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Zeta functions for elements of entropy rank-one actions

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Abstract. An algebraic $\mathbb{Z}^d$-action of entropy rank one is one for which each element has finite entropy. Using the structure theory of these actions due to Einsiedler and Lind, this paper investigates dynamical zeta functions for elements of the action. An explicit periodic point formula is obtained leading to a uniform parameterization of the zeta functions that arise in expansive components of an expansive action, together with necessary and sufficient conditions for rationality in a more general setting.

1. *Introduction*

Let X be a compact metric space and $T : X \to X$ a map. Write

$$F_j(T) = \{x \in X : T^j(x) = x\}$$

for the set of points fixed by T^j and $|F_j(T)|$ for its cardinality. The dynamical *zeta function* of T is defined by the formal expression

$$\zeta_T(z) = \exp\left(\sum_{j=1}^{\infty} \frac{|F_j(T)|}{j} z^j\right).$$

This is a generalization of the Weil zeta function which arises from the Fröbenius map $x \to x^q$ acting simultaneously on each coordinate of a variety over the algebraic closure of a finite field $\mathbb{F}_q$. Such functions are well established as a tool in the study of orbit-growth properties of dynamical systems (see [**13**] for an overview) and can be traced back to the seminal study of periodic points by Artin and Mazur [**1**].

This paper concentrates on zeta functions of automorphisms on a compact abelian group X which arise from an algebraic $\mathbb{Z}^d$-action α. Throughout, X is assumed to be metrizable. The elemental zeta functions considered here should not be confused with the zeta function of the action, in the sense of Lind [**9**].

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There is an established methodology for understanding the dynamical system (X, α) based on the realization of the Pontryagin dual of X as a module $M = \widehat{X}$ over the ring of Laurent polynomials $R_d = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$, where $u_1, \ldots, u_d$ are commuting variables. This approach is covered in detail in Schmidt's monograph [**14**]. A central idea is to decompose α in terms of certain *prime actions*. These arise from R_d-modules of the form $R_d/\mathfrak{p}$, where $\mathfrak{p} \subset R_d$ is a prime ideal associated with the module M, more briefly $\mathfrak{p} \in \mathrm{Ass}(M)$. When M is Noetherian $\mathrm{Ass}(M)$ is finite. Typically, dynamical properties of $\alpha = \alpha_M$ are dually manifested as algebraic properties of the module M and *vice versa*. This leads to a mutual exchange of terminology. For example, the module M is said to be ergodic if α_M is ergodic; the action α_M is Noetherian if M is Noetherian.

This setting has enough scope to encompass both well-known phenomena, such as the rationality of zeta functions of ergodic toral automorphisms [**15**, §1.4] and more delicate irrational examples, such as [**7**].

In what follows, attention is restricted to actions α_M for which each element $\alpha_M^{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^d$, has finite entropy. Such actions are called *entropy rank-one* actions. One of the advantages of focusing on these actions is that if M is Noetherian and $\alpha_M^{\mathbf{n}}$ is ergodic then $|F_j(\alpha_M^{\mathbf{n}})| < \infty$ for all $j \in \mathbb{N}$. Moreover, using the adelic description of prime actions due to Einsiedler and Lind [**5**], we deduce the following periodic point formula.

THEOREM 3.2. *Let α_M be a Noetherian algebraic $\mathbb{Z}^d$-action of entropy rank one. If the automorphism $\alpha_M^{\mathbf{n}}$ is ergodic then the number of points of period $j \in \mathbb{N}$ for $\alpha_M^{\mathbf{n}}$ is finite and there are constants $m(\mathfrak{p})$, independent of j and $\mathbf{n}$, such that*

$$|F_j(\alpha_M^{\mathbf{n}})| = \prod_{\mathfrak{p} \in \mathrm{Ass}(M)} |F_j(\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}})|^{m(\mathfrak{p})}. \tag{1}$$

In §4, this formula is used to calculate zeta functions for elements of an action. Moreover, for many actions these functions invite meaningful comparison. In particular, there is a uniform parameterization of the zeta functions arising in an expansive component of an expansive action. A precise definition of an expansive component is given in §2. This is analogous to a phenomenon noticed for Markov shifts by Boyle and Lind in [**2**, Corollary 8.7]. The article [**2**] describes the notion of expansive subdynamics in general and this motivates the approach taken in §4.

To proceed with the statement of main results, some notation is helpful. For $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, a ring D and units $\xi_1, \ldots, \xi_d \in D$, let $\xi^{\mathbf{n}} = \xi_1^{n_1} \cdots \xi_d^{n_d}$. Given a prime ideal $\mathfrak{p} \subset R_d$, from now on $\xi_{\mathfrak{p}}^{\mathbf{n}}$ will denote the image of $u^{\mathbf{n}} = u_1^{n_1} \cdots u_d^{n_d} \in R_d$ in the domain $R_d/\mathfrak{p}$ under the natural quotient map. Where no confusion will arise, abbreviate $\zeta_{\alpha_M^{\mathbf{n}}}$ simply to $\zeta_{\mathbf{n}}$. Finally, $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

Closely related to the concept of entropy rank, is that of expansive rank (see [**6**] for the details). An algebraic $\mathbb{Z}^d$-action α has *expansive rank one* if there exists $\mathbf{n} \in \mathbb{Z}^d$ such that $\alpha^{\mathbf{n}}$ is expansive. Note that

$$\alpha \text{ has expansive rank one} \implies \alpha \text{ has entropy rank one.}$$

See [**5**, Proposition 4.3] or [**2**, Theorem 6.3] for a proof.

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THEOREM 4.4. *Let α_M be an ergodic algebraic $\mathbb{Z}^d$-action of expansive rank one. For an expansive component C of α, there are finitely many vectors $\Theta_1, \ldots, \Theta_r \in \overline{\mathbb{Q}}^d$ and indices $\lambda_1, \ldots, \lambda_r \in \{-1, 1\}$, depending only on M and C, such that for each $\mathbf{n} \in C \cap \mathbb{Z}^d$, $\zeta_{\mathbf{n}}$ is a rational function of the form*

$$\zeta_{\mathbf{n}}(z) = \prod_{k=1}^{r} (1 - \Theta_k^{\mathbf{n}} z)^{\lambda_k}.$$

Further to the statement of the theorem, the proof is constructive and calculations are carried out for some illustrative examples.

Finally, the following settles the problem of establishing which ergodic automorphisms in our framework can have a rational zeta function.

THEOREM 4.8. *Let α_M be a Noetherian algebraic $\mathbb{Z}^d$-action of entropy rank one. If $\alpha_M^{\mathbf{n}}$ is ergodic, then $\zeta_{\mathbf{n}}$ is rational if and only if for each $\mathfrak{p} \in \mathrm{Ass}(M)$ the only absolute values $|\cdot|_v$ unbounded on $R_d/\mathfrak{p}$ with $|\xi_{\mathfrak{p}}^{\mathbf{n}}|_v = 1$ are Archimedean.*

2. *Algebraic $\mathbb{Z}^d$-actions*

The following construction is explained in full detail in [**14**]. Let X be a compact abelian group, α a $\mathbb{Z}^d$-action by automorphisms $\alpha^{\mathbf{n}}$ on X and $M = \widehat{X}$ the Pontryagin dual of X. Then M is countable, discrete and becomes an R_d-module by identifying the dual automorphism $\widehat{\alpha^{\mathbf{n}}}$ with multiplication by $u^{\mathbf{n}}$ and extending this in a natural way to polynomials $f \in R_d$. That is, for $a \in M$ and $f = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_{\mathbf{n}} u^{\mathbf{n}} \in R_d$ set

$$fa = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_{\mathbf{n}} u^{\mathbf{n}} a.$$

Here all but finitely many $c_{\mathbf{n}} \in \mathbb{Z}$ are zero.

