On touching circles

JOHN R. SILVESTER

1. *Introduction*

The circles $C_1, \ldots, C_n$ form a *chain* of length n, or an *n-chain*, if C_i touches C_{i+1}, for $i = 1, \ldots, n-1$, and the chain is *closed* if also C_n touches C_1. If C_i touches C_{i+1} at Q_i, for $i = 1, \ldots, n$ (subscripts being interpreted modulo n), then $Q_1, \ldots, Q_n$ (assumed distinct) are the *contact points* of the chain. A *cyclic* chain is a chain for which all the circles touch another circle S, the *base* circle of the chain, and if C_i touches S at P_i, then $P_1, \ldots, P_n$ are the *base points* of the chain.

In this Article we shall prove a theorem about the contact points $\{Q_i\}$ of a closed 4-chain $\{C_i\}$. We shall show that *either* $\odot Q_1Q_2Q_3 = \odot Q_3Q_4Q_1$ (so that $Q_1Q_2Q_3Q_4$ is a cyclic quadrilateral) *or* $\odot Q_1Q_2Q_3 \perp \odot Q_3Q_4Q_1$ (the circles are orthogonal), depending on the nature of the contacts, internal or external, between adjacent circles of the chain. If it happens that C_2 also touches C_4, then we can regard C_1, C_2, C_3 as a cyclic 3-chain with base circle C_4, and in this case the Q_i all lie on a circle. We shall apply this to the seven circles theorem of Evelyn, Money-Coutts and Tyrrell to obtain a number of extra circles that meet in various special points. The methods used will include inversion, vector algebra, and coordinate geometry.

2. 4-*chains*

Let $\{C_i\}$ be a closed 4-chain with contact points $\{Q_i\}$. Various different-looking configurations are possible, and examples are shown in Figures 1 and 2. In Figure 1, the contact points appear to lie on a circle, shown dashed, and in Figure 2 the contact points lie by threes on two circles, also shown dashed, that appear to be orthogonal.

Proposition 1: Let $\{C_I\}$ be a closed 4-chain with contact points $\{Q_i\}$. Then either (a) $Q_1Q_2Q_3Q_4$ is cyclic or (b) $\odot Q_1Q_2Q_3 \perp \odot Q_3Q_4Q_1$.

Proof: First invert with respect to a circle centre Q_3, so that, using the same labels for the inverted items as for the originals, we now have two parallel lines C_3 and C_4, with a circle C_1 touching the line C_4 at Q_4, a circle C_2 touching the line C_3 at Q_2, and the two circles touching at Q_1. Now there is a central dilation f (an enlargement) with centre Q_1 such that $f(C_2) = C_1$, and f will have positive scale factor if the circles touch internally or negative scale factor if they touch externally. Then $f(C_3) = C_3'$ is a tangent to C_1, parallel to C_3, and touching it at $f(Q_2) = Q_2'$. If Q_2'' is the point on C_1 diametrically opposite Q_2', then the tangent C_2'' to C_1 at Q_2'' is also parallel to C_3 and we must have either $Q_4 = Q_2'$ or $Q_4 = Q_2''$. See Figure 3.

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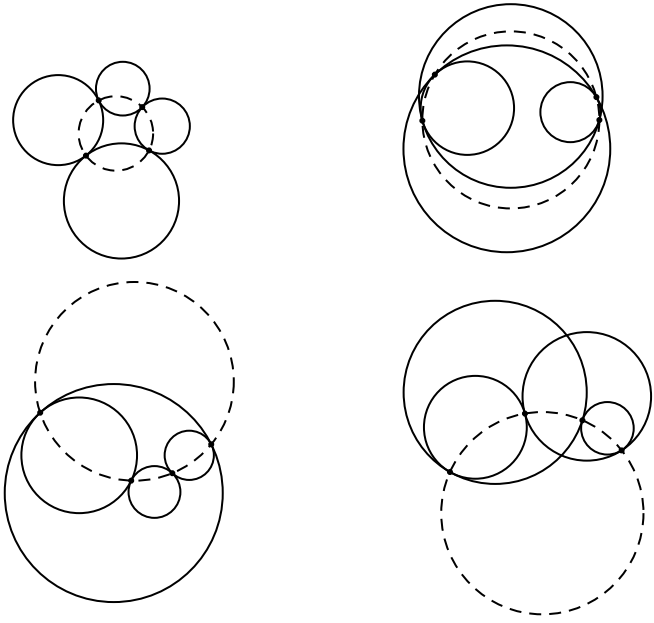


