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Subsemigroups of Stone–Čech compactifications

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1. *Introduction*

The Stone–Čech compactification $\beta\mathbb{N}$ of the discrete space $\mathbb{N}$ of positive integers is a very large topological space; for example, any countable discrete subspace of the growth $\mathbb{N}^* = \beta\mathbb{N}\backslash\mathbb{N}$ has a closure which is homeomorphic to $\beta\mathbb{N}$ itself ([**23**], §3·5). Now $\mathbb{N}$, while hardly inspiring as a discrete topological space, has a rich algebraic structure. That $\beta\mathbb{N}$ also has a semigroup structure which extends that of $(\mathbb{N}, +)$ and in which multiplication is continuous in one variable has been apparent for about 30 years. (Civin and Yood [**3**] showed that βG was a semigroup for each discrete group G, and any mathematician could then have spotted that $\beta\mathbb{N}$ was a subsemigroup of $\beta\mathbb{Z}$.) The question which now appears natural was explicitly raised by van Douwen [**6**] in 1978 (in spite of the recent publication date of his paper), namely, does $\mathbb{N}^*$ contain subspaces simultaneously algebraically isomorphic and homeomorphic to $\beta\mathbb{N}$? Progress on this question was slight until Strauss [**22**] solved it in a spectacular fashion: the image of any continuous homomorphism from $\beta\mathbb{N}$ into $\mathbb{N}^*$ must be finite, and so the homomorphism cannot be injective. This dramatic advance is not the end of the story. It is still not known whether that image can contain more than one point. Indeed, what appears to be one of the most difficult questions about the algebraic structure of $\beta\mathbb{N}$ is whether it contains any non-trivial finite subgroups.

This paper does not address that problem. Its purpose is to explore the power of the methods introduced by Strauss in [**22**] by extending them to new situations. There seem to be two main ingredients to Strauss's proof. The first looks essential. It is that the semigroup under consideration should be a Stonean space, that is compact (including Hausdorff) and extremally disconnected, or at least should share most of the topological properties of such a space. The particular way in which this property is used is through what is quoted as Lemma 2·1 below, and it is one of Strauss's vital insights to have recognized the key role which this lemma plays in the decipherment of the algebraic structure of $\beta\mathbb{N}$. The lemma applies only to countable sets and the present paper therefore restricts itself to considering Stone–Čech compactifications βS of discrete countable semigroups S.

The second ingredient is that the numbers in $\mathbb{N}$ can be considered as being represented by digits (popularly to the base 10, though of course the base is not relevant here). In extending her work we have been led to inquire which semigroups

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can be 'digitized'. One of our answers we believe to be of interest in itself. We do not expect the reader to be impressed by our digitization of a direct sum of finite groups in Section 3 (which indeed is trivial) or of $\mathbb{Z}(p^\infty)$ in §5 (which is saved from triviality only by the need to circumvent a technical problem). However, our treatment of $\mathbb{Q}$ is worth more attention. We require that the digital expansions of the elements of $\mathbb{Q}$ should be finite and as a result we are forced to look not only at variable bases, but at negative bases. Thus our presentation of the theory involves the possibility of arithmetic to negative bases, a feature which, as far as we are aware, occurs in the literature only as a hint in [**16**]. A consequence of the digitization of $\mathbb{Q}$ is that it has the structure of an oid [**17**] with immediate corollaries about the algebraic structure of $\beta\mathbb{Q}$.

The results we obtain apply to semigroups S which are a little more general than commutative and cancellative, and some of them are new even when $S = \mathbb{N}$. A typical conclusion is this. If T is a second discrete semigroup, $\phi: \beta T \to \beta S$ is a continuous homomorphism, $\phi(T)$ is a countable group and $\phi(T)\backslash S \neq \varnothing$, then $\phi(\beta T) = \phi(T)$ is finite. If $\phi(T)$ is a singly generated semigroup and the other hypotheses hold, then the image $\phi(\beta T)$ is again finite. If $\phi(T)$ is an arbitrary countable commutative semigroup, we can say only that the minimal ideal of $\phi(\beta T)$ is a finite group. It is more natural to express our conclusions in terms of compact subsemigroups of βS which satisfy the topological condition that their topological centres are dense (definition in Section 2 below), a property which is enjoyed by every continuous homomorphic image of the Stone–Čech compactification of a discrete semigroup. In particular, we find that a compact right-zero subsemigroup of $\beta S\backslash S$ must be finite.

We should like to thank the referee for a careful reading of the paper which saved us from some insidious errors.

2. *Preliminaries*

We refer the reader to [**23**] for general information about the Stone–Čech compactification βX of a topological space X. In fact nearly all the spaces X we wish to compactify are discrete and in that case βX is compact (including Hausdorff) and extremally disconnected, or Stonean. That this topological property is a basic factor in describing some of the algebraic structure of Stone–Čech compactifications was recognized by Dona Strauss. The following lemma appears as Lemma 1 in each of her papers [**21**], [**22**], and as Lemma 16·15 in [**5**]; its proof can be found in any of these places.