Let $\mathfrak{p} \subset R_d$ be a prime ideal. The *coheight* of $\mathfrak{p}$, denoted by $\mathrm{coht}(\mathfrak{p})$, is the maximum length r of a chain

$$\mathfrak{p} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r$$

of prime ideals. Hence, if $\mathfrak{p}$ is maximal then $\mathrm{coht}(\mathfrak{p}) = 0$. The coheight of $\mathfrak{p}$ coincides with the Krull dimension of the domain $R_d/\mathfrak{p}$. A prime ideal $\mathfrak{p}$ is *associated* with M if it is the annihilator of some element $a \in M$. That is, there exists $a \in M$ such that

$$\{f \in R_d : fa = 0\} = \mathfrak{p}.$$

Using the language of modules and associated primes, ergodicity is characterized as follows.

LEMMA 2.1. *Let α_M be an algebraic $\mathbb{Z}^d$-action.*

(1) *The automorphism $\alpha_M^{\mathbf{n}}$ is ergodic if and only if $(u^{k\mathbf{n}} - 1)x \neq 0$ for all $k \in \mathbb{N}$ and non-zero $x \in M$. Equivalently, no associated prime of M contains a polynomial of the form $u^{k\mathbf{n}} - 1$, with $k \in \mathbb{N}$.*

(2) *The action α_M is ergodic if and only if $\alpha_M^{\mathbf{n}}$ is ergodic for some $\mathbf{n} \in \mathbb{Z}^d$.*

Proof. See [**14**, Lemma 1.2 and Theorem 6.5]. □

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A Noetherian R_d-module M has a finite set of associated primes $\mathrm{Ass}(M)$ and admits a *prime filtration*, that is a chain of submodules

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M,$$

where for each $1 \leq k \leq n$ we have $M_k/M_{k-1} \cong R_d/\mathfrak{q}_k$ for a prime ideal $\mathfrak{q}_k \subset R_d$ which is either an associated prime of M or contains such a prime. Each minimal element of $\mathrm{Ass}(M)$ always appears in such a filtration with a fixed multiplicity [**5**, Lemma 8.2].

Noetherian modules hold particular significance for entropy rank-one actions. In general, there is a problem classifying those modules which correspond to entropy rank-one actions. The difficulty relates to non-Noetherian modules and the solution of Lehmer's problem, which has been open for over seventy years. A discussion of the issues involved is included in [**5**]. By concentrating on Noetherian modules, Lehmer's problem is avoided and the following structure theorem applies.

PROPOSITION 2.2. *Let α_M be a Noetherian algebraic $\mathbb{Z}^d$-action. Then the following hold.*

(1) *α_M has entropy rank one if and only if each of the associated prime actions $\alpha_{R_d/\mathfrak{p}}$ has entropy rank one. Equivalently, for each prime $\mathfrak{p} \in \mathrm{Ass}(M)$, $\mathrm{coht}(\mathfrak{p}) \leq 1$.*

(2) *α_M has expansive rank one if and only if each of the associated prime actions $\alpha_{R_d/\mathfrak{p}}$ has expansive rank one.*

(3) *If α_M is expansive then α_M has expansive rank one if and only if α_M has entropy rank one.*

Proof. This is a combination of [**5**, Proposition 4.4, Proposition 6.1, Theorems 7.1 and 7.2]. □

If α_M is a Noetherian algebraic $\mathbb{Z}^d$-action of entropy rank one, by Proposition 2.2 it follows that each associated prime $\mathfrak{p}$ has $\mathrm{coht}(\mathfrak{p}) = 1$ or is maximal. In the latter case $R_d/\mathfrak{p}$ is a field. The structure of R_d then forces $\mathfrak{p}$ to contain a rational prime p and the images of $u_1, \ldots, u_d$ present themselves as algebraic elements over the prime subring $\mathbb{F}_p$ in $R_d/\mathfrak{p}$. Hence $R_d/\mathfrak{p}$ will be finite. Since no automorphism of a finite field can be ergodic, the following is clear from Lemma 2.1.

LEMMA 2.3. *Suppose that α_M is a Noetherian algebraic $\mathbb{Z}^d$-action of entropy rank one and α_M is ergodic. Then* $\mathrm{coht}(\mathfrak{p}) = 1$ *for every* $\mathfrak{p} \in \mathrm{Ass}(M)$.

When $\mathrm{coht}(\mathfrak{p}) = 1$, two possibilities arise according to the characteristic of $R_d/\mathfrak{p}$. Let K be the field of fractions of $R_d/\mathfrak{p}$. If $R_d/\mathfrak{p}$ has characteristic $p > 0$ then it must contain an element t which is transcendental over $\mathbb{F}_p$. Furthermore, K will be a finite extension of $\mathbb{F}_p(t)$. If $R_d/\mathfrak{p}$ has characteristic zero then $K|\mathbb{Q}$ must be algebraic. Thus K turns out to be a global field. This enables standard adelic machinery to be applied to the prime actions $\alpha_{R_d/\mathfrak{p}}$ as described in [**5**]. A more general treatment of the adelic framework may be found in [**12**].

We now briefly outline the adelic description of a prime algebraic $\mathbb{Z}^d$-action $\alpha_{R_d/\mathfrak{p}}$ where $\mathrm{coht}(\mathfrak{p}) = 1$. Let K denote the field of fractions of $R_d/\mathfrak{p}$. The *places* of K, denoted by $\mathcal{P}(K)$, are given by equivalence classes of absolute values on K. Each place v leads to a distinct completion K_v of K. Such a field K_v is locally compact and is a *local field*. When v is non-Archimedean, the ultrametric inequality holds and the field K_v

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contains a *local ring of integers* $R_v = \{x \in K_v : |x|_v \leq 1\}$ with unique maximal ideal $\mathfrak{m}_v = \{x \in K_v : |x|_v < 1\}$ and finite *residue field* $F_v = R_v/\mathfrak{m}_v$.

When $\mathrm{char}(K) = p > 0$, a place is called *infinite* if it extends the infinite place $|\cdot|_\infty$ on $\mathbb{F}_p(t)$ given by $|f/g|_\infty = p^{\deg(f)-\deg(g)}$. In this setting all places are non-Archimedean. When $\mathrm{char}(K) = 0$, the *infinite* places are the Archimedean ones. A place v is *unbounded* on $D \subset K$ if $|D|_v$ is an unbounded subset of $\mathbb{R}$. Set

$$\mathcal{S}_\mathfrak{p} = \{v \in \mathcal{P}(K) : v \text{ is unbounded on } R_d/\mathfrak{p}\}.$$

Note that $\mathcal{S}_\mathfrak{p}$ contains the infinite places of K. Furthermore, $\mathcal{S}_\mathfrak{p}$ is finite because $R_d/\mathfrak{p}$ is finitely generated. The *adele ring* of $R_d/\mathfrak{p}$ is

$$\mathbb{A}_{R_d/\mathfrak{p}} = \prod_{v \in \mathcal{S}_\mathfrak{p}} K_v.$$

Multiplication and addition in this ring are defined coordinatewise. There is a natural embedding $\iota : K \to \mathbb{A}_{R_d/\mathfrak{p}}$ given by $\iota(a) = (a_v)$ where $a_v = a$ for all $v \in \mathcal{S}_\mathfrak{p}$. Einsiedler and Lind [**5**] establish the following important structural result for prime actions.

PROPOSITION 2.4. *Let $\mathfrak{p}$ be a prime ideal of R_d with $\mathrm{coht}(\mathfrak{p}) = 1$. Then there exists a diagonal $\mathbb{Z}^d$-action β on $\mathbb{A}_{R_d/\mathfrak{p}}$ and a discrete cocompact β-invariant subgroup $\Lambda \subset \mathbb{A}_{R_d/\mathfrak{p}}$ such that the quotient action on $\mathbb{A}_{R_d/\mathfrak{p}}/\Lambda$ is algebraically conjugate to $\alpha_{R_d/\mathfrak{p}}$.*

Proof. This is [**5**, Theorem 6.4]. □

Explicitly, the action β is given by $\beta^\mathbf{n}((x_v)) = \iota(\xi_\mathfrak{p}^\mathbf{n})(x_v)$ where $\xi_\mathfrak{p}^\mathbf{n}$ is as defined in §1. Where no confusion will arise, the embedding ι is suppressed and $\beta^\mathbf{n}$ is described simply as $(x_v) \mapsto (\xi_\mathfrak{p}^\mathbf{n} x_v)$.