FIGURE 1

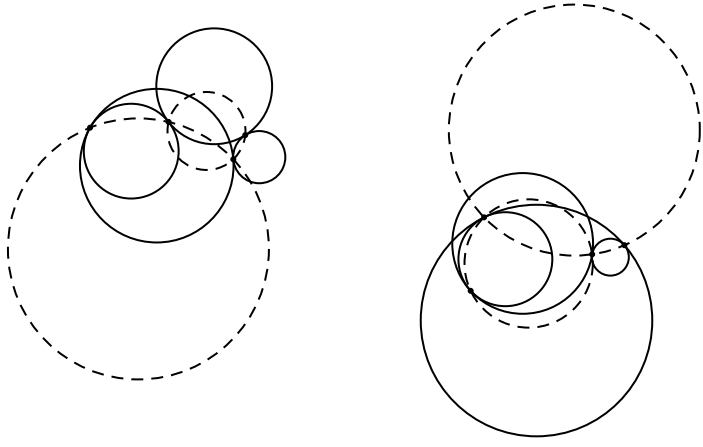


FIGURE 2

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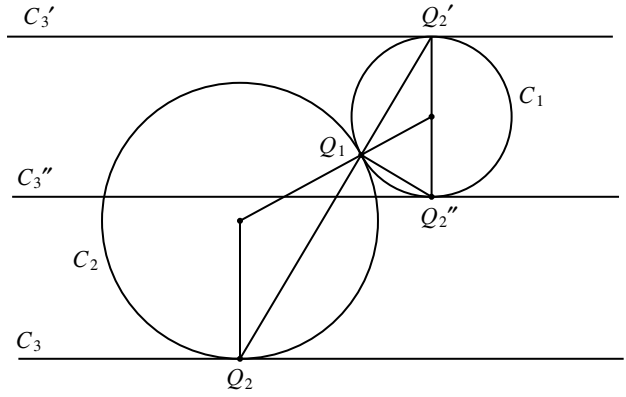


FIGURE 3

If $Q_4 = Q_2'$ then $f(Q_2) = Q_4$, so that the points Q_1, Q_2, Q_4 are collinear. Inverting back, we have type (a): $Q_1Q_2Q_3Q_4$ is cyclic. On the other hand, if $Q_4 = Q_2''$, then $Q_1Q_2 \perp Q_4Q_1$, so inverting back, we have type (b): $\odot Q_1Q_2Q_3 \perp \odot Q_3Q_4Q_1$.

Note that for type (b), we must also have $\odot Q_2Q_3Q_4 \perp \odot Q_4Q_1Q_2$. It is easy to see that an equivalent condition is that one pair of opposite angles of the quadrilateral $Q_1Q_2Q_3Q_4$ add up to 90°, and so the other pair add up to 270°. So Proposition 1 could be restated thus: the opposite angles of the quadrilateral of contact points of a closed 4-chain add up to $90n°$, for $n = 1$, 2 or 3. If we represented the points by complex numbers, then another equivalent statement would be that the cross-ratio of the contact points, in the appropriate order, is either real or pure imaginary.

Next, an easy special case:

Proposition 2: Let $\{C_i\}$ be a closed 4-chain with contact points $\{Q_i\}$. If also C_2 touches C_4 at a point P, then $Q_1Q_2Q_3Q_4$ is cyclic.

Proof: Invert with respect to a circle centre P. In the inverted diagram, C_2 and C_4 are parallel lines with the circles C_1 and C_3 touching both, so that $Q_1Q_2Q_3Q_4$ is a rectangle, and is therefore cyclic. See Figure 4. Inverting back gives the result.

Looking again at Figures 1 and 2, observe that for the examples in Figure 1, when the contact points lie on a circle, the number of internal contacts, and the number of external contacts, are both even, whereas for the examples in Figure 2, both are odd. In order to handle the general case, when two circles C_i, C_j touch, we define the *contact index* ε_{ij} to be +1 if the contact is external, and −1 if it is internal.

