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Binary integral quadratic forms over $\mathbf{R}(t)$

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We use the notations and definitions of the introduction to the previous paper ((1)), and say that 2 lattices K and L on a quadratic space over $\mathbf{R}(t)$ with its Dedekind set $\mathbf{C}$ are in the same genus if, for all $\mathfrak{p} \in \mathbf{C}$, the lattices $K_{\mathfrak{p}}$ and $L_{\mathfrak{p}}$ are in the same class. We mean to count the number of classes of lattices in a given genus over a binary space: we shall find that it is either 1 or $2^{\aleph_0}$.

1. *Preliminaries.* It is possible to develop a theory, rather analogous to the usual one in $\mathbf{R}$, of continued fractions in F_∞, which have numerators 1 and denominators from $\mathbf{R}[t]$; those not terminating being in $F_\infty - \mathbf{R}(t)$, those terminating being in $\mathbf{R}(t)$; and we leave the reader to verify:

THEOREM 1·1. *Let* $\alpha \in F_\infty$ *and* $\pi, \rho \in \mathbf{R}[t]$; *if* $|\alpha - (\pi/\rho)| < 1/|\rho|^2$, *then* π/ρ *is a convergent to* α.

We can also develop a theory of quadratic residues in $\mathbf{R}[t]$, defining the generalized Legendre symbol $\{\alpha/\beta\}$ to be 1 or -1 according as $x^2 \equiv \alpha \bmod \beta$ has or has not a root; and again we invite the reader to verify:

LEMMA 1·2. *Suppose* $(\alpha, \beta) = 1$; *then*

$$\{\alpha/\beta\} = 1 \quad \textit{iff} \quad \forall\, \xi \in \mathbf{R} \quad (\beta(\xi) = 0 \Rightarrow \alpha(\xi) > 0).$$

2. *Reduction of primitive symmetric* 2×2 *matrices.* We call the non-singular matrix $\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$ over $\mathbf{R}[t]$ *primitive* if $(\alpha, \beta, \gamma) = 1$. We define $\Delta = \alpha\gamma - \beta^2$, and call the matrix *definite* if $-\Delta \notin \dot{F}_\infty^2$, and *indefinite* if $-\Delta \in \dot{F}_\infty^2$. (By $\dot{F}_\infty$ we mean $F_\infty - 0$.) Thus the matrix is indefinite iff $\partial\Delta$, the degree of Δ, is even and Δ has leading coefficient negative.

We are looking for results about equivalence classes of these matrices under the relation of congruence. We say that A is congruent to B iff there is a unimodular P over $\mathbf{R}[t]$ such that $P'AP = B$. The following identities

$$\begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \alpha\delta + \beta \\ \alpha\delta + \beta & \alpha\delta^2 + 2\beta\delta + \gamma \end{pmatrix} \quad (\delta \in \mathbf{R}[t]),$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \gamma & \beta \\ \beta & \alpha \end{pmatrix},$$

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} \lambda^2\alpha & \lambda\mu\beta \\ \lambda\mu\beta & \mu^2\gamma \end{pmatrix} \quad (\lambda, \mu \in \dot{\mathbf{R}})$$

show that in the definite case we may suppose $\partial\beta < \partial\alpha \leqslant \partial\gamma$, and in the indefinite case $\partial\beta < \partial\alpha < \partial\gamma$; and in either case, that Δ, α, γ have leading coefficients ± 1 (such

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polynomials we call *monic*) and that β has leading coefficients positive. Such matrices we call *reduced*.

We call the set $$\{\alpha\lambda^2+2\beta\lambda\mu+\gamma\mu^2 \neq 0: \lambda,\mu\in\mathbf{R}[t]\}$$ the set of *values* of $\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}$; it is invariant under congruence.

THEOREM 2·1. *Let* $\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}$ *be reduced definite; and suppose* $\partial\alpha < \partial\gamma$; *then* α *is the unique monic value of least degree of* $\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}$.