We conclude this section by assembling some important facts concerning the dynamical property of expansiveness and the related concept of expansive subdynamics (see [**2**] and [**6**] for details). An algebraic $\mathbb{Z}^d$-action α on a compact abelian group X is *expansive* if there exists a neighbourhood U of 0_X such that

$$\bigcap_{\mathbf{n} \in \mathbb{Z}^d} \alpha^\mathbf{n}(U) = \{0_X\}.$$

[**14**, Theorem 6.5] gives algebraic criteria for expansiveness in terms of the *complex variety* of a prime ideal $\mathfrak{p} \subset R_d$, which is defined by

$$V_\mathbb{C}(\mathfrak{p}) = \{z \in (\mathbb{C}^\times)^d : f(z) = 0 \text{ for all } f \in \mathfrak{p}\}.$$

Let $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$. The following algebraic classifications of expansiveness will be useful.

PROPOSITION 2.5. *Let α_M be an algebraic $\mathbb{Z}^d$-action.*

(1) *α_M is expansive if and only if M is Noetherian and for each $\mathfrak{p} \in \mathrm{Ass}(M)$, $V_\mathbb{C}(\mathfrak{p}) \cap \mathbb{S}^d = \varnothing$.*

(2) *Given a submodule $N \subset M$, α_M is expansive if and only if both α_N and $\alpha_{M/N}$ are expansive.*

(3) *If $\mathfrak{p} \in \mathrm{Ass}(M)$ has $\mathrm{coht}(\mathfrak{p}) = 1$ then the automorphism $\alpha^\mathbf{n}_{R_d/\mathfrak{p}}$ is expansive if and only if $|\xi_\mathfrak{p}^\mathbf{n}|_v \neq 1$ for all $v \in \mathcal{S}_\mathfrak{p}$.*

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Proof. For the first two statements see [**14**, Theorem 6.5 and Corollary 6.15]. The final statement is a simpler version of [**10**, Theorem 4.2.4]. □

For entropy rank-one actions, there is also an unexpected relationship between expansiveness and ergodicity.

PROPOSITION 2.6. *Let α_M be an ergodic algebraic $\mathbb{Z}^d$-action of entropy rank one. If the automorphism $\alpha_M^{\mathbf{n}}$ is expansive then it is also ergodic.*

Proof. Since the automorphism $\alpha_M^{\mathbf{n}}$ is expansive, the action α_M is expansive and Proposition 2.5 implies that M is Noetherian. Hence Lemma 2.3 shows that $\mathrm{coht}(\mathfrak{p}) = 1$ for all $\mathfrak{p} \in \mathrm{Ass}(M)$. For a contradiction, suppose that $\alpha_M^{\mathbf{n}}$ is not ergodic. Then by Lemma 2.1 there exists $k \in \mathbb{N}$ such that $u^{k\mathbf{n}} - 1 \in \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}(M)$. Hence $\xi_{\mathfrak{p}}^{\mathbf{n}}$ is a kth root of unity in $R_d/\mathfrak{p}$ and so there exists $v \in \mathcal{S}_{\mathfrak{p}}$ for which $|\xi_{\mathfrak{p}}^{\mathbf{n}}|_v = 1$. Hence, by Proposition 2.5(3), $\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}}$ is not expansive. But $\alpha_M^{\mathbf{n}}$ is expansive and since $R_d/\mathfrak{p}$ is isomorphic to a submodule of M, Proposition 2.5(2) shows that $\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}}$ should also be expansive. □

Following [**2**], let H be a linear subspace of $\mathbb{R}^d$ of dimension $1 \leq k \leq d$, more briefly a *k-plane*. An algebraic $\mathbb{Z}^d$-action α on a compact abelian group X is *expansive along H* if there is a neighbourhood U of 0_X and a ball $B(r)$ around 0 in $\mathbb{R}^d$ such that

$$\bigcap_{\mathbf{n} \in (H + B(r)) \cap \mathbb{Z}^d} \alpha^{\mathbf{n}}(U) = \{0_X\}.$$

Denote the set of k-planes along which α is non-expansive by $\mathsf{N}_k(\alpha)$. For $d > 1$, a fundamental result of [**2**, Theorem 3.7] is that a k-plane, $1 \leq k \leq d-1$, is non-expansive if and only if it is contained in some non-expansive $(d-1)$-plane. Furthermore, [**6**, Theorem 8.4] shows that $\mathsf{N}_{d-1}(\alpha_M)$ is the union of $\mathsf{N}_{d-1}(\alpha_{R_d/\mathfrak{p}})$ over all primes $\mathfrak{p} \in \mathrm{Ass}(M)$. For expansive rank-one actions, more can be said. Throughout, $\mathbf{e}(1), \ldots, \mathbf{e}(d)$ denote the standard basis vectors in $\mathbb{Z}^d$.

PROPOSITION 2.7. *Suppose α_M is an algebraic $\mathbb{Z}^d$-action of expansive rank one. Then*

$$\mathsf{N}_{d-1}(\alpha_M) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathsf{N}_{d-1}(\alpha_{R_d/\mathfrak{p}})$$

is finite and each non-expansive $(d-1)$-plane is defined by an equation of the form

$$|\xi_{\mathfrak{p}}^{\mathbf{e}(1)}|_v^{l_1} \cdots |\xi_{\mathfrak{p}}^{\mathbf{e}(d)}|_v^{l_d} = 1$$

for some associated prime $\mathfrak{p}$, $v \in \mathcal{S}_{\mathfrak{p}}$ and $\mathbf{l} = (l_1, \ldots, l_d) \in \mathbb{R}^d$. Moreover, every possible equation of this form—as $\mathfrak{p}$ runs through the minimal associated primes of M and as v runs though the places in $\mathcal{S}_{\mathfrak{p}}$—determines a non-expansive $(d-1)$-plane.

Proof. See [**5**, Theorem 7.1 and Theorem 7.2]. □

Therefore, the complement of $\mathsf{N}_1(\alpha_M)$ in $\mathbb{R}^d$, comprising all expansive one-dimensional subspaces, consists of finitely many connected components (in the Grassmannian topology) and in what follows these will be called *expansive components*.

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Proposition 2.7 essentially gives a description of $\mathsf{N}_{d-1}(\alpha_M)$ in terms of a Lyapunov list [**5**, §8] which is a natural extension of the familiar tool used in the study of toral automorphisms. Unfortunately, the Lyapunov list for an entropy rank-one action does not quite carry enough information for calculating zeta functions, as will become clear in §4, so this perspective is not pursued here.

3. *Periodic points*

Consider a prime action of the form $\alpha_{R_d/\mathfrak{p}}$ where $\mathrm{coht}(\mathfrak{p}) = 1$. A straightforward comparison of the S-integer systems in [**3**] with those handled by Proposition 2.4 reveals a common structure. In [**3**] the authors deal with endomorphisms dual to multiplication by elements of integrally closed domains with global field of fractions. Although domains of the form $R_d/\mathfrak{p}$ are not necessarily integrally closed, [**5**, Theorem 6.4] enables elements of the associated action $\alpha_{R_d/\mathfrak{p}}$ to be handled using the same methods as in [**3**]. In particular, there is a formula for the number of periodic points.

LEMMA 3.1. *Let $\alpha_{R_d/\mathfrak{p}}$ be a prime action of entropy rank one. If $\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}$ is ergodic then the number of points of period $j \in \mathbb{N}$ for $\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}$ is finite and*

$$|F_j(\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}})| = \prod_{v \in \mathcal{S}_{\mathfrak{p}}} |\xi_{\mathfrak{p}}^{j\mathbf{n}} - 1|_v.$$

Proof. By Lemma 2.1(2), ergodicity of $\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}$ means Lemma 2.3 also applies. Now use Proposition 2.4 and follow the proof of [**3**, Lemma 5.2]. □

This may be combined with the following to yield a more general periodic point formula.

