u)\,v\,dx\,,\ \forall v \in H_0^1(\Omega)\,, \tag{2.4}$$

where Eq. (2.4) *must be understood as defined in* $\mathscr{D}'(0, T)$.

2.1. Asymptotic behaviour for large t

Theorem 2.2 (Exponential decay)**.** *Let u be the solution of problem* (1.2) *given by Theorem* 2.1. *Then*

$$E(t) \leq E(0)\exp(-\beta t), \quad \text{for all } t > 0\,, \tag{2.5}$$

where β *is a positive constant that depends only on m,* γ *and* λ.

Proof. Taking $v = u$ in Eq. (2.4), we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2\,dx + a(l(u))\int_\Omega |\boldsymbol{\nabla} u|^2 dx + \alpha\int_\Omega |u|^p\,dx = \int_\Omega f(u)u\,dx\,,$$

and using (**Hyp 2**), (**Hyp 3**), Poincare's inequality and dropping the term with α we obtain

$$\frac{d}{dt}\frac{1}{2}\|u(t)\|^2 + 2(m\lambda_1 - \gamma)\frac{1}{2}\|u(t)\|^2 \leq 0\,,$$

and Eq. (2.5) follows readily with $\beta = 2(m\lambda_1 - \gamma)$. □

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Theorem 2.3 (Vanishing in a finite time)**.** *Suppose that* $1 < p < 2$ *and* u *the solution of Eq.* (2.4)*. Then* u *vanishes in a finite time — i.e. there exists* t_f *such that*

$$u(x,t) \equiv 0 \ \text{in}\ \Omega \ \ \text{for} \ \ t > t_f,$$

where t_f *depends only on* u_0*,* p *and* Ω*.*

Proof. Taking $v = u$ in Eq. (2.4), we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2\,dx + a(l(u))\int_\Omega |\nabla u|^2 dx + \alpha\int_\Omega |u|^p\,dx = \int_\Omega f(u)u\,dx\,,$$

and using (**Hyp 2**) and (**Hyp 3**) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2\,dx + m\int_\Omega |\nabla u|^2 dx + \alpha\int_\Omega |u|^p\,dx \le \gamma\int_\Omega |u|^2\,dx\,,$$

so applying Poincare's inequality produces

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2\,dx + \left(m - \frac{\gamma}{\lambda_1}\right)\int_\Omega |\nabla u|^2\,dx + \alpha\int_\Omega |u|^p\,dx \le 0\,.$$

Noting that $\xi = m - \gamma/\lambda_1 > 0$ and taking $C = \min\{\alpha, \xi\}$, we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2\,dx + C\left(\int_\Omega |\nabla u|^2\,dx + \int_\Omega |u|^p\,dx\right) \le 0\,, \tag{2.6}$$

so from the multiplicative inequality we obtain

$$\begin{aligned}\int_\Omega |u|^2\,dx &\le \left(\int_\Omega |\nabla u|^2 dx\right)^\theta \left(\int_\Omega |u|^p\,dx\right)^{2(1-\theta)/p} \\ &\le \left(\int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^p\,dx\right)^{\theta+2(1-\theta)/p}, \ \ \theta \in (0,1)\end{aligned}$$

such that

$$\left(\int_\Omega |u|^2\,dx\right)^\nu \le \int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^p\,dx\,, \tag{2.7}$$

where $0 < \nu = p/[p\theta + 2(1-\theta)] < 1$ since $1 < p < 2$.
Substituting inequality (2.7) into the relation (2.6), we obtain the ordinary differential inequality

$$E'(t) + CE(t)^\nu \le 0\,,$$

so on integrating from 0 to t we have

$$E(t)^{1-\nu} \le E(0)^{1-\nu} - Ct(1-\nu)\,,$$

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whence

$$E(t) = 0 \text{ for } t > t_f = \frac{E(0)^{1-\nu}}{C(1-\nu)}$$

and thus the result. □

Given the hypotheses (**Hyp 1**)–(**Hyp 5**), we will also adopt another hypothesis — viz.

Hyp 6: for all $r \geq 1$,

$$\|u_0\|_{H^{r+1}} + \|u\|_{L^\infty(H^{r+1}(\Omega))} + \|u_t\|_{L^2(H^{r+1}(\Omega))} + \|u_{tt}\|_{L^2(H^1(\Omega))} + \|u_{ttt}\|_{L^2(L^2(\Omega))} \leq C \, .$$

3. Spatial Discretisation

We adopt a regular triangulation $\mathcal{T}_h$ of Ω with elements K, in following classical finite element method theory [10, 11]. Thus let $h = \max_{K\in\mathcal{T}_h}\{h_K\}$ denote the mesh size, where $h_K = diam(K) = \max\{\|\boldsymbol{x}-\boldsymbol{y}\|, \boldsymbol{x}, \boldsymbol{y} \in K\}$, and V_h be the finite dimensional subspace of $H_0^1(\Omega)$, which consists of continuous piecewise polynomials of degree $r \geq 1$ on $\mathcal{T}_h$. Let Π_h be an interpolation operator and $R_h : H_0^1(\Omega) \to V_h$ be a Ritz projection operator defined by

$$\int_\Omega \boldsymbol{\nabla}(u - R_h u)\cdot\boldsymbol{\nabla} w \, dx = 0 \, , \quad \forall w \in H_0^1(\Omega) \, . \tag{3.1}$$

Then we have the following Lemmas.