2·1. LEMMA. *Let Z be a closed subspace of a compact extremally disconnected space (or, more generally, a compact* F*-space) and let $X, Y \subseteq Z$ be countable. If $(\mathrm{cl}_Z X) \cap (\mathrm{cl}_Z Y) \neq \varnothing$ then either $X \cap (\mathrm{cl}_Z Y) \neq \varnothing$ or $(\mathrm{cl}_Z X) \cap Y \neq \varnothing$.* ∎

Our subject is the algebraic structure of βS where S is a discrete semigroup. In most of our examples S will be commutative with an operation naturally denoted by $+$ (the rationals $\mathbb{Q}$ or $\mathbb{Z}(p^\infty)$ and semigroups derived from these). We shall therefore follow the convention of denoting the semigroup operation in βS also by $+$, although this is usually far from commutative [**11**]. The 'sum' $s+t$ of $s, t \in \beta S$ is most conveniently described by the formula

$$s + t = \lim_i \lim_j s_i + t_j,$$

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where (s_i) is any net in S converging to s, and (t_j) in S converges to t (see [**18**], p. 200). Of course it is necessary to prove that this is a good definition, and these proofs usually adopt a different approach altogether ([**3**], theorem 3·4, [**8**], §8, or most simply at the beginning of [**22**]). The order in which the limits are taken is crucial here, and $s+t$ is continuous in s for any fixed t in βS, but is continuous in t only if s is fixed in S (regarded as a subset of βS).

In our notation and terminology about right topological semigroups we shall usually follow the standard text of Berglund, Junghenn and Milnes[**2**]. Thus a compact space T with a semigroup operation $+$ is right topological if $s \mapsto s+t$ is continuous for each $t \in T$. The topological centre of T is

$$\Lambda(T) = \{s \in T : t \mapsto s+t \text{ is continuous}\}.$$

It is easy to prove that if the algebraic centre of a right topological semigroup is dense it is equal to the topological centre. Obviously βS is a compact right topological semigroup with $S \subseteq \Lambda(\beta S)$. Another easy consequence of the limit formula defining the operation $+$ in βS is that if T is compact and right topological and $\phi : S \to \Lambda(T)$ is a homomorphism, then the unique continuous extension $\hat{\phi} : \beta S \to T$ of ϕ is also a semigroup homomorphism.

There is considerable evidence that the proper objects of study in the theory of compact semigroups are those which are right topological and have dense topological centres, in spite of the fact that Ruppert's structure theorem for the minimal ideal ([**19**], [**2**], theorem **1·3·11**) does not require any hypothesis on the topological centre. Signs that this is so could have been detected as early as the fifties, when work on compactifications of transformation groups was beginning to appear (in particular, in the work of Ellis, e.g. [**7**]). In the early seventies, such semigroups played a key role in Namioka's paper [**15**], and Milnes[**14**] showed that the role of the topological centre was crucial for universal mapping theorems. Not surprisingly these semigroups figure prominently in [**2**]. They need a name, and we propose calling them Λ-*semigroups*. A Λ-*group* will be a Λ-semigroup which happens to be a group algebraically.

We shall need a little of the structure theory of compact right topological semigroups. For the details we refer to [**2**], especially Theorem **1·2·16**. Each such semigroup T contains both minimal left ideals L and minimal right ideals R. Naturally, $T+x = L+x = L$ for $x \in L$ and $y+T = y+R = R$ for $y \in R$. The sum $K = L+R$ of any minimal left and any minimal right ideal is the unique minimal two-sided ideal of T, and $K = T+x+T$ for any $x \in K$. On the other hand, $R+L = y+T+x$ for any $y \in R$ and $x \in L$ turns out to be a group; it is a maximal subgroup of K, and though in general there are many such subgroups they are all (algebraically) isomorphic. Also, $R+L = R \cap L$. Since groups have identities, minimal left and minimal right ideals contain idempotents. Since two distinct minimal left ideals are disjoint and each idempotent of R is the identity of some maximal group, R contains as many idempotents as there are minimal left ideals in T. Minimal left ideals are necessarily compact, but this is not true of all the other substructures of T mentioned.

In [**22**], Strauss uses the ordinary expression for a positive integer to the base k in what seems an essential way (the expression in terms of a base, that is, not the base used). The principal tools employed are functions of the integer n defined in terms of the digits of the 'decimal' expression for n. As our intention is to modify her method

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for use in other situations, we shall need to find analogues for algebraic structures other than $\mathbb{N}$. To avoid having to repeat ourselves for each new structure, we give a general description of digitization here.

Let us begin with a semigroup $(S, +)$. For k in an index set, which may be $\{0, 1, 2, \ldots\}$, or $\{\ldots, -1, 0, 1, 2, \ldots\}$, or $\{\ldots, -3, -2, -1\}$, let D_k be a finite set which contains a distinguished element 0. (Typically D_k will be a set of consecutive integers $\{0, 1, \ldots, r_k\}$, but we shall need other possibilities.) We write $\oplus_k D_k$ for the set of all elements either of the form $(d_0, d_1, d_2, \ldots)$ or of the form $(\ldots, d_{-1}, d_0, d_1, d_2, \ldots)$, or of the form $(\ldots, d_{-2}, d_{-1})$, as appropriate, with $d_k \in D_k$ for each k and $d_k = 0$ for all but finitely many k. The support of $(d_k) \in \oplus_k D_k$ is defined to be $\{k : d_k \neq 0\}$, and is denoted by $\operatorname{supp}(d_k)$. We allow ourselves to add elements with disjoint supports in the obvious way: if $\operatorname{supp}(d_k) \cap \operatorname{supp}(d'_k) = \varnothing$, we define $(d_k) + (d'_k)$ to be (d''_k) where $d''_k = d_k$ if $d'_k = 0$ and $d''_k = d'_k$ if $d_k = 0$. A digitization of S is a bijection $\delta : S \to \oplus_k D_k$ with the property that if $s, t \in S$ and $\operatorname{supp}\delta(s) \cap \operatorname{supp}\delta(t) = \varnothing$ then $\delta(s+t) = \delta(s) + \delta(t)$. (In fact, the structure which $\oplus_k D_k$ has makes it an oid in the terminology of [**17**], and it would be natural to refer to δ as an 'oid isomorphism'. However, we do not wish to burden the reader with this nomenclature.) This property of digitizations will be trivial to see in the ones we shall actually construct, and we shall not even mention it explicitly in our proofs.