Proof. Let $\lambda,\mu\in\mathbf{R}[t]$ not both zero, and suppose $\partial\lambda \leqslant \partial\mu$; then
$$\partial(\alpha\lambda^2+2\beta\lambda\mu+\gamma\mu^2) = \partial\gamma+2\partial\mu > \partial\alpha.$$
But suppose $\partial\lambda > \partial\mu$; then $\partial(2\beta\lambda\mu) < \partial(\alpha\lambda^2)$, and since either $\partial\alpha \equiv \partial\gamma \bmod 2$ and α and γ have the same leading coefficients, or $\partial\alpha \not\equiv \partial\gamma \bmod 2$, we have
$$\partial(\alpha\lambda^2+2\beta\lambda\mu+\gamma\mu^2) = \max(\partial\alpha+2\partial\lambda, \partial\gamma+2\partial\mu)$$
so that it attains its minimum of $\partial\alpha$ only when $\partial\lambda = 0$, $\partial\mu = -\infty$; and noting that $\alpha \neq 0$, for otherwise $-\Delta = \beta^2\in\dot{F}^2_\infty$, we have the result.

On F_∞ we define $\alpha > \beta$ iff the leading term of the descending power series in t for $\alpha-\beta$ is positive. (F_∞ is then a complete ordered field.)

THEOREM 2·2. *Let* $\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}$ *be reduced definite; and suppose* $\partial\alpha = \partial\gamma$; *then* (*i*) *in the ordering defined above there is a least and a greatest monic value of least degree, distinct unless* $\partial\Delta = 0$, *which* (*ii*) *may be taken to be* α *and* γ *respectively; in which case* (*iii*) $\partial\beta < \partial(\alpha-\gamma)$ *and* (*iv*) β *is uniquely defined. Conversely, if* $\partial\beta < \partial(\alpha-\gamma)$, *and* $\alpha < \gamma$, *then* α *and* γ *are respectively the unique least and greatest monic values of least degree.*

Proof. The monic values of least degree are $\alpha\lambda^2+2\beta\lambda\mu+\gamma\mu^2$ with $\lambda,\mu\in\mathbf{R}$ and $\lambda^2+\mu^2 = 1$. Let α and γ have second coefficients a and g, and let the coefficient of $t^{\partial\alpha-1}$ in β be b. Then the second coefficients of the monic values of least degree range over the interval
$$\left[\frac{a+g-\sqrt{\{(a-g)^2+4b^2\}}}{2}, \quad \frac{a+g+\sqrt{\{(a-g)^2+4b^2\}}}{2}\right]$$
which reduces to a single point iff $a = g$ and $b = 0$; in which case we repeat the procedure on the third coefficient, and so on until we find that the interval is of positive length: this must happen, since otherwise $\alpha = \gamma$ and $\beta = 0$, contradicting the primitivity, unless $\partial\Delta = 0$. Thus we have (i).

Since for a value of least degree, $\lambda,\mu\in\mathbf{R}$, we have $(\lambda,\mu) = 1$ so that we may pick any value of least degree to stand in the top left-hand corner (see, e.g. (**2**), Section 81: 3); choosing therefore α to be the least monic value of least degree, we have
$$a = \frac{a+g-\sqrt{\{(a-g)^2+4b^2\}}}{2}$$
so that $b = 0$ and
$$g = \frac{a+g+\sqrt{\{(a-g)^2+4b^2\}}}{2}$$
and thus γ is the greatest monic value of least degree. This gives us (ii) and (iii).

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Putting $\begin{pmatrix}\lambda & \nu\\ \mu & \xi\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}\begin{pmatrix}\lambda & \mu\\ \nu & \xi\end{pmatrix} = \begin{pmatrix}\alpha & \beta'\\ \beta' & \gamma\end{pmatrix}$ with $\beta, \beta' \geqslant 0$

and using the fact that

$$\alpha\lambda^2 + 2\beta\lambda\mu + \gamma\mu^2 = \alpha \quad \text{iff} \quad \lambda = \pm 1 \quad \text{and} \quad \mu = 0$$

and similarly for γ, we see that β is uniquely defined, as in (iv).

The converse is obvious.