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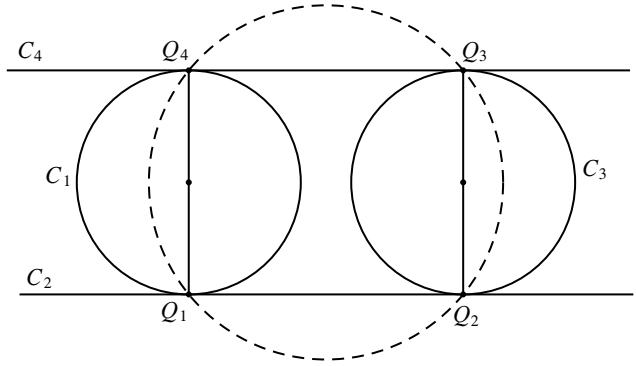


FIGURE 4

Lemma 1: Let C_1, C_2, C_3 be a closed 3-chain. Then $\varepsilon_{12}\varepsilon_{23}\varepsilon_{31} = 1$.

Proof: The result is obvious if all three contacts are external. So now suppose, without loss of generality, that C_1 and C_2 touch internally; specifically, that C_1 is inside C_2. Since C_3 touches both the others, it cannot be inside C_1, nor can it be outside C_2. (Remember that the contact points of a chain are always assumed distinct.) So it must lie between the two, that is, outside C_1 and inside C_2. So $\varepsilon_{12} = -1$, $\varepsilon_{23} = 1$ and $\varepsilon_{31} = -1$. The result follows.

Lemma 2: Let C_1, C_2, C_3 touch each other at the same point, Q. Then $\varepsilon_{12}\varepsilon_{23}\varepsilon_{31} = -1$.

Proof: Let the common tangent at Q be ℓ. If all three circles lie on the same side of ℓ then $\varepsilon_{12} = \varepsilon_{23} = \varepsilon_{31} = -1$ and the result follows. If (say) C_1 and C_2 lie on one side of ℓ and C_3 lies on the other, then $\varepsilon_{12} = -1$ and $\varepsilon_{23} = \varepsilon_{31} = 1$, and the result follows again.

Let $\mathscr{C} = \{C_1, C_2, C_3, C_4\}$ be a closed 4-chain with contact points Q_1, Q_2, Q_3, Q_4. Then there is a second closed 4-chain $\mathscr{C}' = \{C_1, C_2, C_3, C_4'\}$ with contact points Q_1, Q_2, Q_3, Q_4'. To see this, invert with respect to a circle centre Q_3, as in the proof of Proposition 1, where we obtained two tangents to (the inverse of) C_1 parallel to (the inverse of) C_3. These tangents then invert back to C_4 and C_4', in some order.

Proposition 3: One of $\mathscr{C}$, $\mathscr{C}'$ has an even number of internal contacts, and the other has an odd number.

Proof: We have to show that $\varepsilon_{12}\varepsilon_{23}\varepsilon_{34}\varepsilon_{41} = -\varepsilon_{12}\varepsilon_{23}\varepsilon_{34'}\varepsilon_{4'1}$, that is, that $\varepsilon_{34}\varepsilon_{41} = -\varepsilon_{34'}\varepsilon_{4'1}$. Now C_4, C_1, C_4' form a closed 3-chain, so by Lemma 1, $\varepsilon_{41}\varepsilon_{4'1}\varepsilon_{44'} = 1$. Then C_3, C_4, C_4' touch each other at Q_3, so by Lemma 2, $\varepsilon_{34}\varepsilon_{34'}\varepsilon_{44'} = -1$. The result is immediate.

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Proposition 4: One of $\mathscr{C}$, $\mathscr{C}'$ is type (a) and the other is type (b).

Proof: The chains cannot both be type (a), else $Q_1Q_2Q_3Q_4$ and $Q_1Q_2Q_3Q_4'$ are both cyclic, so all five contact points lie on one circle, which is impossible if they are distinct. Also, they cannot both be type (b), else $\odot Q_3Q_4Q_1$ and $\odot Q_3Q_4'Q_1$ are both orthogonal to $\odot Q_1Q_2Q_3$, which means that $Q_1Q_3Q_4Q_4'$ is cyclic, again impossible if the points are distinct. So we must have one chain of each type.

We are now ready to prove our main result about closed 4-chains:

Proposition 5: The closed 4-chain $\mathscr{C}$ is type (a) if, and only if, it has an even number of internal contacts, that is, if, and only if, $\varepsilon_{12}\varepsilon_{23}\varepsilon_{34}\varepsilon_{41} = 1$.