We can turn δ into a proper algebraic isomorphism by using it to define an addition in the whole of $\oplus_k D_k$ by transferring the operation from S. The resulting operation on $\oplus_k D_k$ may not appear very natural when expressed in terms of the sequences of digits (d_k); indeed, even if we are speaking of the semigroup $\mathbb{Z}^+$ with the usual addition, the interpretation of this addition in the usual decimal notation is complicated by the necessity to 'carry' from one place to the next on occasions. We shall face such problems ourselves.

It is important for us to allow the possibility that S is a group, though the existence of an identity in S causes some complications. In keeping with our additive notation, we denote the identity of S, if it has one, by 0. In such cases we require of our digitizations that $\delta(0) = 0$ for all k, and consequently that $\operatorname{supp} 0 = \varnothing$.

For $s \in S$, the kth digit of s will be $d_k(s)$, so that $\delta(s) = (d_k(s))_{k=0}^{\infty}$ (or $(d_k(s))_{k=-\infty}^{\infty}$, or $(d_k(s))_{k=-\infty}^{-1}$). We also define

$$\delta_k : S \to \oplus_{h=0}^{k} D_h, \quad \delta_k(s) = (d_h(s))_{h=0}^{k},$$

or

$$\delta_k : S \to \oplus_{h=-k}^{k} D_h, \quad \delta_k(s) = (d_h(s))_{h=-k}^{k},$$

or

$$\delta_k : S \to \oplus_{h=-k}^{-1} D_h, \quad \delta_k(s) = (d_h(s))_{h=-k}^{-1},$$

and we write $S_k = \delta_k^{-1}(0) \backslash \{0\}$ for each k. Thus each S_k does not contain the identity 0 of S (if it has one).

We give the (finite) sets D_k of digits their discrete topologies so that they become compact. Each function d_k then extends to a unique continuous function $\hat{d}_k : \beta S \to D_k$. Similarly, the map δ extends to a continuous map $\hat{\delta} : \beta S \to \Pi_k D_k$. It is easy to see that for any $s \in \beta S$ the kth digit of $\hat{\delta}(s)$ is $\hat{d}_k(s)$. Of course δ_k extends too, and using the fact that the image of δ_k is finite it is not hard to show that $\hat{\delta}_k^{-1}(0) \backslash \{0\} = \operatorname{cl}_{\beta S} S_k$.

We now state two conjectures. In the generality which we presently have, they cannot be true. The point of the next three sections is to show that, for particular semigroups, they do hold. We state them here to set the pattern for our digitization process.

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2·2. CONJECTURE. *Let e be an idempotent in βS. Then we can digitize S in such a way that $\hat{\delta}_k(e) = 0$ for every k, so that either $e \in \mathrm{cl}_{\beta S} S_k$ for each k or $e = 0$.*

2·3. CONJECTURE. *With a digitization which satisfies Conjecture 2·2 there is a map $\nu: S \to \mathbb{N}$ such that, for each $s \in S$ with $s \neq 0$,*

$$\min\{k: \delta_k(u+s) \neq \delta_k(u)\} \leqslant \nu(s)$$

for every $u \in S$.

3. *Countable direct sums of finite groups*

This brief section deals with a case which is trivial but sets the scene. For each $k \in \mathbb{Z}^+$, let G_k be a finite group (and we do not ask that G_k should be commutative, though we shall denote the group operation by $+$ and the identity by 0). We take

$$S = \bigoplus_{k=0}^{\infty} G_k.$$

Then S is, in effect, already digitized; we simply take $D_k = G_k$ for each k.

A key feature of the present situation is that the maps $\delta_k: S \to \bigoplus_{h=0}^{k} G_h$ are homomorphisms, and therefore the extensions $\hat{\delta}_k$ are homomorphisms from βS to $\bigoplus_{h=0}^{k} G_h$. In particular, therefore, if e is any idempotent in βS, $\hat{\delta}_k(e)$ is idempotent in $\bigoplus_{h=0}^{k} G_h$, and so is the identity 0.

3·1. LEMMA. *Conjecture 2·2 holds for S.* ∎

Conjecture 2·3 is no harder to prove. If $s \in S$ and $s \neq 0$, we write $\nu(s) = \min\{k: d_k(s) \neq 0\}$. Then, as d_k (which is simply the projection onto G_k) is a homomorphism, when $k = \nu(s)$ we have for each $u \in S$

$$d_k(u+s) = d_k(u) + d_k(s) \neq d_k(u),$$

so that also $\delta_k(u+s) \neq \delta_k(u)$. We have shown

3·2. LEMMA. *Conjecture 2·3 holds for S.* ∎

4. *Arithmetic to negative bases and $\mathbb{Q}$*

In this section we shall establish Conjectures 2·2 and 2·3 for $(\mathbb{Q}, +)$. This part of our work is, in the main, elementary and is probably of more general interest. The problem is to digitize $\mathbb{Q}$ in a reasonable way. Now it is common enough to express positive integers to bases other than 10, and variable bases are a well-documented possibility (Hewitt and Ross [**10**] discuss $\mathfrak{a}$-adic expansions in their §10). However, all the bases we have found in the literature use positive numbers – except one. In ([**16**], proposition 3·13), what is essentially an expansion to the base (-2) is used; the numbers which can be represented in the form

$$\sum_{n=0}^{\infty} d_n(-2)^n \quad (0 \leqslant d_n < 2)$$

are all the elements of $\mathbb{Z}$. We shall obtain a more general form of this result (using variable negative bases to digitize $\mathbb{Z}$) and then go on to obtain a digitization for $\mathbb{Q}$. In order to prove that our conjectures hold for this digitization, we shall need some elementary facts about how arithmetic in $\mathbb{Q}$ is interpreted in these expansions. We wish to thank Dr A. J. Douglas and Dr G. T. Vickers for their help with this material.