Lemma 3.1 (cf. Ref. [10]). *If $u \in H^{r+1}(\Omega) \cap H_0^1(\Omega)$, then*

$$\|u - \Pi_h u\| + h\,\|\boldsymbol{\nabla}(u - \Pi_h u)\| \leq C h^{r+1}\|u\|_{H^{r+1}} \, , \tag{3.2}$$

$$\|u - R_h u\| + h\,\|\boldsymbol{\nabla}(u - R_h u)\| \leq C h^{r+1}\|u\|_{H^{r+1}} \, , \tag{3.3}$$

where C is a positive constant that does not depend on h and r.

Lemma 3.2 (cf. Ref. [12]). *For all $p \in (1,\infty)$ and $\tau \geq 0$, there exists a generic constant $C = C(p,d)$ such that for all $\xi, \eta \in \mathbb{R}^d$ with $d \geq 1$ we have*

$$\left||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right| \leq C|\xi-\eta|^{1-\tau}\big(|\xi|+|\eta|\big)^{p-2+\tau} \, , \tag{3.4}$$

$$\big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\big)\cdot(\xi-\eta) \geq C|\xi-\eta|^{2+\tau}\big(|\xi|+|\eta|\big)^{p-2-\tau} \, . \tag{3.5}$$

The spatial discrete problem based on Theorem 2.1 is as follows. Find $u_h \in V_h$ for $t \geq 0$ such that

$$\begin{aligned} &\int_\Omega (u_h)_t v_h \, dx + a(l(u_h)) \int_\Omega \boldsymbol{\nabla} u_h \cdot \boldsymbol{\nabla} v_h \, dx + \alpha \int_\Omega |u_h|^{p-2} u_h v_h \, dx = \int_\Omega f(u_h) v_h \, dx, \\ &u_h(x,0) = \Pi_h u_0 \, , \end{aligned} \tag{3.6}$$

for all $v_h \in V_h$. We then have the following theorem.

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Theorem 3.1. *The semidiscrete problem* (3.6) *admits a unique solution* $u_h \in V_h$*; and if* u *is the solution of the parabolic problem* (1.2)*, then*

$$\|u-u_h\| \le Ch^{r+1} \ \text{ for all } \ t\in(0,T], \tag{3.7}$$

where C *does not depend on* h*.*

Proof. Note that (3.6) is equivalent to a system of ordinary differential equations of first order, so the existence of the solution follows from Caratheodory's theorem. The proof of uniqueness is similar to that of Theorem 4 in Ref. [6], and therefore omitted here.

To prove the convergence, let us write $u_h-u=(u_h-R_hu)+(R_hu-u)=\Theta_h+\eta$. From Lemma 3.1, we have

$$\|\eta\| \le Ch^{r+1}\|u\|_{H^{r+1}} .$$

Now to estimate Θ_h, we have

$$\begin{aligned}
&\int_\Omega (\Theta_h)_t v_h\,dx + a(l(u_h))\int_\Omega \nabla\Theta_h\cdot\nabla v_h\,dx\\
&=\int_\Omega (u_h)_t v_h dx + a(l(u_h))\int_\Omega \nabla u_h\cdot\nabla v_h\,dx - \int_\Omega (R_hu)_t v_h\,dx - a(l(u_h))\int_\Omega \nabla(R_hu)\cdot\nabla v_h\,dx\\
&=\int_\Omega (u-R_hu)_t v_h\,dx + \int_\Omega \big(f(u)-f(u_h)\big)v_h\,dx + \alpha\int_\Omega \big(|u|^{p-2}u-|u_h|^{p-2}u_h\big)v_h\,dx\\
&\quad+\big(a(l(u))-a(l(u_h))\big)\int_\Omega \nabla u\cdot\nabla v_h\,dx.
\end{aligned}$$

If we take $v_h=\Theta_h$, on using the hypotheses (**Hyp 2**), (**Hyp 3**), (**Hyp 4**) and Lemmas 3.1 and 3.2 we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Theta_h\|^2 + m\|\nabla\Theta_h\|^2\\
&\le \|\eta_t\|\|\Theta_h\| + \gamma\|u-u_h\|\|\Theta_h\| + C_1(u,u_h)\|u-u_h\|\|\Theta_h\| + C_2(M,u)\|u-u_h\|\|\nabla\Theta_h\|\,,
\end{aligned} \tag{3.8}$$

where $C_1(w)=C(\|w\|_{L^\infty(0,T,L^2(\Omega))})$ and $C_2(M,u)=C(M,\|\nabla u\|_{L^\infty(0,T,L^2(\Omega))})$ are constants. Then applying the Poincare inequality we have