Proof: First we show that, if $\varepsilon_{12}\varepsilon_{23}\varepsilon_{34}\varepsilon_{41} = 1$, then $\mathscr{C}$ is type (a). Let C_i have vector equation $(\mathbf{x} - \mathbf{a}_i).(\mathbf{x} - \mathbf{a}_i) = r_i^2$, where (for reasons that will emerge) we could take the non-zero real number r_i to be either positive or negative.

Take $r_1 > 0$, and change the sign of r_2 if necessary so that $\operatorname{sign}(r_1r_2) = \varepsilon_{12}$. So if $\varepsilon_{12} = 1$ then r_1 and r_2 have the same sign, and C_1 and C_2 touch externally, so the distance between their centres is $|r_1 + r_2|$. But if $\varepsilon_{12} = -1$ then r_1 and r_2 have opposite signs, and C_1 and C_2 touch internally, so the distance between their centres is $|r_1 + r_2|$ in this case also.

Now change the signs of either r_3 or r_4 or both, if necessary, so that $\operatorname{sign}(r_2r_3) = \varepsilon_{23}$ and $\operatorname{sign}(r_3r_4) = \varepsilon_{34}$. Then

$$\operatorname{sign}(r_1r_4) = \operatorname{sign}((r_1r_2)(r_2r_3)(r_3r_4)) = \varepsilon_{12}\varepsilon_{23}\varepsilon_{34} = \varepsilon_{41}.$$

So now we have

$$(\mathbf{a}_i - \mathbf{a}_{i+1}).(\mathbf{a}_i - \mathbf{a}_{i+1}) = (r_i + r_{i+1})^2 \tag{1}$$

for $1 \leqslant i \leqslant 4$.

Subtracting the first of the equations (1) from the fourth gives

$$(\mathbf{a}_4 - \mathbf{a}_2).(\mathbf{a}_4 + \mathbf{a}_2 - 2\mathbf{a}_1) = (r_4 - r_2)(r_4 + r_2 + 2r_1) \tag{2}$$

and subtracting the second from the third gives

$$(\mathbf{a}_4 - \mathbf{a}_2).(\mathbf{a}_4 + \mathbf{a}_2 - 2\mathbf{a}_3) = (r_4 - r_2)(r_4 + r_2 + 2r_3). \tag{3}$$

Then subtract $r_1 \times$ (3) from $r_3 \times$ (2):

$$(r_3 - r_1).(\mathbf{a}_4.\mathbf{a}_4 - \mathbf{a}_2.\mathbf{a}_2) - 2(\mathbf{a}_4 - \mathbf{a}_2).(r_3\mathbf{a}_1 - r_1\mathbf{a}_3) = (r_3 - r_1)(r_4^2 - r_2^2). \tag{4}$$

Now the radical axis m of C_2 and C_4 is obtained by subtracting their equations, that is,

$$(\mathbf{x} - \mathbf{a}_4).(\mathbf{x} - \mathbf{a}_4) - (\mathbf{x} - \mathbf{a}_2).(\mathbf{x} - \mathbf{a}_2) = r_4^2 - r_2^2,$$

or

$$\mathbf{a}_4.\mathbf{a}_4 - \mathbf{a}_2.\mathbf{a}_2 - 2(\mathbf{a}_4 - \mathbf{a}_2).\mathbf{x} = r_4^2 - r_2^2. \tag{5}$$

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Let X be given by $\mathbf{x} = \dfrac{r_3\mathbf{a}_1 - r_1\mathbf{a}_3}{r_3 - r_1}$. Substituting in (5) and multiplying through by $r_3 - r_1$ gives (4) again, and so we deduce that X lies on m. But X is one of the centres of similitude of C_1 and C_3, external if $r_1r_3 > 0$ and internal otherwise. Put $r_2' = -r_2$. Now Q_1 is a centre of similitude of C_1 and C_2, the external one if C_1 and C_2 touch internally, that is, if $r_1r_2 < 0$, or $r_1r_2' > 0$, and the internal one otherwise; and likewise Q_2 is a centre of similitude of C_2 and C_3, the external one if $r_2'r_3 > 0$ and the internal one otherwise. But if three circles are taken two at a time, the six centres of similitude lie by threes on four lines, each line containing just one or all three external centres of similitude. Since $(r_1r_2')(r_2'r_3)(r_3r_1) > 0$, we deduce that Q_1Q_2 passes through X, and likewise so does Q_3Q_4. So X has the same power with respect to C_2 and C_4, that is, $(XQ_1)(XQ_2) = (XQ_3)(XQ_4)$, from which we have that Q_1, Q_2, Q_3, Q_4 lie on a circle S, so that $\mathscr{C}$ is type (a). See Figure 5. (Notice that X is the radical centre of C_2, C_4 and S.)