THEOREM 3.2. *Let α_M be a Noetherian algebraic $\mathbb{Z}^d$-action of entropy rank one. If the automorphism $\alpha^{\mathbf{n}}_M$ is ergodic then the number of points of period $j \in \mathbb{N}$ for $\alpha^{\mathbf{n}}_M$ is finite and there are constants $m(\mathfrak{p})$, independent of j and $\mathbf{n}$, such that*

$$|F_j(\alpha^{\mathbf{n}}_M)| = \prod_{\mathfrak{p} \in \mathrm{Ass}(M)} |F_j(\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}})|^{m(\mathfrak{p})}. \tag{2}$$

For the duration of the proof, fix $j \in \mathbb{N}$, set $b = u^{j\mathbf{n}} - 1$ and let $\mathfrak{b}$ be the principal ideal generated by b.

LEMMA 3.3. *For any R_d-module N, whenever $N/\mathfrak{b}N$ or $F_j(\alpha^{\mathbf{n}}_N)$ is finite*

$$|F_j(\alpha^{\mathbf{n}}_N)| = |N/\mathfrak{b}N|.$$

Proof. This is a simplification of [**8**, Lemma 7.2]. □

LEMMA 3.4. *Let $L \subset N$ be submodules of M with $N/L \cong R_d/\mathfrak{q}$ for a non-trivial prime ideal $\mathfrak{q} \subset R_d$. Assume that $\alpha^{\mathbf{n}}_M$ is ergodic.*

(1) *If $b \in \mathfrak{q}$ then $\mathfrak{b}N \cap L = \mathfrak{b}N$ and $N/L \cong \mathfrak{b}N/\mathfrak{b}L$.*
(2) *If $b \notin \mathfrak{q}$ then $\mathfrak{b}N \cap L = \mathfrak{b}L$. If in addition $\mathfrak{q}$ is assumed to be maximal then $N/L \cong \mathfrak{b}(N/L)$.*

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Proof. Since N/L is a cyclic module there exists $a \in N \setminus L$ such that $N = R_d a + L$ with $fa \in L$ if and only if $f \in \mathfrak{q}$.

Assume $b \in \mathfrak{q}$. Let $x \in \mathfrak{b}N$ and write $x = b(fa + l)$, $f \in R_d$, $l \in L$. Then $x = f(ba) + bl \in L$. Hence $\mathfrak{b}N \cap L = \mathfrak{b}N$.

To see that $N/L \cong \mathfrak{b}N/\mathfrak{b}L$, let $\phi : N \to \mathfrak{b}N/\mathfrak{b}L$ be defined by multiplication by b, followed by the natural quotient map. Clearly $L \subset \ker(\phi)$. To see that $\ker(\phi) = L$, assume for a contradiction that $\ker(\phi) \setminus L \neq \varnothing$. Then there exists $f \in R_d \setminus \mathfrak{q}$ and $l, l' \in L$ such that

$$b(fa + l) = bl' \Rightarrow b(fa + l - l') = 0.$$

Now $fa + l - l' \neq 0$ (since $fa \notin L$) and this element of M is annihilated by b. But by Lemma 2.1 this contradicts the ergodicity assumption on $\alpha^{\mathbf{n}}_M$.

Now assume $b \notin \mathfrak{q}$. Let $x \in \mathfrak{b}N \cap L$ and again write $x = b(fa + l)$, $f \in R_d$, $l \in L$. Then $fba = x - bl \in L$ and hence $fb \in \mathfrak{q}$. Since $b \notin \mathfrak{q}$, the primality of $\mathfrak{q}$ means $f \in \mathfrak{q}$, in which case $fa + l \in L$. Hence $x \in \mathfrak{b}L$.

To see that $N/L \cong \mathfrak{b}(N/L)$ when $\mathfrak{q}$ is maximal, note $\mathfrak{q} \subsetneq \mathfrak{b} + \mathfrak{q}$ and hence

$$\mathfrak{b}(N/L) \cong \mathfrak{b}(R_d/\mathfrak{q}) = (\mathfrak{b} + \mathfrak{q})/\mathfrak{q} = R_d/\mathfrak{q}. \qquad \square$$

Proof of Theorem 3.2. Let $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M$ be a prime filtration of M. We have $M_k/M_{k-1} \cong R_d/\mathfrak{q}_k$ where $\mathfrak{q}_k$ is an associated prime of M or $\mathfrak{q}_k$ contains such a prime, in which case $\mathfrak{q}_k$ will be maximal as the associated primes all have coheight 1. Our aim is to obtain the inductive formula

$$|M_k/\mathfrak{b}M_k| = \begin{cases} |(M_k/M_{k-1})/(\mathfrak{b}(M_k/M_{k-1}))||M_{k-1}/\mathfrak{b}M_{k-1}| & \text{when } \mathfrak{q}_k \in \operatorname{Ass}(M), \\ |M_{k-1}/\mathfrak{b}M_{k-1}| & \text{otherwise,} \end{cases} \tag{3}$$

where $1 \leq k \leq n$ and $|\cdot|$ denotes the cardinality of the module. When $\mathfrak{q}_k$ is associated with M, $\alpha^{\mathbf{n}}_{R_d/\mathfrak{q}_k}$ is ergodic by Lemma 2.1. Lemma 3.1 ensures that $F_j(\alpha^{\mathbf{n}}_{R_d/\mathfrak{q}_k})$ is finite and Lemma 3.3 shows that

$$\left| \frac{M_k/M_{k-1}}{\mathfrak{b}(M_k/M_{k-1})} \right| = |F_j(\alpha^{\mathbf{n}}_{R_d/\mathfrak{q}_k})|.$$

Therefore, by induction the above will show that each $M_k/\mathfrak{b}M_k$ is finite. Furthermore, (2) will follow by defining $m(\mathfrak{p})$ to be the multiplicity of $\mathfrak{p}$ in the filtration.

To establish (3), proceed as follows. For ease of notation, for the remainder of the proof let $N = M_k$, $L = M_{k-1}$ and $\mathfrak{q} = \mathfrak{q}_k$. Since

$$\frac{N}{\mathfrak{b}N + L} \cong \frac{N/\mathfrak{b}N}{(\mathfrak{b}N + L)/\mathfrak{b}N}$$

it follows that

$$|N/\mathfrak{b}N| = \left| \frac{N/L}{(\mathfrak{b}N + L)/L} \right| \left| \frac{\mathfrak{b}N + L}{\mathfrak{b}N} \right| = \left| \frac{N/L}{\mathfrak{b}(N/L)} \right| \left| \frac{L}{\mathfrak{b}N \cap L} \right|. \tag{4}$$

When $\mathfrak{q}$ is an associated prime of M, by Lemma 2.1, case 2 of Lemma 3.4 may be applied to deliver (3). Hence it remains to consider the case when $\mathfrak{q}$ is not an associated prime of M. Then $\mathfrak{q}$ is forced to be maximal and N/L is finite.

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In case 1 of Lemma 3.4, $\mathfrak{b}$ annihilates N/L and (4) becomes

$$|N/\mathfrak{b}N| = |N/L||L/\mathfrak{b}N| = |N/L|\left|\frac{L/\mathfrak{b}L}{\mathfrak{b}N/\mathfrak{b}L}\right|.$$

The inductive formula follows, since in this instance $\mathfrak{b}N/\mathfrak{b}L \cong N/L$. In case 2 of Lemma 3.4, equation (4) becomes

$$|N/\mathfrak{b}N| = \left|\frac{N/L}{\mathfrak{b}(N/L)}\right||L/\mathfrak{b}L|.$$

Again (3) is obtained, since here $\mathfrak{b}(N/L) \cong N/L$. □

Remark 3.5. The proof of Theorem 3.2 relies heavily on the structure of entropy rank-one actions. Any primes, appearing in the chosen filtration, that strictly contain an associated prime are forced to be maximal. Together with the ergodicity assumption, this effectively removes potential contributions to the periodic point formula from non-associated primes appearing in the chosen filtration.