$$\frac{1}{2}\frac{d}{dt}\|\Theta_h\|^2 + m\|\nabla\Theta_h\|^2 \le C\left\{\|\eta_t\|^2+\|\eta\|^2\right\} + m\|\nabla\Theta_h\|^2, \tag{3.9}$$

and integrating with respect to time on the interval $[0,t]$ we obtain

$$\|\Theta_h\|^2 \le \|\Theta_h(0)\|^2 + C\int_0^T \left(\|\eta_t(s)\|^2+\|\eta(s)\|^2\right)ds\,,$$

so the inequality (3.7) follows from Lemma 3.1. □

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4. Fully Discrete Problem

We now proceed to the fully discrete approximation. Let $\{t_n | \, t_n = n\delta;\, 0 \le n \le N\}$ be a uniform partition of $[0,T]$ with time step $\delta = T/N$. We write $t_{n-\frac{1}{2}} = (t_n + t_{n-1})/2$ and $w^n = w(x,t_n)$, and for the sequence of functions $\{w^n\}_{n=0}^N$ define

$$\begin{aligned}
\bar{\partial} w^n &= \frac{w^n - w^{n-1}}{\delta}\,, \\
\bar{w}^n &= \frac{1}{2}(w^n + w^{n-1})\,, \quad n = 1,2,\cdots,N\,, \\
\hat{w}^n &= \frac{1}{2}(3w^{n-1} - w^{n-2})\,, \quad n = 2,\cdots,N\,.
\end{aligned}$$

A linearised Crank-Nicolson Galerkin finite element method associated with Eq. (2.4) is then as follows.

Find $u_h^n \in V_h$ such that

$$\begin{aligned}
&\int_\Omega \bar{\partial} u_h^n \, v_h \, dx + a(l(\hat{u}_h^n)) \int_\Omega \boldsymbol{\nabla} \bar{u}_h^n \cdot \boldsymbol{\nabla} v_h \, dx + \alpha \int_\Omega |\hat{u}_h^n|^{p-2} \bar{u}_h^n \, v_h \, dx = \int_\Omega f(\hat{u}_h^n) \, v_h \, dx\,, \\
&u_h^0 = \Pi_h u_0, \quad n = 1,\cdots,N\,,
\end{aligned} \tag{4.1}$$

for all $v_h \in V_h$, where $\hat{u}_h^1$ is the solution of the following equation:

$$\begin{aligned}
&\int_\Omega \frac{\hat{u}_h^1 - u_h^0}{\delta/2} v_h \, dx + a(l(u_h^0)) \int_\Omega \boldsymbol{\nabla} \hat{u}_h^1 \cdot \boldsymbol{\nabla} v_h \, dx + \alpha \int_\Omega |u_h^0|^{p-2} \hat{u}_h^1 \, v_h \, dx \\
&= \int_\Omega f(u_h^0) \, v_h \, dx\,.
\end{aligned} \tag{4.2}$$

Theorem 4.1. *Let $p \ge 2$. If u is the solution of the parabolic problem* (1.2) *and u_h^n is the solution of Eqs.* (4.1)-(4.2)*, then*

$$\|u^n - u_h^n\| \le C(h^{r+1} + \delta^2)\,, \tag{4.3}$$

where C is a generic constant that does not depend on h and δ.

Proof. Let us decompose $u_h^n - u^n$ as

$$u_h^n - u^n = (u_h^n - R_h u^n) + (R_h u^n - u^n) = \Theta_n + \eta_n\,.$$

The estimation of η_n is then given by Lemma 3.1 as follows.

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To estimate Θ_n, we use

$$\begin{aligned}
&\int_\Omega \bar{\partial}\Theta_n v_h\,dx + a(l(\hat{u}_h^n))\int_\Omega \nabla\bar{\Theta}_n\cdot\nabla v_h\,dx + \alpha\int_\Omega |\hat{u}_h^n|^{p-2}\bar{\Theta}_n v_h\,dx\\
&= \int_\Omega \bar{\partial}u_h^n v_h\,dx + a(l(\hat{u}_h^n))\int_\Omega \nabla\bar{u}_h^n\cdot\nabla v_h\,dx + \alpha\int_\Omega |\hat{u}_h^n|^{p-2}\bar{u}_h^n v_h\,dx\\
&\quad - \int_\Omega \bar{\partial}R_h u^n v_h\,dx - a(l(\hat{u}_h^n))\int_\Omega \nabla\bar{R}_h u^n\cdot\nabla v_h\,dx - \alpha\int_\Omega |\hat{u}_h^n|^{p-2}\bar{R}_h u^n v_h\,dx\\
&= \int_\Omega f(\hat{u}_h^n)v_h dx - \int_\Omega (u_t)^{n-1/2}v_h\,dx - a(l(u^{n-1/2}))\int_\Omega \nabla u^{n-1/2}\cdot\nabla v_h\,dx\\
&\quad - \alpha\int_\Omega |u^{n-1/2}|^{p-2}u^{n-1/2}v_h\,dx + \int_\Omega \left((u_t)^{n-1/2} - \bar{\partial}R_h u^n\right)v_h dx\\
&\quad + a(l(u^{n-1/2}))\int_\Omega \nabla(u^{n-1/2} - \bar{R}_h u^n)\cdot\nabla v_h\,dx + \left(a(l(u^{n-1/2})) - a(l(\hat{u}_h^n))\right)\int_\Omega \nabla\bar{R}_h u^n\cdot\nabla v_h\,dx\\
&\quad + \alpha\int_\Omega \left(|u^{n-1/2}|^{p-2}u^{n-1/2} - |\bar{R}_h u^n|^{p-2}\bar{R}_h u^n\right)v_h\,dx + \alpha\int_\Omega \left(|\bar{R}_h u^n|^{p-2} - |\hat{u}_h^n|^{p-2}\right)\bar{R}_h u^n v_h\,dx.
\end{aligned}$$