THEOREM 2·3. *Let* $\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}$ *be reduced indefinite; then there are at most* $\aleph_0$ *polynomials* $\alpha' \in \mathbf{R}[t]$ *such that there exist* $\beta', \gamma' \in \mathbf{R}[t]$ *with* $\begin{pmatrix}\alpha' & \beta'\\ \beta' & \gamma'\end{pmatrix}$ *reduced and congruent to* $\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}$.

Proof. Note that $\partial\alpha < \frac{1}{2}\partial\Delta$; if α' is as required, $\exists\, \lambda, \mu \in \mathbf{R}[t]$ with $(\lambda, \mu) = 1$ and $\alpha' = \alpha\lambda^2 + 2\beta\lambda\mu + \gamma\mu^2$; and α' is monic and $\partial\alpha' < \frac{1}{2}\partial\Delta$.

Note too that

$$2(\partial\lambda - \partial\mu) = \partial\gamma - \partial\alpha$$

since otherwise $\partial\alpha' \geqslant \partial\gamma > \frac{1}{2}\partial\Delta$.

Since

$$\frac{\alpha'}{\mu^2} = \left(\frac{\lambda}{\mu} + \frac{\beta}{\alpha} - \frac{\sqrt{(-\Delta)}}{\alpha}\right)\left(\frac{\lambda}{\mu} + \frac{\beta}{\alpha} + \frac{\sqrt{(-\Delta)}}{\alpha}\right)$$

and

$$\partial\left(\frac{\beta}{\alpha}\right) < 0 < \partial\left(\frac{\sqrt{(-\Delta)}}{\alpha}\right),$$

we have, for $\epsilon = \pm 1$ as appropriate,

$$\begin{aligned}\partial\left(\frac{\lambda}{\mu} + \frac{\beta}{\alpha} + \epsilon\frac{\sqrt{(-\Delta)}}{\alpha}\right) &= \max\left(\partial\lambda - \partial\mu, \tfrac{1}{2}\partial\Delta - \partial\alpha\right)\\ &= \tfrac{1}{2}\partial\Delta - \partial\alpha.\end{aligned}$$

Thus

$$\partial\left(\frac{\lambda}{\mu} + \frac{\beta}{\alpha} - \epsilon\frac{\sqrt{(-\Delta)}}{\alpha}\right) = \partial\alpha' - \tfrac{1}{2}\partial\Delta - 2\partial\mu < -2\partial\mu$$

so that by Theorem 1·1 λ/μ is a convergent to $[\epsilon\sqrt{(-\Delta)} - \beta]/\alpha$; but λ/μ is in its lowest terms, and α' is monic, so that there are at most $\aleph_0$ possible α'.

3. *Class number results.* Here $\mathbf{C} = 2^{\aleph_0}$ and by scaling we restrict ourselves to genera of scale $\mathfrak{o}$ (in the notation of (2)).

THEOREM 3·1. *There are at most a finite number of genera of scale* $\mathfrak{o}$ *and volume* $\mathfrak{v}$ *on a binary quadratic space* V *over* $\mathbf{R}(t)$.

Proof. Let L be a lattice on V of scale $\mathfrak{o}$ and volume $\mathfrak{v}$; then if $\mathfrak{p} \in \mathbf{C}$,

$$L_{\mathfrak{p}} \cong \langle \pm 1\rangle \perp \langle \pm \pi^r\rangle \quad (r \geqslant 0)$$

by (1), Section 3; and we see that

$$r = \operatorname{ord}_{\mathfrak{p}} \mathfrak{v}.$$

Thus if $\mathfrak{p} \in \mathbf{C} - \mathbf{R}$, then $\operatorname{cls} L_{\mathfrak{p}}$ is fixed, by (1), Prop. 3·1; and if $\mathfrak{p} \in \mathbf{R}$, again $\operatorname{cls} L_{\mathfrak{p}}$ is fixed, by (1), Section 1 and Prop. 3·1, remembering that

$$V_{\mathfrak{p}} \cong \langle \pm 1\rangle \perp \langle \pm \pi^r\rangle;$$

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unless $V_\mathfrak{p}$ is isotropic and r is a positive multiple of 2: for then we could have

$$L_\mathfrak{p} \cong \langle 1\rangle \perp \langle -\pi^r\rangle \quad \text{or} \quad L_\mathfrak{p} \cong \langle -1\rangle \perp \langle \pi^r\rangle.$$

But for all except a finite number of $\mathfrak{p}$, we have $r = 0$; which gives the result.