4. *Zeta functions*

Our aim is to now combine Lemma 3.1 and Theorem 3.2 to enable explicit calculation of dynamical zeta functions for elements of a given action. Notation is made simpler by using characters and lists. Throughout this section, a *character* is a homomorphism $\chi : \mathbb{Z}^d \to \mathbb{C}^\times$. The group of all such characters is isomorphic to the punctured affine space $(\mathbb{C}^\times)^d$. A *real character* is one with real image. By a *list* we mean a finite sequence of the form $L = \langle \chi_1, \ldots, \chi_n \rangle$ which allows for multiplicities. The sum $\sum_{J \subset L}$ infers that 2^n summands always appear. For combinatorial purposes it is useful to know whether a list has an even or an odd number of elements. Hence, set

$$\omega(L) = (-1)^n.$$

Finally, when L is a list of characters, use the notation

$$\chi_L = \chi_1 \chi_2 \cdots \chi_n$$

to denote the product over all elements of L, with the understanding that $\chi_\varnothing \equiv 1$.

Let $\mathfrak{p} \subset R_d$ be a prime ideal with $\operatorname{coht}(\mathfrak{p}) = 1$ and let K be the global field of fractions of $R_d/\mathfrak{p}$. Assume that $\operatorname{char}(R_d/\mathfrak{p}) = 0$ so all infinite places are Archimedean. These may be obtained from the embeddings of K into $\mathbb{C}$, which are uniquely determined by the embeddings of $R_d/\mathfrak{p}$ into $\mathbb{C}$. In particular, a point $z \in V_{\mathbb{C}}(\mathfrak{p})$ determines a ring homomorphism into $\mathbb{C}$ via the substitution map $f + \mathfrak{p} \mapsto f(z)$ which is well defined since $g(z) = 0$ for all $g \in \mathfrak{p}$. The map is injective because $R_d/\mathfrak{p}$ has Krull dimension one (a non-trivial kernel would imply the existence of an injective ring homomorphism of a finite field into $\mathbb{C}$). Note that each $z \in V_{\mathbb{C}}(\mathfrak{p})$ induces a character on $\mathbb{Z}^d$ in an obvious way, there are finitely many such characters and the coordinates of these are all algebraic numbers. More generally, any place v on a domain of the form $R_d/\mathfrak{p}$ induces a real character on $\mathbb{Z}^d$ via the values taken by $|\xi_{\mathfrak{p}}^{\mathbf{e}(i)}|_v$, $i = 1, \ldots, d$. This method will always be used when constructing characters using non-Archimedean places.

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The following lists of characters will be important. For a prime $\mathbb{Z}^d$-action $\alpha_{R_d/\mathfrak{p}}$ with $\mathrm{coht}(\mathfrak{p}) = 1$, let $\mathcal{W}(R_d/\mathfrak{p})$ be the list of characters induced by the non-Archimedean $v \in \mathcal{S}_{\mathfrak{p}}$ and let $\mathcal{V}(R_d/\mathfrak{p})$ be the list of characters induced by the distinct complex embeddings of $R_d/\mathfrak{p}$. Note that $\mathcal{V}(R_d/\mathfrak{p}) = \varnothing$ when $\mathrm{char}(R_d/\mathfrak{p}) > 0$. For a general $\mathbb{Z}^d$-action α_M of entropy rank one set

$$\mathcal{W}(M) = \bigcup \mathcal{W}(R_d/\mathfrak{p}),$$
$$\mathcal{V}(M) = \bigcup \mathcal{V}(R_d/\mathfrak{p}),$$

where the union of lists is taken over all associated primes $\mathfrak{p}$ of M with $\mathrm{coht}(\mathfrak{p}) = 1$, incorporating their multiplicity $m(\mathfrak{p})$.

LEMMA 4.1. *Let α_M be an algebraic $\mathbb{Z}^d$-action of entropy rank one and suppose that $\alpha_M^{\mathbf{n}}$ is ergodic. If for each $\mathfrak{p} \in \mathrm{Ass}(M)$ the only places $v \in \mathcal{S}_{\mathfrak{p}}$ with $|\xi_{\mathfrak{p}}^{\mathbf{n}}|_v = 1$ are Archimedean, then*

$$|F_j(\alpha_M^{\mathbf{n}})| = \sum_{L \subset \mathcal{V}(M)} \omega(L) \chi_{\mathcal{T}} \chi_{\mathcal{R}}^{(\pm)} \chi_{\mathcal{R} \cap L'} \chi_{\mathcal{R}' \cap L}(j\mathbf{n}), \tag{5}$$

where $\mathcal{R}$ is the list of real characters χ in $\mathcal{V}(M)$ with $|\chi(\mathbf{n})| > 1$, $\mathcal{T}$ is the list of characters χ in $\mathcal{W}(M)$ with $\chi(\mathbf{n}) > 1$, L' and $\mathcal{R}'$ denote the respective complements of L and $\mathcal{R}$ in $\mathcal{V}(M)$ and for any real character χ, $\chi^{(\pm)}$ is the character given by $\chi^{(\pm)}(\mathbf{n}) = \chi(\mathbf{n})/|\chi(\mathbf{n})|$.

Proof. According to Lemma 3.1 and Theorem 3.2, $|F_j(\alpha_M^{\mathbf{n}})|$ is a product of expressions of the form $|\xi_{\mathfrak{p}}^{j\mathbf{n}} - 1|_v$ where $\mathfrak{p}$ runs through the associated primes of M and v runs through all places of $\mathcal{S}_{\mathfrak{p}}$. To eliminate the exponents $m(\mathfrak{p})$ arising from Theorem 3.2, simply take the product over a list which incorporates this multiplicity.

Let $v \in \mathcal{W}(M)$ be a non-Archimedean place arising from some $\mathfrak{p} \in \mathrm{Ass}(M)$ and χ the character induced by v. By assumption $|\xi_{\mathfrak{p}}^{\mathbf{n}}|_v \neq 1$. Therefore, by the ultrametric inequality

$$|\xi_{\mathfrak{p}}^{j\mathbf{n}} - 1|_v = \begin{cases} \chi(j\mathbf{n}) & \text{when } \chi \in \mathcal{T}, \\ 1 & \text{otherwise.} \end{cases}$$

This accounts for the factor $\chi_{\mathcal{T}}$ in (5).

Hence, it remains to consider the product over any Archimedean places v. If v is a complex place of $R_d/\mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}(M)$, then it is given by the product of a conjugate pair of embeddings of $R_d/\mathfrak{p}$ into $\mathbb{C}$. Hence there are a pair of characters $\chi, \overline{\chi}$ in $\mathcal{V}(\alpha_{R_d/\mathfrak{p}})$ with $|\xi_{\mathfrak{p}}^{j\mathbf{n}} - 1|_v = (1 - \chi(j\mathbf{n}))(1 - \overline{\chi}(j\mathbf{n}))$. If v is real and induces the character χ note that $\chi(j\mathbf{n}) \neq 1$. Moreover,

$$|\xi_{\mathfrak{p}}^{j\mathbf{n}} - 1|_v = \begin{cases} \chi^{(\pm)}(j\mathbf{n})\chi(j\mathbf{n})(1 - \chi(-j\mathbf{n})) & \text{when } \chi \in \mathcal{R}, \\ 1 - \chi(j\mathbf{n}) & \text{otherwise.} \end{cases}$$

Therefore, the product of $|\xi_{\mathfrak{p}}^{j\mathbf{n}} - 1|_v$ over the list of Archimedean places v and associated primes $\mathfrak{p}$ is

$$\chi_{\mathcal{R}}^{(\pm)}(j\mathbf{n})\chi_{\mathcal{R}}(j\mathbf{n}) \prod_{\chi \in \mathcal{V}(M)} 1 - \chi(\lambda_\chi j\mathbf{n}), \tag{6}$$