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We begin with any sequence

$$b_0 = 1, b_1, \dots$$

of positive integers with $b_r \geqslant 2$ for $r \geqslant 1$. We write, for each r,

$$f_r = b_0 b_1 \dots b_r,$$

the product of the first $r+1$ terms of the sequence.

4·1. THEOREM. *Each $n \in \mathbb{Z}$ has a unique expression in the form*

$$n = \sum_{r=0}^{\infty} d_r(-1)^r f_r, \quad 0 \leqslant d_r < b_{r+1} = f_{r+1}/f_r,$$

with $d_r = 0$ for all except finitely many r.

Proof. For each $k \geqslant 0$ write

$$F_k = \left\{ \sum_{r=0}^{k} d_r(-1)^r f_r : 0 \leqslant d_r < b_{r+1} \right\}.$$

We shall show that F_k is a block of consecutive integers in $\mathbb{Z}$.

First, the largest number in F_{2k} is

$$M_{2k} = (b_1 - 1)f_0 + (b_3 - 1)f_2 + \dots + (b_{2k+1} - 1)f_{2k},$$

while the smallest is

$$m_{2k} = -(b_2 - 1)f_1 - (b_4 - 1)f_3 - \dots - (b_{2k} - 1)f_{2k-1}.$$

Their difference is

$$\begin{aligned} M_{2k} - m_{2k} &= (b_1 f_0 - f_0) + (b_2 f_1 - f_1) + (b_3 f_2 - f_2) + \dots + (b_{2k+1} f_{2k} - f_{2k}) \\ &= f_{2k+1} - f_0 = f_{2k+1} - 1, \end{aligned}$$

since $b_{r+1} f_r = f_{r+1}$ for all r. In the same way we see that the difference between the largest number M_{2k+1} in F_{2k+1} and the smallest, m_{2k+1}, is

$$M_{2k+1} - m_{2k+1} = f_{2k+2} - 1.$$

We now prove the uniqueness of our expressions. Suppose

$$n = \sum_{r=0}^{2k} d_r(-1)^r f_r = \sum_{r=0}^{2k} d'_r(-1)^r f_r;$$

we may assume there are the same number of terms in each sum by inserting zero coefficients. Then

$$\begin{aligned} m_{2k-1} + d_{2k}(-1)^{2k} f_{2k} &\leqslant \sum_{r=0}^{2k} d_r(-1)^r f_r = \sum_{r=0}^{2k} d'_r(-1)^r f_r \\ &\leqslant M_{2k-1} + d'_{2k}(-1)^{2k} f_{2k}, \end{aligned}$$

so that

$$d_{2k} f_{2k} \leqslant M_{2k-1} - m_{2k-1} + d'_{2k} f_{2k} = f_{2k} - 1 + d'_{2k} f_{2k}.$$

This is possible only if $d_{2k} \leqslant d'_{2k}$. By symmetry, $d_{2k} = d'_{2k}$. We cancel the terms with suffix $2k$, and continue in the same way to find $d_r = d'_r$ for all r.

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The total number of integers in F_k is now seen to be (the number of choices for d_0) × (the number of choices for d_1) × ... × (the number of choices for d_k) = $b_1 b_2 \dots b_{k+1} = f_{k+1}$. The number of integers between the smallest in F_k, m_k, and the largest, M_k, is $M_k - m_k + 1 = f_{k+1}$. Thus F_k consists of all these consecutive integers.

The set $\bigcup_{k=0}^{\infty} F_k$ consists of all the expressions we are considering and consists of consecutive integers. Since, for any n, it contains f_{2n}, and $f_{2n} \to \infty$, and it also contains $-f_{2n+1}$, and $-f_{2n+1} \to -\infty$, $\bigcup_{k=0}^{\infty} F_k$ is the whole of $\mathbb{Z}$. ▌

We needed Theorem 4·1 in the general form given in order to obtain easily our digitization of $\mathbb{Q}$. We could have produced an equally general result for $\mathbb{Q}$, but that we do not require.

4·2. THEOREM. *Each $q \in \mathbb{Q}$ has a unique expression in the form*

$$\dots + d_{-s} \cdot \frac{(-1)^s}{(s+1)!} + \dots + d_{-2} \cdot \frac{1}{3!} + d_{-1} \cdot \frac{(-1)}{2!} + d_0 + d_1 \cdot (-1) \cdot 2! + d_2 \cdot 3! + \dots + d_r \cdot (-1)^r \cdot (r+1)! + \dots,$$

where $0 \leqslant d_{-s} \leqslant s$ for each $s > 0$ and $0 \leqslant d_r \leqslant r+1$ for each $r \geqslant 0$, and where $d_r = d_{-s} = 0$ for all but finitely many r, s. Conversely, each such expression represents a rational number.

Proof. The converse is trivial.

To find the expression for a given $q \in \mathbb{Q}$, take any odd integer $p \in \mathbb{N}$ which is so large that $p!q$ is an integer. Apply Theorem 4·1 with the sequence $(b_n)_{n=0}^{\infty}$ taken to be

$$1, p, (p-1), (p-2), \dots, 4, 3, 2, 2, 3, 4, \dots, r, (r+1), \dots.$$

Then (f_n) is the sequence

$$1, p, p(p-1), \dots, p(p-1)\dots 3, p!, p!2, \dots, p!r!, p!(r+1)!, \dots.$$

Theorem 4·1 produces integer coefficients (c_n) such that

$$p!q = c_0 - c_1 \cdot p + c_2 \cdot p(p-1) - \dots - c_{p-2} \cdot p(p-1)\dots 3 + c_{p-1} \cdot p! - c_p \cdot p! \cdot 2 + c_{p+1} \cdot p! \cdot 3! - \dots$$

(where the fact that p is odd was used in determining the signs). Dividing by $p!$ and changing the notation for the coefficients gives

$$q = d_{-(p-1)} \cdot \frac{1}{p!} + d_{-(p-2)} \cdot \frac{(-1)}{(p-1)!} + \dots + d_{-1} \cdot \frac{(-1)}{2!} + d_0 + d_1 \cdot (-1) 2! + d_2 \cdot 3! + \dots.$$

The fact that the coefficients d_r lie in the prescribed ranges is easily checked from Theorem 4·1 and the expansion for $p!q$.