This theorem prompts the following sequence of definitions: for all except a finite number of $\xi \in \mathbf{R}$ we define $s(V;\xi)$ to be the signature of V at ξ as in (1), Section 2. We define

$$A(\Delta, V) = \{\mathfrak{p} \in \mathbf{R}: V_\mathfrak{p} \quad \text{isotropic and} \quad \operatorname{ord}_\mathfrak{p} \Delta > 0\}$$

where $\Delta \in \mathbf{R}^\cdot(t)$.

Next we define for θ: $A(\Delta, V) \to \{1, -1\}$ the function ϕ: $\mathbf{R} \to \{1, -1, 0\}$ by

$$\begin{aligned} \phi(\xi) &= \tfrac{1}{2}s(V;\xi) && (\Delta(\xi) \neq 0 \quad \text{and} \quad s(V;\xi) \text{ defined}) \\ &= \theta(\xi) && (\xi \in A(\Delta, V)) \\ &= 0 && (\text{otherwise}) \end{aligned}$$

and define the genus $g(\Delta, V, \theta)$ to be the set of lattices on V, of scale $\mathfrak{o}$ and volume $\Delta\mathfrak{o}$, and with for all $\mathfrak{p} \in A(\Delta, V)$

$$L_\mathfrak{p} \cong \langle \theta\mathfrak{p}\rangle \perp \langle -(\theta\mathfrak{p})\pi^{2r}\rangle \quad (r > 0)$$

If this set is not empty, it coincides with the genus as ordinarily defined.

Our next theorem investigates this set:

THEOREM 3·2. *If the genus $g(\Delta, V, \theta)$ is not empty, then*

(i) $\Delta \in \mathbf{R}[t]$.

(ii) $\Delta/dV \in \pm\,\mathbf{R}^\cdot(t)^2$.

(iii) *ϕ changes between 1 and -1 only across an interval of positive length on which it is 0.*

Conversely, if (i), (ii) *and* (iii) *hold,*

(iv) $\partial\Delta \geqslant 3 \Rightarrow \exists\ \mathbf{C}$ *classes in the genus* $g(\Delta, V, \theta)$.

(v) $\partial\Delta \leqslant 2 \Rightarrow \exists$ *just* 1 *proper class in the genus* $g(\Delta, V, \theta)$.

Proof. (i) and (ii) are well-known results (see, e.g. (2), Section 82 E): and we shall suppose the sign of Δ adjusted so that $\Delta/dV \in \mathbf{R}^\cdot(t)^2$, and thus

$$A(\Delta, V) = \{\xi \in \mathbf{R}: \Delta = (t-\xi)^{2q}\rho, q > 0, \rho(\xi) < 0\}.$$

For (iii), suppose that ϕ changes abruptly between 1 and -1 at a; then $\Delta(\xi) > 0$ for ξ near a, so that $\Delta(a) \geqslant 0$. Suppose

$$\exists\ L \in g(\Delta, V, \theta) \quad \text{with} \quad L \cong \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix};$$

since V represents α and γ, (1), Section 2, shows that $(t-a)|\alpha, \gamma$; also $\{-\Delta/\alpha\} = 1$ so that by Lemma 1·2 $(t-a)|\Delta$ and thus $(t-\alpha)|\beta$ and the scale of L is not $\mathfrak{o}$, a contradiction.