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where $\lambda_\chi = -1$ for $\chi \in \mathcal{R}$ and $\lambda_\chi = 1$ otherwise. The symmetric polynomial expansion of the product in (6) is

$$\sum_{L \subset \mathcal{V}(M)} \omega(L) \prod_{\chi \in L} \chi(\lambda_\chi j\mathbf{n}) = \sum_{L \subset \mathcal{V}(M)} \omega(L) \chi_{\mathcal{R}\cap L}(-j\mathbf{n}) \chi_{\mathcal{R}'\cap L}(j\mathbf{n}). \tag{7}$$

Furthermore $\chi_{\mathcal{R}}(j\mathbf{n})\chi_{\mathcal{R}\cap L}(-j\mathbf{n}) = \chi_{\mathcal{R}\cap L'}(j\mathbf{n})$, which together with (7) transforms (6) to

$$\sum_{L \subset \mathcal{V}(M)} \omega(L) \chi_{\mathcal{R}}^{(\pm)} \chi_{\mathcal{R}\cap L'} \chi_{\mathcal{R}'\cap L}(j\mathbf{n}). \qquad \square$$

Remarks 4.2. If the conditions of Lemma 4.1 are satisfied, it is clear from (5) that $\zeta_{\mathbf{n}}$ will be rational. It also follows from (5) that the radius of convergence $r_{\mathbf{n}}$ of $\zeta_{\mathbf{n}}$ is governed by the dominant term in the sum, that is the one given by $L = \langle \chi \in \mathcal{V}(M) : |\chi(\mathbf{n})| > 1 \rangle$. By [**5**, Proposition 8.5]

$$\log |\chi_{\mathcal{T}} \chi_L(\mathbf{n})| = h(\alpha_M^{\mathbf{n}})$$

where $h(\alpha_M^{\mathbf{n}})$ is the entropy of $\alpha_M^{\mathbf{n}}$. Hence $r_{\mathbf{n}} = \exp(-h(\alpha_M^{\mathbf{n}}))$.

Lemma 4.1 may always be applied when $\alpha_M^{\mathbf{n}}$ is expansive and ergodic. To see this, note that by Proposition 2.5(2), for algebraic $\mathbb{Z}^d$-actions the property of expansiveness passes to submodules. Hence if $\alpha_M^{\mathbf{n}}$ is expansive then $\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}}$ is also expansive for each $\mathfrak{p} \in \mathrm{Ass}(M)$. Proposition 2.5(3) then shows that $|\xi_{\mathfrak{p}}^{\mathbf{n}}|_v \neq 1$ for all $v \in \mathcal{S}_{\mathfrak{p}}$ and $\mathfrak{p} \in \mathrm{Ass}(M)$.

However, it is also clear that expansiveness is not a necessary condition for rationality of the zeta function, as the following partially hyperbolic $\mathbb{Z}^2$-action from [**4**] shows quite dramatically.

Example 4.3. Let $f(x) = x^6 - 2x^5 - 5x^4 - 3x^3 - 5x^2 - 2x + 1$ and consider the algebraic number field $\mathbb{Q}(a)$ corresponding to $f(x)$, where a is a complex root of f. Then a and $b = 2a^5 - 6a^4 - 3a^3 - 6a^2 - 6a$ are fundamental units and $M = \mathbb{Z}[a]$ becomes an R_2-module via the substitutions $u_1 \mapsto a$, $u_2 \mapsto b$. Since M has $\mathbb{Z}$-basis $1, a, a^2, a^3, a^4, a^5$, α_M is a $\mathbb{Z}^2$-action on $\widehat{M} \cong \mathbb{T}^6$. Also, M is a domain and so it has one associated prime $\mathfrak{p} \subset R_2$ (simply the kernel of the substitution map) and this has $\mathrm{coht}(\mathfrak{p}) = 1$. Therefore, α_M has entropy rank one. By construction, the only places unbounded on $M \cong R_2/\mathfrak{p}$ are Archimedean: one complex and four real. The complex place w has $|a|_w = |b|_w = 1$ and so $\alpha_M^{\mathbf{n}}$ is non-expansive for all $\mathbf{n} \in \mathbb{Z}^2$. However, the multiplicative independence of a and b means that $\alpha_M^{\mathbf{n}}$ is ergodic for every $\mathbf{n} \in \mathbb{Z}^2$. Thus Lemma 4.1 applies and $\zeta_{\mathbf{n}}$ is rational for every $\mathbf{n} \in \mathbb{Z}^2$.

The following theorem should be compared with the results of Smale [**15**, Proposition 4.15] and Boyle and Lind [**2**, Corollary 8.7].

THEOREM 4.4. *Let α_M be an ergodic algebraic $\mathbb{Z}^d$-action of expansive rank one. For an expansive component C of α, there are finitely many vectors $\Theta_1, \ldots, \Theta_r \in \overline{\mathbb{Q}}^d$ and indices $\lambda_1, \ldots, \lambda_r \in \{-1, 1\}$, depending only on M and C, such that for each $\mathbf{n} \in C \cap \mathbb{Z}^d$, $\zeta_{\mathbf{n}}$ is a rational function of the form*

$$\zeta_{\mathbf{n}}(z) = \prod_{k=1}^{r} (1 - \Theta_k^{\mathbf{n}} z)^{\lambda_k}.$$

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Proof. First note that since α_M is ergodic, Proposition 2.6 shows that any expansive element is also ergodic. Moreover, since expansive rank one $\Rightarrow$ entropy rank one, Lemma 4.1 may be applied to any expansive automorphism $\alpha_M^{\mathbf{n}}$. Let $\mathbf{n}, \mathbf{m} \in C \cap \mathbb{Z}^d$. Using the notation of Lemma 4.1, for a sublist $L \subset \mathcal{V}(M)$ set $\theta(L) = (\theta_1(L), \ldots, \theta_d(L)) \in \overline{\mathbb{Q}}^d$, where for $i = 1, \ldots, d$,

$$\theta_i(L) = \chi_{\mathcal{T}} \chi_{\mathcal{R}}^{(\pm)} \chi_{\mathcal{R} \cap L'} \chi_{\mathcal{R}' \cap L}(\mathbf{e}_i).$$

Then

$$\begin{aligned}
\zeta_{\mathbf{n}}(z) &= \exp\left(\sum_{j=1}^{\infty} \frac{|F_j(\alpha_M^{\mathbf{n}})|}{j} z^j \right) \\
&= \exp\left(\sum_{L \subset \mathcal{V}(M)} \omega(L) \sum_{j=1}^{\infty} \frac{(\theta(L)^{\mathbf{n}} z)^j}{j} \right) \\
&= \exp\left(\sum_{L \subset \mathcal{V}(M)} \log(1 - \theta(L)^{\mathbf{n}} z)^{-\omega(L)} \right) \\
&= \prod_{L \subset \mathcal{V}(M)} (1 - \theta(L)^{\mathbf{n}} z)^{-\omega(L)}.
\end{aligned}$$

After appropriate relabelling of the vectors $\theta(L)$ and corresponding indices $\omega(L)$, this gives the required form of the zeta function for $\alpha_M^{\mathbf{n}}$.