To show uniqueness, suppose that q is given by each of two expressions of the form stipulated in the theorem. As the coefficients d_{-s} are zero when s is large enough, we may assume that both expressions are of the form given immediately above, that is, both begin with the same term in $1/p!$ with p odd (notice that we are not assuming $d_{-(p-1)} \neq 0$). Multiplying the two expressions by $p!$ now gives two expansions for the integer $p!q$ which must have identical coefficients by the uniqueness part of Theorem 4·1. ▌

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For the rest of this section we shall find it convenient to write the expansions of Theorem 4·2 in the form

$$q = \sum_{r=-\infty}^{\infty} d_r(-1)^r f_r$$

so that $f_r = 1/(-r+1)!$ if $r < 0$ and $f_r = (r+1)!$ if $r \geqslant 0$. Then also $0 \leqslant dr < f_{r+1}/f_r$ for each r. Also, in conformity with the notation used in Theorem 4·1 (but not in the proof of Theorem 4·2), we shall write $b_{r+1} = f_{r+1}/f_r$ for every r.

Our digitization of $\mathbb{Q}$ therefore assigns to each rational q a doubly-infinite sequence

$$(\ldots, d_{-2}, d_{-1}, d_0, d_1, d_2, \ldots),$$

and we shall next make some elementary remarks about how addition in $\mathbb{Q}$ is reflected in the behaviour of these sequences. To be more precise, let q have the sequence (d_r) of digits, while q' has (d'_r). We would expect the digits (d''_r) of $q+q'$ to come essentially from adding the digits of q and q' but, mindful of what happens with ordinary arithmetic, we might expect there to be 'carrying' from some places to others. With the order in which we have written our digit sequence above, carrying is always from left to right. We shall show that what can be carried into any place from earlier places is either 0, 1 or -1, and that the circumstances in which $+1$ is carried are relatively rare. We see what happens in the rth place under each of these three possibilities.

Case (*i*). When 0 is carried from previous places to the rth, the contribution to the sum $q+q'$ from the rth digits is $d_r(-1)^r f_r + d'_r(-1)^r f_r = (d_r + d'_r)(-1)^r f_r$, and we need to express this in terms of digits. If $0 \leqslant d_r + d'_r < b_{r+1}$, that is already done (that is, $d''_r = d_r + d'_r$). Now in any case, $0 \leqslant d_r, d'_r < b_{r+1}$, so that $0 \leqslant d_r + d'_r \leqslant 2b_{r+1} - 2$. If $b_{r+1} \leqslant d_r + d'_r$, then we can write

$$\begin{aligned} d_r(-1)^r f_r + d'_r(-1)^r f_r &= (d_r + d'_r - b_{r+1})(-1)^r f_r + b_{r+1}(-1)^r f_r \\ &= d''_r(-1)^r f_r + (-1)(-1)^{r+1} f_{r+1}, \end{aligned}$$

where $d''_r = d_r + d'_r - b_{r+1} < b_{r+1}$. This can be interpreted as saying that (-1) is carried from the rth place to the $(r+1)$th. (Of course, an expansion with a (-1) in the $(r+1)$th place is not allowed; what we are saying is that speaking of carrying (-1) is a convenient way of understanding the formula just written down, and indeed a convenient way of actually doing the arithmetic.) Notice that in both situations $d''_r = (d_r + d'_r) \bmod b_{r+1}$.

Case (*ii*). When 1 is carried to the rth place we need to consider

$$d_r(-1)^r f_r + d'_r(-1)^r f_r + 1 \,.\, (-1)^r f_r.$$

Since $0 \leqslant d_r + d'_r + 1 \leqslant 2b_{r+1} - 1$, the conclusions are similar to those of Case (i): $d''_r = (d_r + d'_r + 1) \bmod b_{r+1}$ and either 0 or (-1) is carried to the $(r+1)$th place.

Case (*iii*). If (-1) is carried to the rth place, we must look at

$$d_r(-1)^r f_r + d'_r(-1)^r f_r + (-1)(-1)^r f_r.$$

Now usually $0 \leqslant d_r + d'_r - 1 \leqslant 2b_{r+1} - 3$, and our conclusions are as before. But it is possible that $d_r = d'_r = 0$, and then $d_r + d'_r - 1 = -1$. We must use instead

$$\begin{aligned} (-1)(-1)^r f_r &= (b_{r+1} - 1)(-1)^r f_r - b_{r+1}(-1)^r f_r \\ &= (b_{r+1} - 1)(-1)^r f_r + (-1)^{r+1} f_{r+1}. \end{aligned}$$

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Thus $d''_r = (d_r + d'_r - 1) \bmod b_{r+1} = (-1) \bmod b_{r+1}$, and $+1$ is carried to the $(r+1)$th place.

We should like to emphasize the conditions under which $+1$ can be carried to the $(r+1)$th place. Both digits d_r and d'_r in the rth place must be zero, and (-1) must have been carried into the rth place. (Indeed, it is the fact that (-1) is not a permitted digit which ensures that there is further carrying to the $(r+1)$th place.) Moreover, when this happens, the digit d''_r in the rth place of the sum is $b_{r+1}-1$.