Applying the definition of Ritz projection operator (3.1), we have

$$\begin{aligned}
&\int_\Omega \bar{\partial}\Theta_n v_h dx + a(l(\hat{u}_h^n))\int_\Omega \nabla\bar{\Theta}_n\cdot\nabla v_h\,dx + \alpha\int_\Omega |\hat{u}_h^n|^{p-2}\bar{\Theta}_n v_h\,dx\\
&= \int_\Omega \left(f(\hat{u}_h^n) - f(u^{n-1/2})\right)v_h\,dx + \int_\Omega ((u_t)^{n-1/2} - \bar{\partial}R_h u^n)v_h\,dx\\
&\quad + a(l(u^{n-1/2}))\int_\Omega \nabla(u^{n-1/2} - \bar{u}^n)\cdot\nabla v_h\,dx + a(l(u^{n-1/2})) - a(l(\hat{u}_h^n))\int_\Omega \nabla\bar{u}^n\cdot\nabla v_h\,dx\\
&\quad + \alpha\int_\Omega (|u^{n-1/2}|^{p-2}u^{n-1/2} - |\bar{R}_h u^n|^{p-2}\bar{R}_h u^n)v_h\,dx + \alpha\int_\Omega (|\bar{R}_h u^n|^{p-2} - |\hat{u}_h^n|^{p-2})\bar{R}_h u^n v_h\,dx.
\end{aligned} \tag{4.4}$$

Setting $v_h = \bar{\Theta}_n$ and applying Holder's inequality, it follows that

$$\begin{aligned}
&\frac{1}{2}\bar{\partial}\|\Theta_n\|^2 + a(l(u^{n-1/2}))\|\nabla\bar{\Theta}_n\|^2 + \alpha\int_\Omega |\hat{u}_h^n|^{p-2}|\bar{\Theta}_n|^2\,dx\\
&\le \int_\Omega \left|f(\hat{u}_h^n) - f(u^{n-1/2})\right||\bar{\Theta}_n|\,dx + \left\|(u_t)^{n-1/2} - \bar{\partial}R_h u^n\right\|\|\bar{\Theta}_n\|\\
&\quad + a(l(u^{n-1/2}))\left\|\nabla(u^{n-1/2} - \bar{u}^n)\right\|\|\nabla\bar{\Theta}_n\| + \left|(a(l(u^{n-1/2})) - a(l(\hat{u}_h^n)))\right|\|\nabla\bar{u}^n\|\|\nabla\bar{\Theta}_n\|\\
&\quad + \alpha\int_\Omega \left|\left(|u^{n-1/2}|^{p-2}u^{n-1/2} - |\bar{R}_h u^n|^{p-2}\bar{R}_h u^n\right)\right||\bar{\Theta}_n|\,dx\\
&\quad + \alpha\int_\Omega \left|\left(|\bar{R}_h u^n|^{p-2} - |\hat{u}_h^n|^{p-2}\right)\right||\bar{R}_h u^n||\bar{\Theta}_n|\,dx.
\end{aligned}$$

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Now applying Poincare's inequality, (**Hyp 2**), (**Hyp 3**) and Lemma 3.2 we arrive at

$$\begin{aligned}
&\frac{1}{2}\bar{\partial}\left\|\Theta_n\right\|^2+m\left\|\boldsymbol{\nabla}\bar{\Theta}_n\right\|^2\\
\leq&\Big\{\gamma\left\|u^{n-1/2}-\hat{u}_h^n\right\|+C\left\|(u_t)^{n-1/2}-\bar{\partial}R_hu^n\right\|+M\left\|\boldsymbol{\nabla}(u^{n-1/2}-\bar{u}^n)\right\|\\
&+C_1(A,u)\left\|u^{n-1/2}-\hat{u}_h^n\right\|+C_2(u)\left\|u^{n-1/2}-\bar{R}_hu^n\right\|+C_3(u,u_h)\left\|\bar{R}_hu^n-\hat{u}_h^n\right\|\Big\}\left\|\boldsymbol{\nabla}\bar{\Theta}_n\right\|,
\end{aligned}$$