The lists $\mathcal{R}$ and $\mathcal{T}$ are as defined in Lemma 4.1 and so are implicitly dependent on $\mathbf{n}$. However, $\mathcal{R}$ and $\mathcal{T}$ remain unchanged if defined in terms of $\mathbf{m}$. To see this, consider the following. A character $\chi \in \mathcal{R} \cup \mathcal{T}$ extends to a character $\chi^* : \mathbb{R}^d \to \mathbb{R}^\times$ by defining $\chi^*(\kappa \mathbf{e}_i) = |\chi(\mathbf{e}_i)|^\kappa$ for $i = 1, \ldots, d$ and $\kappa \in \mathbb{R}$. If $\chi^*(\mathbf{m}) \leq 1$, since C is connected, continuity of χ^* means the intermediate value theorem applies. Therefore, since by assumption $\chi^*(\mathbf{n}) > 1$, there exists $\mathbf{l} \in C$ such that $\chi^*(\mathbf{l}) = 1$. However, χ^* is induced by a place in $\mathcal{S}_{\mathfrak{p}}$ for some $\mathfrak{p}$ associated with M. Hence by Proposition 2.7, $\mathbf{l}$ defines a non-expansive hyperplane intersecting the expansive component C. This contradiction proves the claim. Hence the stated formula, with $\mathbf{m}$ in place of $\mathbf{n}$, is valid for the zeta function $\zeta_{\mathbf{m}}$. □

Example 4.5. (Ledrappier's example) Following [**5**, Example 6.8], Ledrappier's example corresponds to the R_2-module $R_2/\mathfrak{p}$, where $\mathfrak{p} = (2, 1 + u_1 + u_2)$. Furthermore, $R_2/\mathfrak{p} \cong \mathbb{F}_2[t, (1 + t)^{-1}]$, where the isomorphism is given by $u_1 \mapsto t$, $u_2 \mapsto 1 + t$. The set of places $\mathcal{S}_{\mathfrak{p}}$ comprises the usual infinite place of $\mathbb{F}_2(t)$ together with the places induced by the discrete valuation rings obtained by localizing $\mathbb{F}_2[t]$ at the prime ideals (t) and $(1 + t)$. These non-Archimedean places give $\mathcal{W}(M) = \{(2, 2), (\frac{1}{2}, 1), (1, \frac{1}{2})\}$ and since $\mathrm{char}(R_2/\mathfrak{p}) > 0$, $\mathcal{V}(M) = \varnothing$. There are six expansive components for this example given by the regions of $\mathbb{R}^2$ bounded by the three intersecting lines $|t|_v^{n_1} |1 + t|_v^{n_2} = 1$, $v \in \mathcal{S}_{\mathfrak{p}}$. Starting in the first quadrant and working anticlockwise, label these $C_1, \ldots, C_6$.

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For $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$,

$$|F_j(\alpha_M^{\mathbf{n}})| = \prod_{v \in \mathcal{S}_\mathfrak{p}} |t^{jn_1}(1+t)^{jn_2} - 1|_v$$
$$= \begin{cases} 2^{j(|n_1|+|n_2|)}, & \mathbf{n} \in C_1 \cup C_4, \\ 2^{j|n_2|}, & \mathbf{n} \in C_2 \cup C_5, \\ 2^{j|n_1|}, & \mathbf{n} \in C_3 \cup C_6. \end{cases}$$

Hence the zeta functions are given by

$$\zeta_{\mathbf{n}}(z) = \begin{cases} 1/(1 - 2^{|n_1|+|n_2|}z), & \mathbf{n} \in C_1 \cup C_4, \\ 1/(1 - 2^{|n_2|}z), & \mathbf{n} \in C_2 \cup C_5, \\ 1/(1 - 2^{|n_1|}z), & \mathbf{n} \in C_3 \cup C_6. \end{cases}$$

Example 4.6. (Commuting solenoidal automorphisms) Consider the $\mathbb{Z}^2$-action which is dual to $\times 2$ and $\times 3$ on $M = \mathbb{Z}[\frac{1}{6}]$. That is, the dynamical system corresponding to the R_2-module $R_2/\mathfrak{p}$ where $\mathfrak{p} = (u_1 - 2, u_2 - 3)$. Then, $\mathcal{S}_\mathfrak{p} = \{2, 3, \infty\}$, $\mathcal{V}(M) = \{(2, 3)\}$ and $\mathcal{W}(M) = \{(\frac{1}{2}, 1), (1, \frac{1}{3})\}$. Note that $\widehat{M}$ is locally isomorphic to $\mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3$. There are 6 expansive components for this action, which are the regions of $\mathbb{R}^2$ bounded by the three intersecting lines $|2|_v^{n_1}|3|_v^{n_2} = 1$, $v \in \mathcal{S}_\mathfrak{p}$. As in the previous example, starting in the first quadrant and working anticlockwise, label these $C_1, \ldots, C_6$. For $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$,

$$|F_j(\alpha_M^{\mathbf{n}})| = |2^{jn_1}3^{jn_2} - 1|_\infty |2^{jn_1}3^{jn_2} - 1|_2 |2^{jn_1}3^{jn_2} - 1|_3$$
$$= \begin{cases} 2^{j|n_1|}3^{j|n_2|} - 1, & \mathbf{n} \in C_1 \cup C_4, \\ 3^{j|n_2|} - 2^{j|n_1|}, & \mathbf{n} \in C_2 \cup C_5, \\ 2^{j|n_1|} - 3^{j|n_2|}, & \mathbf{n} \in C_3 \cup C_6. \end{cases}$$

Therefore, the following parameterization of zeta functions is obtained.

$$\zeta_{\mathbf{n}}(z) = \begin{cases} (1 - z)/(1 - 2^{|n_1|}3^{|n_2|}z), & \mathbf{n} \in C_1 \cup C_4, \\ (1 - 2^{|n_1|}z)/(1 - 3^{|n_2|}z), & \mathbf{n} \in C_2 \cup C_5, \\ (1 - 3^{|n_2|}z)/(1 - 2^{|n_1|}z), & \mathbf{n} \in C_3 \cup C_6. \end{cases}$$

One of the key ideas in the theory of expansive subdynamics is that of a phase transition in dynamical behaviour when passing from one expansive component to another. The example above clearly exhibits this phenomenon for the parameterization of zeta functions. However, careful examination of the formulae for the number of periodic points reveals that this need not always be the case. For example, consider a product of two of the systems in Example 4.6, that is

$$M = R_2/\mathfrak{p} \times R_2/\mathfrak{p},$$

where $\mathfrak{p} = (u_1 - 2, u_2 - 3)$. The set of expansive components is identical to that above. However, for $\mathbf{n} = (n_1, n_2)$ lying in either of the adjacent components C_2, C_3, the number of points of period j for $\alpha_M^{\mathbf{n}}$ is

$$3^{2jn_2} - 2.2^{j|n_1|}3^{jn_2} + 2^{2j|n_1|}.$$

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TABLE 1. Essential data for $\zeta_{\mathbf{n}}$ in Example 4.7.

Zeros		Poles	
Θ	Multiplicity	Θ	Multiplicity
$(1-\sqrt{2}, 2+\sqrt{3})$	1	$(-1, 1)$	2
$(1-\sqrt{2}, 2-\sqrt{3})$	1	$(1, 1)$	2
$(-1+\sqrt{2}, 2-\sqrt{3})$	1	$(-1, (2-\sqrt{3})^2)$	1
$(-1+\sqrt{2}, 2+\sqrt{3})$	1	$(-1, (2+\sqrt{3})^2)$	1
$(1+\sqrt{2}, 2-\sqrt{3})$	1	$((1-\sqrt{2})^2, 1)$	1
$(1+\sqrt{2}, 2+\sqrt{3})$	1	$((1+\sqrt{2})^2, 1)$	1
$(-1-\sqrt{2}, 2+\sqrt{3})$	1		
$(-1-\sqrt{2}, 2-\sqrt{3})$	1		

Hence, the parameterization of zeta functions in both C_2 and C_3 is

$$\zeta_{\mathbf{n}}(z) = \frac{(1-2^{|n_1|}3^{n_2}z)^2}{(1-2^{2|n_1|}z)(1-3^{2n_2}z)}.$$

The breakdown of any significant phase transition is complete in the following example.