We can now see by induction that our three cases cover all possibilities. Indeed, the cases describe the inductive steps, and we have only to find a point for the induction to begin. But we can find an $r \in \mathbb{Z}$ such that $d_s = d'_s = 0$ for all $s < r$, and in these circumstances 0 is carried into the rth place.

Our next lemma is a technical one which will allow us to obtain information about the idempotents in $\beta\mathbb{Q}$. Before we prove it, we see what some notation in §2 means in our present context. First, for $k \in \mathbb{Z}$, $D_k = \{0, 1, \ldots, b_{k+1}-1\}$. For $q \in \mathbb{Q}$, the kth digit of q is $d_k(q) \in D_k$, so that $q = \sum_{k=-\infty}^{\infty} d_k(q)(-1)^k f_k$. The map $\delta_k : \mathbb{Q} \to \oplus_{h=-k}^{k} D_k$ is given for $k > 0$ by

$$\delta_k(q) = (d_{-k}(q), d_{-k+1}(q), \ldots, d_k(q)),$$

and $\mathbb{Q}_k = \delta_k^{-1}(0)$.

4·3. Lemma. *Let $q, q' \in \mathbb{Q}$ and $k \in \mathbb{N}$ be such that $\delta_k(q) = \delta_k(q') = \delta_k(q+q')$. Then $\delta_{k-2}(q) = 0$.*

Proof. Write $d_r = d_r(q) = d_r(q') = d_r(q+q')$ for $-k \leqslant r \leqslant k$. We know that in calculating the sum $q+q'$ using digits, either 0, 1, or -1 is carried from the previous places into the rth. Suppose $r \geqslant -k+1$. Then if 0 is carried we need $d_r + d_r \equiv d_r \pmod{b_{r+1}}$ and since $0 \leqslant d_r < b_{r+1}$ this requires $d_r = 0$. Then also 0 is carried to the $(r+1)$th place, so $0 = d_r = d_{r+1} = \ldots = d_k$.

If 1 is carried to the rth place then (as described just before this lemma) $d_r = d_r(q) = d_r(q') = 0$, and also $d_r = d_r(q+q') = b_{r+1}-1$. Thus, this case cannot occur.

If (-1) is carried to the rth place then we need $d_r + d_r - 1 \equiv d_r \pmod{b_{r+1}}$, which leads to $d_r = 1$. Then 0 is carried to the $(r+1)$th place, and so by the first paragraph of this proof $d_{r+1} = d_{r+2} = \ldots = d_k = 0$. The smallest value of r we can take here is $r = -k+1$. Thus in all our cases we have found $d_{-k+2} = \ldots = d_{k-2} = 0$, that is $\delta_{k-2}(q) = 0$. ∎

Now at last we can establish our conjectures for $\mathbb{Q}$. We remind the reader that our $\beta\mathbb{Q}$ is produced from the *discrete* group $\mathbb{Q}$ (and not using its usual topology).

4·4. Lemma. *Conjecture 2·2 holds for $\mathbb{Q}$.*

Proof. Of course, we use the digitization proved above. We shall prove that for any idempotent e in $\beta\mathbb{Q}$, $\hat{\delta}_k(e) = 0$ for all $k \in \mathbb{N}$. Fix e. The digits of e are $\hat{d}_k(e)$, $-\infty < k < \infty$. Recall that $\hat{\delta} : \beta\mathbb{Q} \to \Pi_k D_k = D$ say. The subset V of D determined by

$$V = \{(d_r) \in D : d_r = \hat{d}_r(e), -k-2 \leqslant r \leqslant k+2\}$$

is an open neighbourhood of $\hat{\delta}(e)$ in D, so that $\hat{\delta}^{-1}(V)$ is an open neighbourhood of e in $\beta\mathbb{Q}$.

Take a net (e_i) in $\mathbb{Q}$ such that $e_i \to e$ in $\beta\mathbb{Q}$. Since $e+e = e \in \hat{\delta}^{-1}(V)$, we can choose i_1 so large that both $e_{i_1} + e \in \hat{\delta}^{-1}(V)$ (using right continuity) and $e_{i_1} \in \hat{\delta}^{-1}(V)$. Then, using

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the fact that $e_{i_1} \in \Lambda(\beta\mathbb{Q})$, we can find i_2 so large that both $e_{i_1}+e_{i_2} \in \hat{\delta}^{-1}(V)$ and $e_{i_2} \in \hat{\delta}^{-1}(V)$. These inclusions say

$$\delta_{k+2}(e_{i_1}) = \delta_{k+2}(e_{i_2}) = \delta_{k+2}(e_{i_1}+e_{i_2}).$$

Lemma 4·3 tells us that $\delta_k(e_{i_1}) = 0$, and the definition of V yields $\hat{d}_r(e) = 0$ for $-k \leqslant r \leqslant k$, i.e. $\hat{\delta}_k(e) = 0$. ∎

4·5. LEMMA. *Conjecture* 2·3 *holds for* $\mathbb{Q}$.

Proof. We use the notation of the conjecture. Take $s \in \mathbb{Q}$. Let $r_1 < r_2 < \ldots < r_m$ be the suffixes r_i for which $d_{r_i}(s) \neq 0$. Then for all $r < r_1$ we have $d_r(s) = 0$, so that for every $u \in \mathbb{Q}$ we have $d_r(u) = d_r(u)+d_r(s) = d_r(u+s)$. As there is no carrying into the r_1th place from earlier places we also have

$$d_{r_1}(u+s) = (d_{r_1}(u)+d_{r_1}(s)) \bmod b_{r_1+1} \neq d_{r_1}(u)$$

because $0 < d_{r_1}(s) < b_{r_1+1}$.