where $C_1(A,u)=C(A,\|\boldsymbol{\nabla}u\|_{L^\infty(0,T,L^2(\Omega))})$, $C_2(u)=C(\|u\|_{L^\infty(0,T,L^\infty(\Omega))})$, and $C_3(u,u_h)=C(\|u\|_{L^\infty(0,T,L^\infty(\Omega))},\|u_h\|_{L^\infty(0,T,L^\infty(\Omega))})$, such that

$$\begin{aligned}
\frac{1}{2}\bar{\partial}\|\Theta_n\|^2\leq&C\Big\{\left\|(u_t)^{n-1/2}-\bar{\partial}R_hu^n\right\|+\left\|\boldsymbol{\nabla}(u^{n-1/2}-\bar{u}^n)\right\|+\left\|u^{n-1/2}-\hat{u}_h^n\right\|\\
&+\left\|v_h^{n-1/2}-\hat{v}_h^n\right\|+\left\|u^{n-1/2}-\bar{R}_hu^n\right\|+\left\|\bar{R}_hu^n-\hat{u}_h^n\right\|\Big\}^2. \qquad (4.5)
\end{aligned}$$

From numerical differentiation and interpolation theory, and on invoking Lemma 3.1, we can prove that

$$\begin{aligned}
\left\|(u_t)^{n-1/2}-\bar{\partial}R_hu^n\right\|\leq&\left\|(u_t)^{n-1/2}-\bar{\partial}u^n\right\|+\left\|\bar{\partial}\eta_n\right\|\\
\leq&C\delta^2\|u_{ttt}\|_{L^\infty(0,T,L^2(\Omega))}+Ch^{r+1}\|u_t\|_{L^\infty(0,T,H^{r+1}(\Omega))}, \qquad (4.6)\\
\left\|\boldsymbol{\nabla}(u^{n-1/2}-\bar{u}^n)\right\|\leq&C\delta^2\|\boldsymbol{\nabla}u_{tt}\|_{L^\infty(0,T,L^2(\Omega))}, \qquad (4.7)\\
\left\|u^{n-1/2}-\bar{R}_hu^n\right\|\leq&\left\|u^{n-1/2}-\bar{u}^n\right\|+\left\|\bar{u}^n-\bar{R}_hu^n\right\|\\
\leq&C\delta^2\|u_{tt}\|_{L^\infty(0,T,L^2(\Omega))}+Ch^{r+1}\|u\|_{L^\infty(0,T,H^{r+1}(\Omega))}, \qquad (4.8)\\
\left\|u^{n-1/2}-\hat{u}_h^n\right\|\leq&\left\|u^{n-1/2}-\hat{u}^n\right\|+\|\hat{\eta}_n\|+\left\|\hat{\Theta}_n\right\|\\
\leq&C\delta^2\|u_{tt}\|_{L^\infty(0,T,L^2(\Omega))}+Ch^{r+1}\|u\|_{L^\infty(0,T,H^{r+1}(\Omega))}\\
&+C\left(\|\Theta_{n-1}\|+\|\Theta_{n-2}\|\right)\quad\text{for } n\geq 2, \qquad (4.9)\\
\left\|\bar{R}_hu^n-\hat{u}_h^n\right\|\leq&\left\|\bar{R}_hu^n-u^{n-1/2}\right\|+\left\|u^{n-1/2}-\hat{u}_h^n\right\|. \qquad (4.10)
\end{aligned}$$

From the inequalities (4.6)-(4.10), we therefore have

$$\|\Theta_n\|^2\leq C\delta\left(\|\Theta_{n-1}\|^2+\|\Theta_{n-2}\|^2\right)+C\delta\left(\delta^2+h^{r+1}\right)^2. \qquad (4.11)$$

Iterating and noting that $\sum_{k=1}^{n}\delta\leq T$ for all $1\leq n\leq N$, we then obtain

$$\|\Theta_n\|^2\leq C\left(\|\Theta_0\|^2+\|\Theta_1\|^2\right)+C\left(\delta^2+h^{r+1}\right)^2. \qquad (4.12)$$

Further, from the definition of u_h^0 and Lemma 3.1 we have

$$\|\Theta_0\|\leq Ch^{r+1}\|u_0\|_{L^\infty(0,T,H^{r+1}(\Omega))};$$

and from (4.5) we have

$$\begin{aligned}
\frac{1}{2}\bar{\partial}\|\Theta_1\|^2\leq&C\left\{\left\|(u_t)^{1/2}-\bar{\partial}R_hu^1\right\|+\left\|\boldsymbol{\nabla}(u^{1/2}-\bar{u}^1)\right\|+\left\|u^{1/2}-\hat{u}_h^1\right\|\right.\\
&\left.+\left\|u^{1/2}-\bar{R}_hu^1\right\|+\left\|\bar{R}_hu^1-\hat{u}_h^1\right\|\right\}^2, \qquad (4.13)
\end{aligned}$$