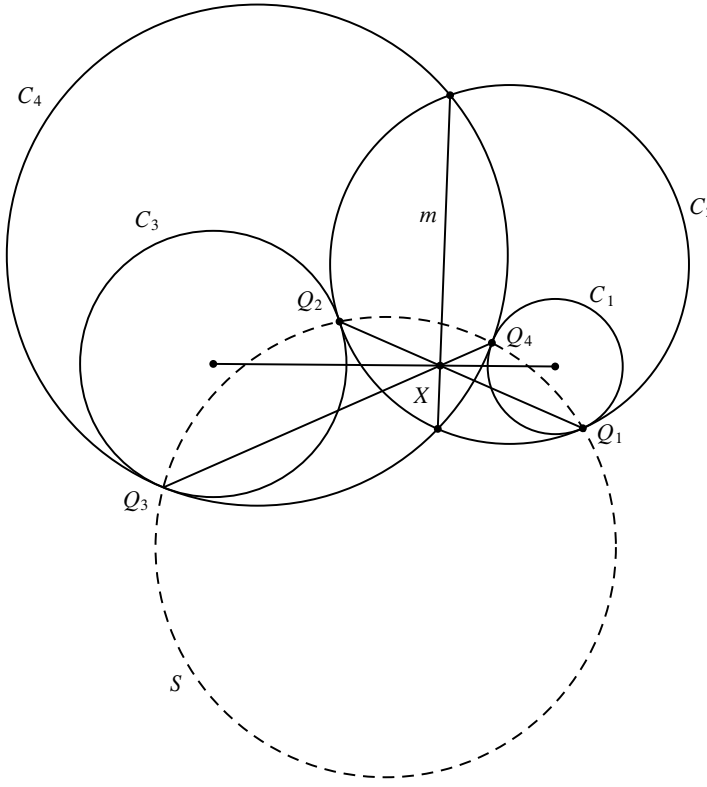


FIGURE 5

Now suppose $\mathscr{C}$ has an odd number of internal contacts. By Proposition 3, $\mathscr{C}'$ has an even number of internal contacts, so by the above argument it is type (a). Then by Proposition 4, $\mathscr{C}$ must be type (b), and we have finished.

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3. *The seven circles theorem*

The seven circles theorem of Evelyn, Money-Coutts and Tyrrell concerns a closed cyclic chain of length 6, and says that, if a certain extra condition is satisfied, then the lines P_1P_4, P_2P_5, P_3P_6 joining opposite base points meet at a point X, or are all parallel. See Figure 6. Conversely, if $P_1, \ldots, P_6$ lie on a circle S and the lines P_1P_4, P_2P_5, P_3P_6 meet at a point X or are all parallel, then any cyclic chain with base circle S and having these six points (in order) as base points will be closed. Choose a different radius for C_1 (with C_1 outside or inside S) and it is always possible to complete the chain with these base points. (And here, as usual, a straight line is regarded as a special sort of circle, with its centre is at infinity.) Note also that X can be inside or outside S, or (in the parallel case) at infinity. For details, see [1] and [2].