For the rest of the proof we need a few lemmas:

LEMMA 3·3. *Suppose* (i), (ii) *and* (iii) *of the theorem are satisfied; and let $\alpha \in \mathbf{R}[t]$ have the properties*:

$$(\alpha, \Delta) = 1$$

$$\forall\, \xi \in \mathbf{R} \quad (\phi(\xi) \neq 0 \Rightarrow \operatorname{sign}\alpha(\xi) = \phi(\xi)).$$

Then $\exists \beta, \gamma \in \mathbf{R}[t]$ and $L \in g(\Delta, V, \theta)$ with $L \cong \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$ and $\partial\beta < \partial\alpha$.

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Proof. It is clear that V represents α.

Suppose that $\alpha(\xi) = 0$; then $\phi(\xi) = 0$ so that $\Delta(\xi) \leqslant 0$; but $(\alpha, \Delta) = 1$ so that $\Delta(\xi) < 0$ and by Lemma 1·2 $\{-\Delta/\alpha\} = 1$. Thus $\exists \beta, \gamma \in \mathbf{R}[t]$ with $\partial\beta < \partial\alpha$, such that $\alpha\gamma - \beta^2 = \Delta$.

Now
$$V \cong \begin{pmatrix} \alpha & 0 \\ 0 & dV/\alpha \end{pmatrix} \cong \begin{pmatrix} \alpha & 0 \\ 0 & \Delta/\alpha \end{pmatrix} \cong \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$$
trivially. Thus $\exists L$ on V with $L \cong \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$, and since $(\alpha, \Delta) = 1$, L has scale $\mathfrak{o}$ and volume $\Delta\mathfrak{o}$.

Note now that
$$\forall \xi \in A(\Delta, V), \quad \operatorname{sign} \alpha(\xi) = \theta(\xi)$$
so that L_ξ represents $\theta(\xi)$ and thus
$$L_\xi \cong \langle \theta(\xi) \rangle \perp \langle -\theta(\xi)\, \pi^{2r} \rangle \quad (r > 0)$$
for otherwise, L_ξ would not represent $\theta(\xi)$, but only $-\theta(\xi)$.

Hence $L \in g(\Delta, V, \theta)$ and the lemma is proved.

LEMMA 3·4. *Suppose* (i), (ii) *and* (iii) *of the theorem are satisfied; then* $\exists \alpha \in \mathbf{R}[t]$ *such that the conditions of Lemma* 3·3 *hold, and such that* $0 < \partial\alpha \leqslant \frac{1}{2}\partial\Delta$.

Proof. Divide $\mathbf{R}$ into intervals of positive length by the points at which ϕ changes, except for the isolated zeros of ϕ: and pick a finite set of elements a_i thus:

(1) If there is a finite interval within which $\phi = 0$ and across which ϕ changes between 1 and -1, pick one element from within each such interval.

(2) If there is no such interval, but there is an infinite interval on which $\phi = 0$, pick a_1 within it.

(3) If (1) and (2) are impossible, pick $a_1, a_2 \in \mathbf{C} - \mathbf{R}$ with $a_1 = \bar{a}_2$ and $\Delta(a_1) \neq 0$.

Then $\alpha = \epsilon \prod_i (t - a_i)$, with $\epsilon = \pm 1$ as appropriate, is as required.

LEMMA 3·5. *Suppose that case* (1) *of Lemma* 3·4 *holds, and the construction yields* $\partial\alpha = \frac{1}{2}\partial\Delta$. *Let* $\beta \in \mathbf{R}[t]$ *with* $\partial\beta < \frac{1}{2}\partial\Delta$ *and* $(\beta, \Delta) = 1$; *then* $\exists\, \epsilon \in \mathbf{R}$, *with* $\epsilon > 0$, *such that* $\forall \lambda \in \mathbf{R}$ *with* $0 < \lambda < \epsilon$ $\exists$ *monic* $\alpha, \gamma \in \mathbf{R}[t]$ *satisfying the conditions of Lemma* 3·3 *and with*
$$\alpha\gamma = \Delta + \lambda^2\beta^2,$$
$$\partial\alpha = \partial\gamma,$$
$$\partial(\alpha - \gamma) = \partial\alpha - 1.$$

Proof. Divide $\mathbf{R}$ into intervals as in Lemma 3·4.