We define $\nu(s) = |r_1|$. Since $d_{r_1}(u+s) \neq d_{r_1}(u)$, we also have $\delta_{|r_1|}(u+s) \neq \delta_{|r_1|}(u)$, and so Conjecture 2·3 holds. ∎

At this point we shall make a slight detour (until the end of this section) in order to obtain results about the minimal ideal of $\beta\mathbb{Q}$. For this brief passage, we shall assume the reader has some acquaintance with the ideas in [**17**]. We remarked in Section 2 that a digitization map is an oid isomorphism, so that we have shown that $\mathbb{Q}$ is an oid. It does appear unlikely to us that $\mathbb{Q}$ is a standard oid in the sense of [**16**], meaning not only that $\mathbb{Q}$ can be described using a countable family of digit sets D_k (which is a requirement of our present theory in any case) but that digitization can be carried out in such a way that card $D_k = 2$ for all k. On the other hand, $\mathbb{Q}$ is not the most general type of oid discussed in [**17**]. The basic idea in oid theory is to consider the set $H = \bigcap_{k=-\infty}^{\infty} \mathrm{cl}_{\beta\mathbb{Q}} \mathbb{Q}_k = \bigcap_k \hat{\delta}_k^{-1}(0)$. This is a semigroup (which is easy to prove) and, according to [**17**], has the properties claimed for $\beta\mathbb{Q}$ in the next theorem.

4·6. THEOREM. *$\beta\mathbb{Q}$ has 2^c minimal left ideals, 2^c minimal right ideals, and every maximal group in its minimal ideal contains a free group on 2^c generators.*

Proof. We adopt an approach similar to that used to obtain these conclusions for $\beta\mathbb{N}$ in [**11**]. All we need to do is show that if L is a minimal left ideal of $\beta\mathbb{Q}$, then $L \cap H$ is a minimal left ideal of H. The corresponding result for right ideals follows by symmetry (for no continuity is involved) and then (for example) the maximal subgroup $L \cap R$ of the minimal ideal of $\beta\mathbb{Q}$ contains $(L \cap H) \cap (R \cap H)$, a maximal subgroup of the minimal ideal of H, which we know contains a free group of the required size.

Now $L \cap H$ is a left ideal in H and therefore contains a minimal left ideal L_H. Let e be any idempotent in L_H. Then e is in L, so is a right identity for L. Thus $L \cap H = (L \cap H)+e \subseteq H+e \subseteq L_H$. Thus $L \cap H = L_H$. ∎

Perhaps we should comment on what has been achieved here. The point is not simply that $\beta\mathbb{Q}$ contains copies of the free group on 2^c generators, for that is immediate from the fact that $\mathbb{Q}$ contains copies of $\mathbb{N}$ and the results of [**11**]. That this free group can be found in the minimal ideal of $\mathbb{Q}$ is indeed no great surprise, but is new as far as we know.

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5. $\mathbb{Z}(p^\infty)$

The reader who is paying close attention will have noticed that, while the form of Conjecture 2·2 makes it appear that the digitization of S could depend on the idempotent e in βS, the actual digitizations produced in Sections 3 and 4 were independent of such influences. Now at last we encounter a case in which we shall require this flexibility. Indeed, the group $\mathbb{Z}(p^\infty)$, where p is any prime number, appears to come already digitized; it can be described as the set of all real numbers which can be written in the form

$$\ldots + d_{-k}\cdot\frac{1}{p^k} + \ldots + d_{-2}\cdot\frac{1}{p^2} + d_{-1}\cdot\frac{1}{p} = \sum_{k=-\infty}^{-1} d_k\, p^k,$$

where $0 \leqslant d_k \leqslant p-1$ for each k and all but finitely many d_k are zero, with the group operation $+$ taken to be addition modulo 1. We naturally take $D_k = \{0, 1, \ldots, p-1\}$ for each k, and we do indeed have a natural digitization.

We shall first see that Conjecture 2·2 is not always satisfied with this digitization. To do this we identify the circle group $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$ and let $\psi\colon \mathbb{Z}(p^\infty) \to \mathbb{T}$ be the natural embedding of $\mathbb{Z}(p^\infty)$ into $\mathbb{T}$. The homomorphism ψ maps to the compact group $\mathbb{T}$, so it has a unique extension to a homomorphism $\hat{\psi}\colon \beta\mathbb{Z}(p^\infty) \to \mathbb{T}$. We can give an alternative description of $\hat{\psi}$; the digit maps d_k extend to continuous maps $\hat{d}_k\colon \beta\mathbb{Z}(p^\infty) \to \{0, 1, \ldots, p-1\}$, and then, for $z \in \beta\mathbb{Z}(p^\infty)$,

$$\hat{\psi}(z) = \sum_{k=-\infty}^{-1} \hat{d}_k(z)\, p^k,$$

the infinite series converging in $\mathbb{T}$ in the usual sense.

Fix an idempotent e in $\beta\mathbb{Z}(p^\infty)$. Then $\hat{\psi}(e)$ is an idempotent in $\mathbb{T}$, so that $\hat{\psi}(e) = 0$. However, this does not imply that $\hat{d}_k(e) = 0$ for all k as we should wish. In fact there are two alternatives: either (a) $\hat{d}_k(e) = 0$ for all k, or else (b) $\hat{d}_k(e) = p-1$ for all k. What is worse, alternative (b) can occur. It is not hard to see that

$$\{z \in \beta\mathbb{Z}(p^\infty)\colon \hat{d}_k(z) = p-1 \text{ for all } k\}$$

is a compact subsemigroup of $\beta\mathbb{Z}(p^\infty)$ and must therefore contain an idempotent. We have no option but to re-digitize.