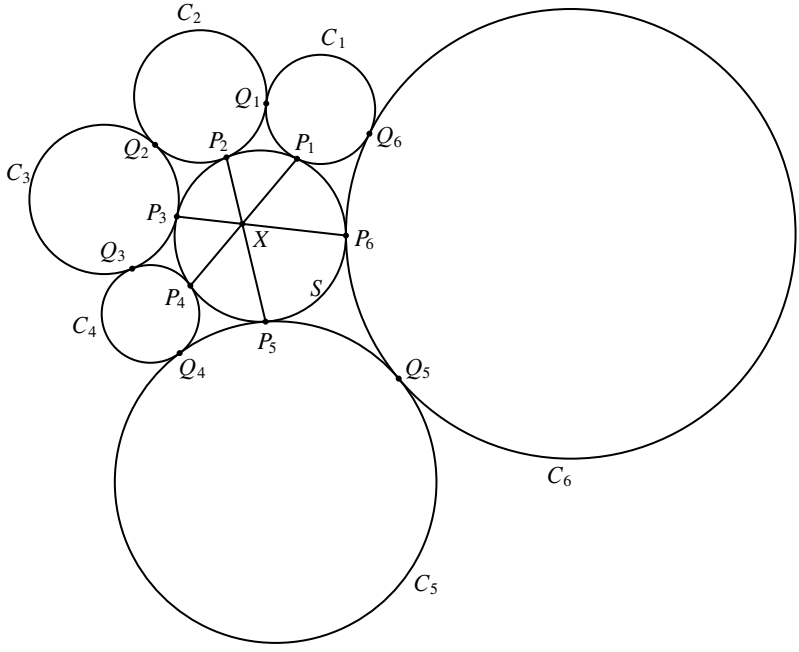


FIGURE 6

Now C_1, C_2, C_3, S form a closed 4-chain, with contact points Q_1, Q_2, P_3, P_1, and since also C_2 touches S (at P_2), then by Proposition 2, the four contact points lie on a circle, which we denote C_{123}. Likewise we obtain five other circles, C_{234}, C_{345}, C_{456}, C_{561} and C_{612}. Next, write $C_{i\,i+3}$ for the circle through P_i and P_{i+3} orthogonal to S. We now show that there are three points $P_{i\,i+3}$ each lying on five of the nine circles just described.

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Proposition 6: C_{14}, C_{123}, C_{234}, C_{456} and C_{561} meet in a point P_{14} (that is, P_{14} lies on the circles whose labels have either first or last subscript equal to either 1 or 4); likewise C_{25}, C_{234}, C_{345}, C_{561} and C_{612} meet in a point P_{25}; and C_{36}, C_{345}, C_{456}, C_{612} and C_{123} meet in a point P_{36}. Further, $S' = \odot P_{14}P_{25}P_{36}$ is orthogonal to C_{14}, C_{25} and C_{36}. See Figure 7.

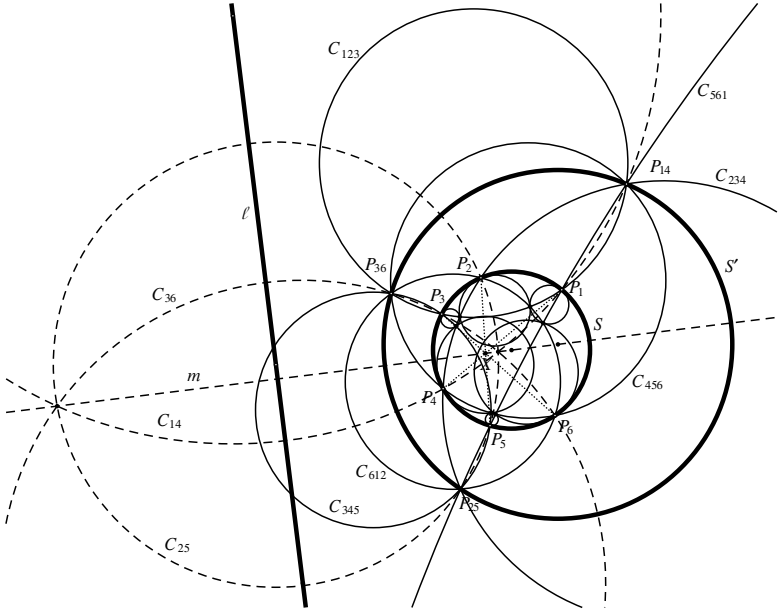


FIGURE 7

Proof: Invert with respect to a circle centre P_6. In the inverted diagram, S and C_6 are parallel lines; C_{456} and C_{621} are lines; the centres of C_{14} and C_{25} lie on S, and C_{36} is a line perpendicular to S. See Figure 8; but see also the remark at the end of the article.