ϕ is non-zero on or at $\frac{1}{2}\partial\Delta + 1$ intervals or isolated points.

To each infinite such interval there corresponds at least 1 real root of Δ, viz. the endpoint of the interval; to each finite such interval there correspond in the same way at least 2 real roots of Δ; and to each such isolated point there corresponds a repeated real root, the isolated point itself, of Δ, lying in $A(\Delta, V)$.

It follows by counting that:

$\Delta > 0$ (since for $|\xi|$ large, $\phi(\xi) \neq 0$),

all the roots of Δ are real,

Δ is cube-free,

any double root of Δ lies in $A(\Delta, V)$.

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Therefore, since $\partial\beta < \frac{1}{2}\partial\Delta$, for λ small enough the roots of $\Delta + \lambda^2\beta^2$ are all real and distinct and lie 2 within each interval (necessarily finite) on which $\phi = 0$. Accordingly we construct α as in Lemma 3·4 using the lesser of each such pair of roots as our a_i, and γ, using the greater. This gives the result.

LEMMA 3·6. *Suppose* $\partial\Delta = 4$ *and* $\Delta > 0$; *then* $\exists M \in \mathbf{R}$ *such that* $\forall \lambda \in \mathbf{R}$ *with* $\lambda > M$

$$\Delta + \lambda^2 \text{ has no real roots};$$

and if $\Delta + \lambda^2 = \alpha\gamma$ *for* α, γ *monic* $\in \mathbf{R}[t]$

$$\partial(\alpha - \gamma) = 1.$$

Proof. It is clear that for λ large enough, $\Delta + \lambda^2$ has no real roots; suppose therefore that

$$\Delta = t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0$$

and that $\partial(\alpha - \gamma) < 1$ so that

$$\Delta + \lambda^2 = (t^2 + at + b)(t^2 + at + c)$$

for some $a, b, c \in \mathbf{R}$. Then

$$b + c = a_2 - \tfrac{1}{4}a_3^2,$$
$$bc = a_0 + \lambda^2$$

which for λ sufficiently large are impossible.

For the proof of (iv) of Theorem 3·2, we first use Lemmas 3·3 and 3·4 to construct $\mathbf{C}$ lattices in $g(\Delta, V, \theta)$, whose matrices $\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$ all have different α.

If for these lattices $\partial\alpha < \frac{1}{2}\partial\Delta$, they are all reduced, and either by Theorem 2·1 they are all inequivalent, or by Theorem 2·2 they fall into equivalence classes of at most $\aleph_0$ members each: so that in either case there are $\mathbf{C}$ classes of lattices in $g(\Delta, V, \theta)$.

But if for these lattices $\partial\alpha = \frac{1}{2}\partial\Delta$, then *either*

(*a*) case (1) of Lemma 3·4 holds; *or*

(*b*) $\partial\Delta = 4$ and case (3) of Lemma 3·4 holds, and $\Delta \in \hat{F}^2_\infty$.

In case (*a*) we use Lemma 3·5 with $\beta = 1$ and Lemma 3·3 to construct $\mathbf{C}$ lattices in $g(\Delta, V, \theta)$, with matrices $\begin{pmatrix} \alpha & \lambda \\ \lambda & \gamma \end{pmatrix}$ for $\lambda \in \mathbf{R}$ and $\lambda > 0$, which by Lemma 3·5 and Theorem 2·3 are inequivalent.

In case (*b*) for large $\lambda \in \mathbf{R}$ we take 2 conjugate complex roots of $\Delta + \lambda^2$ for a_1, a_2 in case (3) of Lemma 3·4, and by Lemma 3·3 construct $\mathbf{C}$ lattices in $g(\Delta, V, \theta)$, inequivalent by Lemma 3·6 and Theorem 2·3.

Thus in either case $g(\Delta, V, \theta)$ contains $\mathbf{C}$ inequivalent lattices and (iv) of Theorem 3·2 is proved.