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where $\hat{u}_h^1$ is the solution of Eq. (4.2). From the inequalities (4.6)-(4.10) and the triangle inequality, we can write

$$\frac{1}{2}\bar{\partial}\|\Theta_1\|^2 \le C\left\{\delta^2 + h^{r+1}\right\}^2 + C\left\|R_h u^{1/2} - \hat{u}_h^1\right\|^2. \tag{4.14}$$

Let us denote $\hat{\Theta}_0 = \Theta_0$ and $\hat{\Theta}_1 = \hat{u}_h^1 - R_h\hat{u}^1$, with $\hat{u}^1 = u^{1/2} = u(t_{1/2})$. Then from Eq. (4.2) and the Ritz operator definition (3.1) we have

$$\begin{aligned}
&2\int_\Omega \bar{\partial}\hat{\Theta}_1 v_h\,dx + a(l(u_h^0))\int_\Omega \boldsymbol{\nabla}\hat{\Theta}_1\cdot\boldsymbol{\nabla}v_h\,dx + \alpha\int_\Omega |u_h^0|^{p-2}\hat{\Theta}_1 v_h\,dx \\
&= \int_\Omega \left(f(u_h^0) - f(u^{1/2})\right)v_h\,dx + \int_\Omega \left((u_t)^{1/2} - 2\bar{\partial}R_h u^{1/2}\right)v_h\,dx \\
&\quad + \left(a(l(u^{1/2})) - a(l(u_h^0))\right)\int_\Omega \boldsymbol{\nabla}u^{1/2}\cdot\boldsymbol{\nabla}v_h\,dx \\
&\quad + \alpha\int_\Omega \left(|u^{1/2}|^{p-2}u^{1/2} - |R_h u^{1/2}|^{p-2}R_h u^{1/2}\right)v_h\,dx \\
&\quad + \alpha\int_\Omega \left(|u_h^0|^{p-2} - |R_h u^{1/2}|^{p-2}\right)R_h u^{1/2} v_h\,dx\,.
\end{aligned} \tag{4.15}$$

Choosing $v_h = \hat{\Theta}_1$ and applying (**Hyp 2**), (**Hyp 3**), Poincare's inequality, Holder's inequality and Lemma 3.2, we arrive at

$$\begin{aligned}
&\frac{2}{\delta}\|\hat{\Theta}_1\|^2 - \frac{2}{\delta}\int_\Omega \Theta_0\hat{\Theta}_1 dx + m\|\boldsymbol{\nabla}\hat{\Theta}_1\|^2 \\
&\le C\left\{\left\|(u_t)^{1/2} - 2\bar{\partial}R_h u^{1/2}\right\| + \left\|u^{1/2} - u_h^0\right\| + \left\|u^{1/2} - R_h u^{1/2}\right\| + \left\|R_h u^{1/2} - u_h^0\right\|\right\}\|\boldsymbol{\nabla}\hat{\Theta}_1\|\,,
\end{aligned}$$

or equivalently

$$\begin{aligned}
\frac{1}{\delta}\|\hat{\Theta}_1\|^2 \le \frac{1}{\delta}\|\Theta_0\|^2 + C\Big\{&\left\|(u_t)^{1/2} - 2\bar{\partial}R_h u^{1/2}\right\| + \left\|u^{1/2} - u_h^0\right\| \\
&+ \left\|u^{1/2} - R_h u^{1/2}\right\| + \left\|R_h u^{1/2} - u_h^0\right\|\Big\}^2.
\end{aligned} \tag{4.16}$$

Using numerical differentiation, we can also prove that

$$\begin{aligned}
\left\|(u_t)^{1/2} - 2\bar{\partial}R_h u^{1/2}\right\| &\le \left\|(u_t)^{1/2} - 2\bar{\partial}u^{1/2}\right\| + 2\left\|\bar{\partial}u^{1/2} - \bar{\partial}R_h u^{1/2}\right\| \\
&\le C\delta\|u_t\|_{L^\infty(0,T,L^2(\Omega))} + Ch^{r+1}\|u_t\|_{L^\infty(0,T,H^{r+1}(\Omega))}\,, \\
\left\|u^{1/2} - u_h^0\right\| &\le \left\|u^{1/2} - u_0\right\| + \left\|u_0 - u_h^0\right\| \\
&\le C\delta\|u_t\|_{L^\infty(0,T,L^2(\Omega))} + Ch^{r+1}\|u_0\|_{L^\infty(0,T,H^{r+1}(\Omega))}\,,
\end{aligned}$$

such that we have

$$\|\hat{\Theta}_1\|^2 \le C\delta^3 + h^{2(r+1)}\,, \tag{4.17}$$

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and on combining the inequalities (4.17) and (4.14) we finally obtain

$$\begin{aligned}\|\Theta_1\|^2 &\leq C\|\Theta_0\|^2 + C\delta\{\delta^3 + h^{2(r+1)}\} + C\delta\{\delta^2 + h^{r+1}\}^2 \\ &\leq C\{\delta^2 + h^{r+1}\}^2 \end{aligned} \tag{4.18}$$

to complete the proof. □

5. Numerical Simulations

In this section we present several numerical simulations, to illustrate our theoretical analysis. All computations were performed using Matlab.