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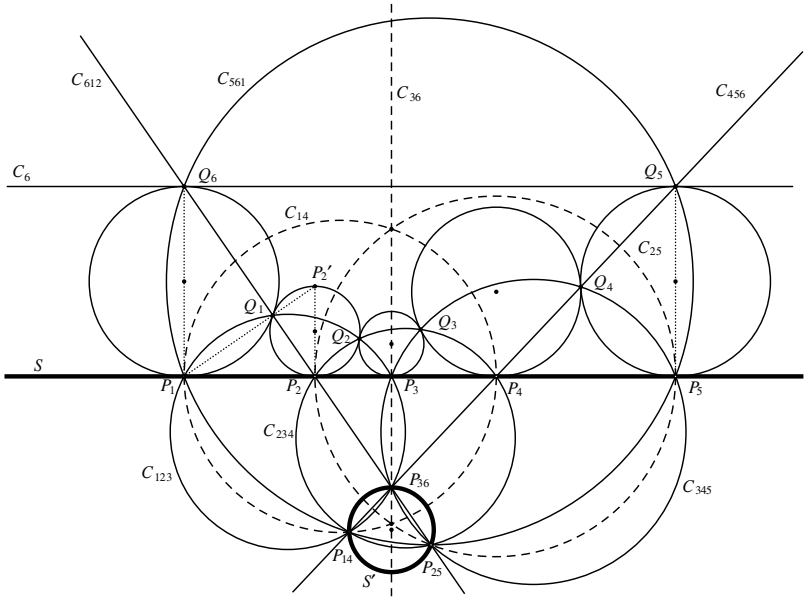


FIGURE 8

Choose coordinates so that S is the x-axis, C_{36} is the y-axis, and $P_3 = (0,\ 0)$; and scale so that C_6 is the line $y = 2$, making C_1 and C_5 unit circles. The fact that, in the original figure, P_1P_4, P_2P_5, P_3P_6 are concurrent or all parallel means that $\{P_1, P_4\}$, $\{P_2, P_5\}$, $\{P_3, P_6\}$ are point pairs of an involution on the circle S, so that in the inverted figure $\{P_1, P_4\}$, $\{P_2, P_5\}$, $\{P_3, \infty\}$ are point pairs of an involution on the line S. Thus, if $P_i = (t_i,\ 0)$ for $i = 1,\ 2,\ 4,\ 5$, then $t_1t_4 = t_2t_5 = a$, say. Put $b = t_1t_5$ and $c = a - b$. Note that a, b and c are preserved if we interchange t_1 with t_5 and t_2 with t_4.

Now $P_4 = (t_4,\ 0)$ and $Q_5 = (t_5,\ 2)$, so C_{456} is the line $2(x - t_4) - (t_5 - t_4)y = 0$, or $2t_1x + cy - 2a = 0$. Interchanging t_1 with t_5 and t_2 with t_4, we see that C_{612} is the line $2t_5x + cy - 2a = 0$, and these two lines meet at $\left(0,\ \frac{2a}{c}\right)$, which is on $x = 0$, that is, on C_{36}. So $P_{36} = \left(0,\ \frac{2a}{c}\right)$, and we need to show that this also lies on C_{123} and C_{345}.

Let the radius of C_i be r_i. The common tangent of C_1 and C_2 has length $2\sqrt{r_1r_2}$; but here $r_1 = 1$, so $4r_2 = (t_2 - t_1)^2$. Now the line P_1Q_1 meets C_2 again at the point P_2' diametrically opposite P_2, that is, at the point $(t_2,\ 2r_2)$, so its slope is $\frac{2r_2}{t_2 - t_1} = \frac{1}{2}(t_2 - t_1)$. Its equation is thus $y = \frac{1}{2}(t_2 - t_1)(x - t_1)$, or $(t_2 - t_1)x - 2y - t_1(t_2 - t_1) = 0$.

Next, the circle C_1 has equation $(x - t_1)^2 + (y - 1)^2 = 1$, or $x^2 + y^2 - 2t_1x - 2y + t_1^2 = 0$. Now P_1Q_1 is the common chord of C_1 and C_{123}, so the equation of C_{123} is a linear combination of those of P_1Q_1 and C_1.