Example 4.7. (Commuting toral automorphisms) The following system is studied in [**5**, Example 6.6]. Let

$$\mathfrak{p} = (u_1^2 - 2u_1 - 1, u_2^2 - 4u_2 + 1) \subset R_2.$$

The roots of the first polynomial are $1 \pm \sqrt{2}$ and of the second $2 \pm \sqrt{3}$. Hence $M = R_2/\mathfrak{p} \cong \mathbb{Z}[\sqrt{2}, \sqrt{3}]$ and $\mathcal{S}_{\mathfrak{p}}$ consists of four infinite places induced by the elements of $G = \mathrm{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3})|\mathbb{Q})$. Therefore

$$\mathcal{V}(M) = \{(\sigma(1+\sqrt{2}), \sigma(2+\sqrt{3})) : \sigma \in G\}$$

and $\mathcal{W}(M) = \varnothing$. Note that the group $\widehat{M} \cong \mathbb{A}_{R_2/\mathfrak{p}}/\Lambda$, as described in Proposition 2.4, is also isomorphic to $\mathbb{T}^4$ and α_M can be described in terms of toral automorphisms (see [**5**] for the matrices). There are four expansive components for this action, which are the regions of $\mathbb{R}^2$ bounded by the two intersecting lines given by

$$|1+\sqrt{2}|_v^{n_1}|2+\sqrt{3}|_v^{n_2} = 1$$

where $v \in \mathcal{S}_{\mathfrak{p}}$ (note there are some duplications). Following the method of Theorem 4.4, the essential data for the corresponding zeta functions is assembled in Table 1. Vectors are grouped according to multiplicity and whether they generate poles or zeros of $\zeta_{\mathbf{n}}$. The parameterization is the same for all four expansive components, although the various sublists $L \subset \mathcal{V}(M)$ generate the vectors $\theta(L)$ in slightly different ways in each pair of components.

THEOREM 4.8. *Let α_M be a Noetherian algebraic $\mathbb{Z}^d$-action of entropy rank one. If $\alpha_M^{\mathbf{n}}$ is ergodic, then $\zeta_{\mathbf{n}}$ is rational if and only if for each $\mathfrak{p} \in \mathrm{Ass}(M)$ the only absolute values $|\cdot|_v$ unbounded on $R_d/\mathfrak{p}$ with $|\xi_{\mathfrak{p}}^{\mathbf{n}}|_v = 1$ are Archimedean.*

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For the proof, the following technical lemma is useful.

LEMMA 4.9. *Let K_v be a non-Archimedean local field and suppose $x \in K_v$ has $|x|_v = 1$ and infinite multiplicative order. Let $p > 0$ be the characteristic of the residue field F_v and $\gamma \in \mathbb{N}$ the multiplicative order of the image of x in F_v. Then $|x^j - 1|_v = 1$ whenever $(\gamma, j) = 1$ and $\gamma \neq 1$. Furthermore, there are constants $0 < C < 1$ and $r_0 \geq 0$ such that whenever $j = k\gamma p^r$ with $(p, k) = 1$ and $r > r_0$,*

$$|x^j - 1|_v = \begin{cases} C^{p^r} & \text{if } \operatorname{char}(K_v) = p, \\ C|p|_v^r & \text{if } \operatorname{char}(K_v) = 0. \end{cases}$$

Proof. Firstly, if $\gamma \neq 1$, $(\gamma, j) = 1 \Leftrightarrow x^j + \mathfrak{m}_v \neq 1 + \mathfrak{m}_v \Leftrightarrow x^j - 1 \notin \mathfrak{m}_v$. Assume $j = k\gamma p^r$ with $(p, k) = 1$, $r \geq 0$ and note $|k|_v = 1$. Write $x^{\gamma p^r} = 1 + y_r$ where $y_r \in \mathfrak{m}_v$. Then

$$|x^j - 1|_v = |(1 + y_r)^k - 1|_v = |ky_r + O(y_r^2)|_v. \tag{8}$$

If $\operatorname{char}(K_v) = p$ then $|ky_r|_v = |y_r|_v$ and the ultrametric inequality means (8) becomes

$$|y_r|_v = |(x^\gamma)^{p^r} - 1|_v = |(x^\gamma - 1)^{p^r}|_v.$$

Hence the result follows by setting $C = |x^\gamma - 1|_v$ and $r_0 = 0$. Now suppose $\operatorname{char}(K_v) = 0$, $r > 0$ and replace k by kp in (8). Then as soon as r is large enough $|y_r|_v < |p|_v$ and by the ultrametric inequality (8) is $|p|_v|y_r|_v$. Let r_0 be the first such r so that for $j = k\gamma p^{r_0+1}$, $|x^j - 1|_v = |p|_v|y_{r_0}|_v$. If $r_1 = r_0 + 1$ then for $j = k\gamma p^{r_1}$, again applying (8) with kp in place of k gives $|\xi^j - 1|_v = |p|_v|y_{r_1}|_v = |p|_v^2|y_{r_0}|_v$. Now use induction and set $C = |y_{r_0} p^{-r_0}|_v$ to obtain the required result. □

Proof of Theorem 4.8. If the only absolute values $|\cdot|_v$ unbounded on $R_d/\mathfrak{p}$ with $|\xi_{\mathfrak{p}}^{\mathbf{n}}|_v = 1$ are Archimedean, then as previously remarked (Remarks 4.2), Lemma 4.1 can be used to show that $\zeta_{\mathbf{n}}$ is rational.

Hence it remains to show that if there exists a place $v \in \mathcal{W}(M)$ arising from a prime $\mathfrak{p} \in \operatorname{Ass}(M)$ with $|\xi_{\mathfrak{p}}^{\mathbf{n}}|_v = 1$ then $\zeta_{\mathbf{n}}$ is irrational. Let a_j be the product of $|\xi_{\mathfrak{p}}^{j\mathbf{n}} - 1|_v$ over the complete sublist J of places $v \in \mathcal{W}(M)$ with the property that $|\xi_{\mathfrak{p}}^{\mathbf{n}}|_v = 1$, $\mathfrak{p}$ being the appropriate prime from which v has arisen. Let b_j be the right-hand side of (5), so that $|F_j(\alpha_M^{\mathbf{n}})|_v = a_j b_j$ by the periodic point formulae already established.

For a contradiction, assume that $\zeta_{\mathbf{n}}$ is rational. By differentiating, it follows that $\sum_{j=1}^{\infty} a_j b_j z^j$ is also rational. Furthermore, $\sum_{j=1}^{\infty} b_j z^j$ is rational and it follows from the Hadamard quotient theorem [**16**, 6.3.1] that $\sum_{j=1}^{\infty} a_j z^j$ is also rational. Hence, following [**11**, §6], the sequence $\{a_j\}$ is a recurrence sequence and cannot take on infinitely many values infinitely often.

Now choose $p \in \{\operatorname{char}(F_v) : v \in J\}$ and let

$$I = \{v \in J : \operatorname{char}(F_v) = p\}.$$

Suppose $v \in J$ arises from a prime $\mathfrak{p} \in \operatorname{Ass}(M)$ and let γ_v be the multiplicative order of the image of $\xi_{\mathfrak{p}}^{\mathbf{n}}$ in F_v. Set $\gamma = \prod_{v \in I} \gamma_v$ and let q be a rational prime which does not divide any of the γ_v, $v \in J$ nor any of the $\operatorname{char}(F_v)$, $v \in J$. For $r, s \in \mathbb{N}$ define $j(r, s) = \gamma p^r q^s$. If $r \in \mathbb{N}$ is fixed, by applying Lemma 4.9 to each factor $|\xi_{\mathfrak{p}}^{j\mathbf{n}} - 1|_v$ where $v \in J$,

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for all $s, s' \in \mathbb{N}$, it follows that $a_{j(r,s)} = a_{j(r,s')}$ and hence this value occurs infinitely often in $\{a_j\}$. But Lemma 4.9 also shows that infinitely many distinct values of $a_{j(r,s)}$ are obtained from sufficiently large distinct $r \in \mathbb{N}$. However, this cannot happen if $\{a_j\}$ is a recurrence sequence. □

We conclude by pointing out the significance of [**7**] in relation to Theorem 4.8 and Example 4.6. In stark contrast to the zeta functions arising in expansive components, for this example the rational non-expansive directions have irrational zeta functions. Further to this observation, using a similar method to [**7**] it is possible to show that each of these has natural boundary at the circle of convergence. This phenomenon is suggested in greater generality by the behaviour of the sequence of periodic points when there are non-Archimedean contributions from 'isometric' places. However, when there is more than one such place, the interplay between singularities and zeros on the circle of convergence appears to be more involved than in [**7**].

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