5.1. Example 1

This example demonstrates the evolution in time of the approximate solution. As stated in Theorem 2.2, the energy decays exponentially; and as stated in Theorem 2.3, when $1 < p < 2$ the solution has finite time extinction. In particular, we considered problem (1.2) in $\Omega = (0,1)^2$, with the coefficient $\alpha = 1$ and

$$f(s) = s(10-s)\,, \quad a(s) = 3 + \cos(s)\,, \quad l(u) = \int_\Omega u\,dx\,.$$

The initial condition was

$$u_0(x,y) = 2xy(1-x)(1-y)\,,$$

and it is notable that all of the hypotheses (**Hyp 1**)–(**Hyp 5**) are satisfied.

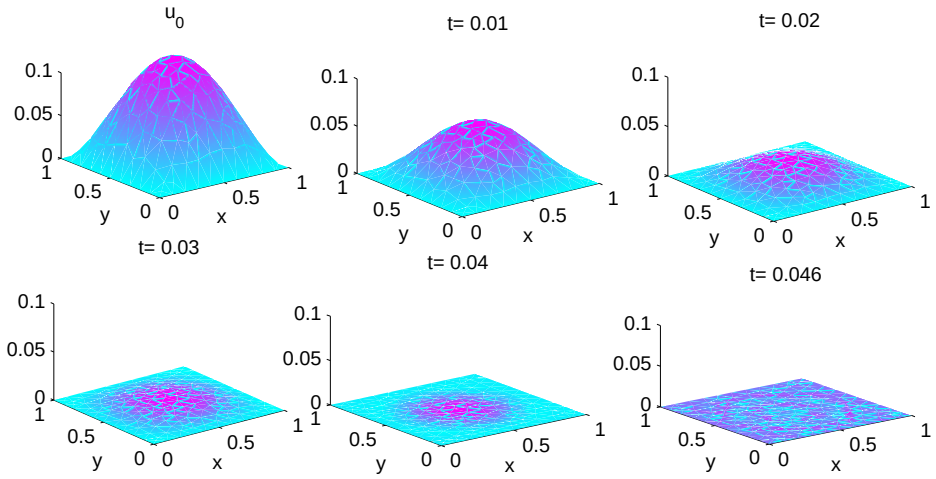


Figure 1: The solution u_h^n at certain times for $p = 1.2$.

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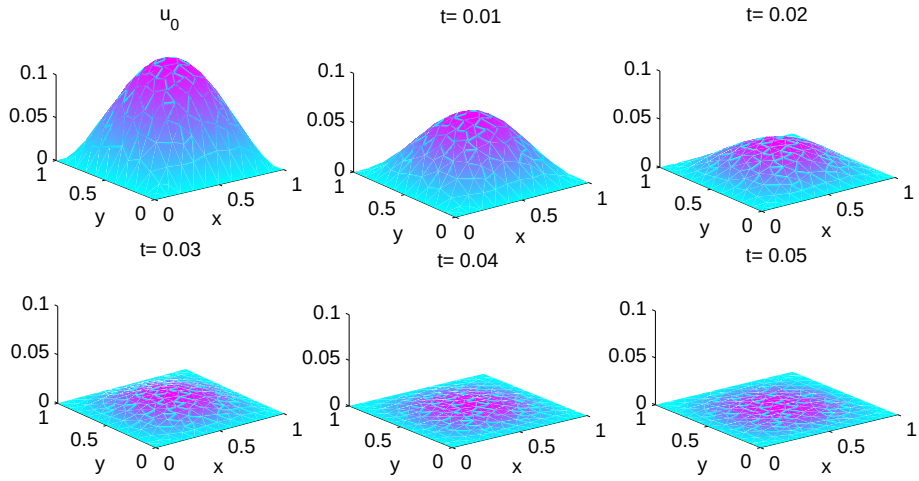


Figure 2: The solution u_h^n at certain times for $p = 3.5$.

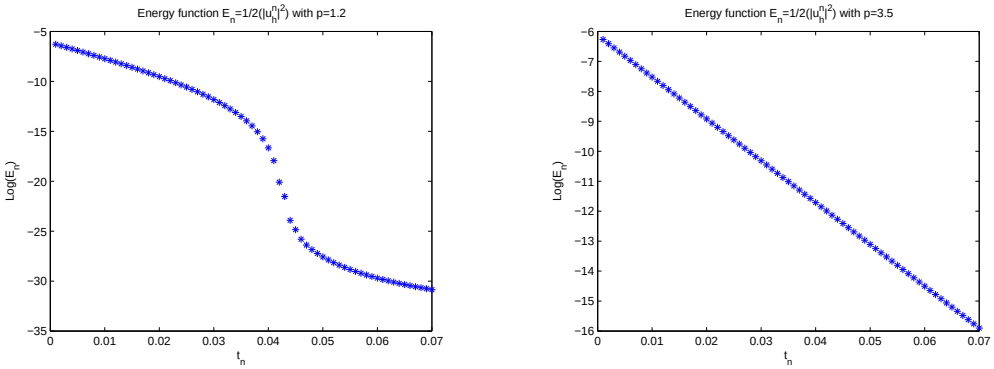


Figure 3: Evolution of the energy function E_n for $p = 1.2$ and $p = 3.5$, respectively.