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Since C_{123} passes through $P_3 = (0, 0)$, its equation is obtained by eliminating the constants between the other two equations. So C_{123} has equation

$$\left(x^2 + y^2 - 2t_1x - 2y + t_1^2\right)(t_2 - t_1) + \left((t_2 - t_1)x - 2y - t_1(t_2 - t_1)\right)t_1 = 0,$$

or $x^2 + y^2 - t_1x - \frac{2ay}{c} = 0$. Interchanging t_1 with t_5 and t_2 with t_4, the equation of C_{345} is $x^2 + y^2 - t_5x - \frac{2ay}{c} = 0$, and it is immediate that both C_{123} and C_{345} pass through $\left(0, \frac{2a}{c}\right)$, that is, through P_{36}, and we have (inverting back) our five circles through P_{36}. A similar argument (using a different inversion) shows that the five specified circles meet at P_{14}, and likewise for P_{25}.

Now we must find S'. The common chord of S' and C_{123}, in the inverted figure, is C_{456}, so S' has equation $x^2 + y^2 - t_1x - \frac{2ay}{c} + \lambda(2t_1x + cy - 2a) = 0$, for some λ. Putting $\lambda = \frac{1}{2}$, we obtain the circle $x^2 + y^2 + \left(\frac{c}{2} - \frac{2a}{c}\right)y - a = 0$ through P_{14} and P_{36}. Since this is unaltered if we interchange t_1 with t_5 and t_2 with t_4, it passes through P_{25} also, so it is S'. Further, since it has no x-term, it is orthogonal to C_{36}, and by a similar argument (using different inversions) it is orthogonal to C_{14} and C_{25} also.

Note that the centres of C_{14}, C_{25} and C_{36}, in the original diagram, all lie on the polar ℓ of X with respect to S. So the radical axis of any two of these circles is perpendicular to ℓ. Now P_1P_4 is the radical axis of C_{14} and S; and P_2P_5 is the radical axis of C_{25} and S, so that the meet of these lines, X, is the radical centre of these three circles, and so lies on the radical axis of C_{14} and C_{25}. Likewise X lies on the radical axis of C_{14} and C_{36}, and that of C_{25} and C_{36}. But we just saw that these three radical axes are all perpendicular to ℓ; so they must coincide, in a line m, which is the diameter of S through X; and C_{14}, C_{25}, C_{36} belong to a coaxal system with radical axis m. S and S' belong to the orthogonal coaxal system, and its radical axis is ℓ. So if X is inside S, then ℓ, S and S' do not meet but C_{14}, C_{25} and C_{36} have common chord m; see Figure 7. But if X is outside S it is the other way around: m, C_{14}, C_{25} and C_{36} do not meet but S and S' have common chord ℓ; see Figure 9. Then, Figure 8 is the diagram for the inverse of the former case; we leave it to the reader to draw a suitable diagram for the inverse of the latter.

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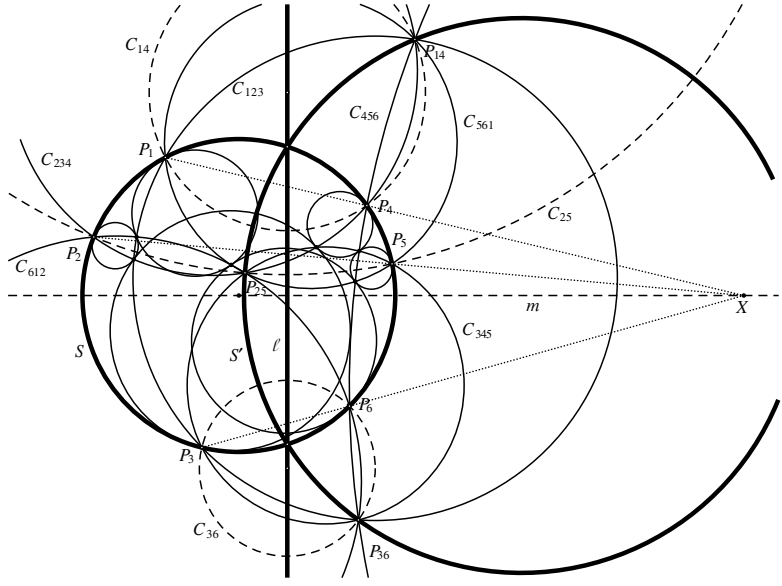


FIGURE 9

References

1. C. J. A. Evelyn, G. B. Money-Coutts and J. A. Tyrrell, *The seven circles theorem and other new theorems*, Stacey International (1974).
2. J. R. Silvester, The seven circles theorem revisited, *Math. Gaz.* **102**, (July 2018) pp. 280-301.

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