Finally, the proof of (v) of Theorem 3·2: it will be in six parts, of each of which the pattern will be:

A. To list all possible reduced lattices in $g(\Delta, V, \theta)$.

B. To show *either* that exactly one of these has its top left-hand entry satisfying the second condition of Lemma 3·3—which phrase we abbreviate to 'satisfying'.

Or that if more than one of these have their top left-hand entry satisfying, then they are all equivalent.

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Then a proof similar to that of Lemma 3·3 (not exactly the same, because we do not need, and do not use, $(\alpha, \Delta) = 1$: we simply do not write down lattices of the wrong scale, to prevent which contingency was the condition $(\alpha, \Delta) = 1$ introduced) shows that a lattice exists; and B, together with the fact that if V represents α, and

$$\exists L \in g(\Delta, V, \theta) \quad \text{with} \quad L \cong \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix},$$

then $$\forall \xi \in \mathbf{R}, \quad s(V; \xi) \neq 0 \Rightarrow \operatorname{sign} \alpha(\xi) = \tfrac{1}{2} s(V; \xi)$$

and $$\xi \in A(\Delta, V) \Rightarrow \operatorname{sign} \alpha(\xi) = \theta(\xi),$$

shows us that it is unique.

1. $\partial \Delta = 0$.

A. $\begin{pmatrix} 1 & 0 \\ 0 & \Delta \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 0 & -\Delta \end{pmatrix}$.

B. $\Delta = 1 \Rightarrow \phi$ is never $0 \Rightarrow$ just one of $1, -1$ satisfies. (Recall (iii) of Theorem 3·2.) $\Delta = -1 \Rightarrow \phi$ is always $0 \Rightarrow 1$ and -1 satisfy; but

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

are immediately equivalent.

2. $\partial \Delta = 1$.

A. $\begin{pmatrix} 1 & 0 \\ 0 & \Delta \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 0 & -\Delta \end{pmatrix}$.

B. $\phi \neq 0$ on an interval; therefore just one of $1, -1$ satisfy.

3. $\partial \Delta = 2$, $\Delta > 0$.

A. $\begin{pmatrix} 1 & 0 \\ 0 & \Delta \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 0 & -\Delta \end{pmatrix} \quad \begin{pmatrix} t-a & 0 \\ 0 & t-b \end{pmatrix} \quad \begin{pmatrix} a-t & 0 \\ 0 & b-t \end{pmatrix}$

(the 2 latter iff Δ has distinct real roots).

B. We make a table:

Successive values of ϕ on $\mathbf{R}$	$1, 0, 1$	$1, 0, -1$	$-1, 0, 1$	$-1, 0, -1$
Precisely which of $1, -1, t-a, a-t$ satisfies	1	$a-t$	$t-a$	-1

4. $\partial \Delta = 2$, $\Delta < 0$, Δ square-free and with no real roots.

A. $\begin{pmatrix} 1 & 0 \\ 0 & \Delta \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 0 & -\Delta \end{pmatrix}$.

B. Let $\Delta = -(t^2 + 2dt + e)$; then

$$(e - d^2)^{-\frac{1}{2}} \begin{pmatrix} t+d & \Delta \\ 1 & -(t+d) \end{pmatrix}$$

takes one into the other; and since $\phi = 0$ everywhere, 1 and -1 satisfy.

5. $\partial \Delta = 2$, $\Delta < 0$, Δ square-free and with real roots.

A. $\begin{pmatrix} 1 & 0 \\ 0 & \Delta \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 0 & -\Delta \end{pmatrix}$.

B. $\phi \neq 0$ somewhere; therefore just one of $1, -1$ satisfy.

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6. $\Delta = -(t-a)^2$.

A. $\begin{pmatrix} \theta(a) & 0 \\ 0 & \theta(a)\Delta \end{pmatrix}$ $\begin{pmatrix} -\theta(a) & 0 \\ 0 & -\theta(a)\Delta \end{pmatrix}$.

B. $\theta(a)$ satisfies, $-\theta(a)$ does not.

The observation, that since all these lattices are diagonal, class and proper class coincide, completes the proof of (v), and thus of Theorem 3·2.

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