As expected, for $p = 1.2$ the solution $u_h^n \equiv 0$ for $t \geq 0.046$ as shown in Fig. 1; and for $p = 3.5$ the solution u_h^n is very small but nonzero for $t \geq 0.046$ as seen in Fig. 2, which even shows the evolution at $t = 0.05$. To illustrate the asymptotic behaviour, the energy function $E_n = 1/2\|u_h^n\|^2$ is plotted in Fig. 3 using a logarithmic scale on the y-axis, showing that E_n decays for $p = 1.2$ and $p = 3.5$, and when $p = 1.2$ the energy function E_n is extinguished at $t \approx 0.046$.

5.2. Example 2

To analyse the convergence rate, we considered the following problem

$$\begin{cases} u_t - a(l(u))\Delta u + \alpha|u|^{p-2}u = f(u) + g \quad \text{in} \quad \Omega \times (0,T] \\ u(x,t) = 0 \qquad \text{on} \quad \partial\Omega \times (0,T] \\ u(x,0) = u_0(x) \quad \text{in} \quad \Omega\,. \end{cases} \tag{5.1}$$

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Table 1: L^2-error for h.

Degree	h	1/5	1/10	1/15	1/20	1/25
1	$\|u-u_h^n\|$	$2.295e-2$	$5.640e-3$	$2.564e-3$	$1.440e-3$	$9.431e-4$
2	$\|u-u_h^n\|$	$3.789e-4$	$4.885e-5$	$1.491e-5$	$6.426e-5$	$3.227e-6$

Table 2: L^2-error for δ.

δ	0.005	0.0025	0.0017	0.0013	0.0001
$\|u-u_h^n\|$	$3.400e-4$	$8.903e-5$	$4.173e-5$	$2.282e-5$	$1.491e-5$

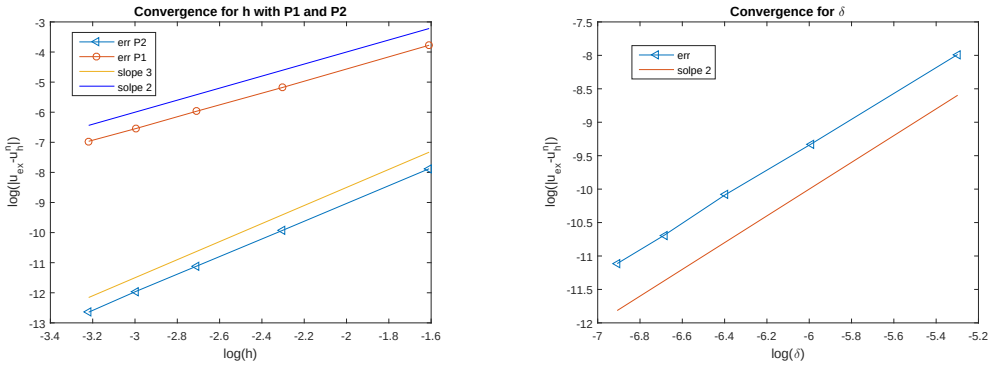


Figure 4: Convergence rate for h and δ, respectively.

We chose $\Omega = (0,1)^2$, the coefficient $\alpha = 1$, $p = 3.5$,

$$f(s) = s(10-s)\,, \quad a(s) = 3+\cos(s)\,, \quad l(u) = \int_\Omega u\,dx\,,$$

and g corresponding to the exact solution

$$u(x,y,t) = 2\left(1+t^2\exp(-t)\right)xy(1-x)(1-y)\,.$$

We solved this problem using the linearised Crank-Nicolson Galerkin finite element method (4.1)-(4.2), with linear and quadratic finite element approximation ($r = 1, 2$ respectively) and a uniform triangulation. The numerical results are presented in Table 1 for time $T = 0.07$ and Table 2 for $h = 1/15$. From Fig. 4 we see that the error estimated in the L^2-norm is $\mathcal{O}(h^{r+1}+\delta^2)$ with $r = 1, 2$ which is consistent with our theoretical analysis.

6. Conclusion

We have presented a linearised Crank-Nicolson Galerkin finite element method with piecewise polynomials of arbitrary degree as basis functions in space, for a nonlocal nonlinear parabolic problem, and proven the optimal L^2 error estimate. We presented some numerical results using Matlab, which are consistent with our theory.

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