5·1. Lemma. *Conjecture* 2·2 *holds for* $\mathbb{Z}(p^\infty)$.

Proof. Fix an idempotent e. If alternative (a) above holds then the conjecture is satisfied, so we need only deal with alternative (b). We thus suppose that $\hat{d}_k(e) = p-1$ for all k. The proof is in fact simple in conception. The map $z \mapsto -z$ in $\mathbb{Z}(p^\infty)$ is an isomorphism for which the image of a block of consecutive $(p-1)$s contains a block of consecutive 0s of almost the same size. We shall simply assign to z the digits which were initially given to $-z$. We sketch a few of the details.

Because $z \mapsto -z$ is an isomorphism, we can obviously obtain a digitization of $\mathbb{Z}(p^\infty)$ from the map

$$z \mapsto \sum_{k=-\infty}^{-1} d_k(-z)\, p^k,$$

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and this has digit maps d'_k defined by $d'_k(z) = d_k(-z)$. Also, we know that, for $z \in \mathbb{Z}(p^\infty)$,

$$\sum_{k=-\infty}^{-1} d_k(-z)\, p^k + \sum_{k=-\infty}^{-1} d_k(z)\, p^k = 0.$$

It is a matter of elementary arithmetic to see that if $d_k(z) = p-1$ for $-K \leqslant k \leqslant -1$, then $d_k(-z) = 0$ at least for $-K+1 \leqslant k \leqslant -1$. Take a net (e_α) in $\mathbb{Z}(p^\infty)$ with $e_\alpha \to e$. Then for each k, $d_k(e_\alpha) \to \hat{d}_k(e)$, so that eventually $d_k(e_\alpha) = p-1$. Given K, we can choose α so large that $d_k(e_\alpha) = p-1$ for all k with $-K \leqslant k \leqslant -1$, so that $d'_k(e_\alpha) = d_k(-e_\alpha) = 0$ for $-K+1 \leqslant k \leqslant -1$. Thus $\hat{d}'_k(e) = 0$ for all k, as required. ∎

5·2. LEMMA. *Conjecture* 2·3 *holds for* $\mathbb{Z}(p^\infty)$.

Proof. We take the digitization for which 5·1 is satisfied. Given $z \in \mathbb{Z}(p^\infty)$ with $z \neq 0$, let $r_1 < r_2 < \ldots < r_m$ be the suffixes for which $d_{r_i}(z) \neq 0$. Write $\nu(z) = |r_1|$. Since $d_r(z) = 0$ for $r < r_1$ and $d_{r_1}(z) \neq 0$, it is easy to see that, in calculating $d_{r_1}(u+z)$, there is no carrying from previous places into the r_1th place, so that

$$d_{r_1}(u+z) = d_{r_1}(u) + d_{r_1}(z) \pmod p \neq d_{r_1}(u)$$

for any $u \in \mathbb{Z}(p^\infty)$. Therefore $\delta_{|r_1|}(u+z) \neq \delta_{|r_1|}(u)$, which proves the lemma. ∎

The reader might wonder whether the conclusions of Theorem 4·6 hold for $\beta(\mathbb{Z}(p^\infty))$, seeing that $\mathbb{Z}(p^\infty)$ is an oid. The answer is that they do. However, the proof presents a slight difficulty; it is not enough to consider the subsemigroup $\bigcap_k \mathrm{cl}_{\beta\mathbb{Z}(p^\infty)} \mathbb{Z}(p^\infty)_k$, for this does not contain all the idempotents of $\beta\mathbb{Z}(p^\infty)$. Instead, we must consider $\hat{\psi}^{-1}(0)$, where $\psi : \mathbb{Z}(p^\infty) \to \mathbb{T}$ is the homomorphism considered earlier in this section. The theory of [17] does not apply directly to $\hat{\psi}^{-1}(0)$, and we shall not give the details of the proof here (which presents no particular difficulty but would take some space).

6. *Countable direct sums*

The purpose of the present section is to discuss Conjectures 2·2 or 2·3 in the context of countable direct sums of semigroups for which they hold. In fact we do not need the full force of the conjectures in this case, and digitizing a countable direct sum, while appearing conceptually quite easy through some kind of diagonalization, is likely to be messy in practice, so we shall establish here variants of the conjectures which will prove enough for us. In this section (and afterwards) we may regard a finite sequence of digits not simply as corresponding to an element of a digitized semigroup, but as being identical to it – indeed, the digits of an element do determine that element uniquely.

Let $S^{(i)} (i = 1, 2, \ldots)$ be semigroups for which Conjectures 2·2 and 2·3 hold. We shall use a superscript$^{(i)}$ to refer to an entity associated with $S^{(i)}$, such as $\delta^{(i)}$ or $\nu^{(i)}$. The projection from $S = \bigoplus_{i=1}^{\infty} S^{(i)}$ to $S^{(i)}$ will be denoted by $\mathrm{pr}^{(i)}$; it has a continuous extension $\widehat{\mathrm{pr}}^{(i)} : \beta S \to \beta S^{(i)}$, and this is a homomorphism.

We now fix an idempotent e in βS. For each i, $\widehat{\mathrm{pr}}^{(i)} e$ is an idempotent in $\beta S^{(i)}$. We use this to fix a digitization of $S^{(i)}$ for which Conjectures 2·2 and 2·3 hold. We now define maps $\delta_k : S \to S$ for $k = 1, 2, 3, \ldots$ by writing for $s = (s^{(1)}, s^{(2)}, \ldots) \in S$,

$$\delta_k(s) = (\delta_k^{(1)}(s^{(1)}), \delta_k^{(2)}(s^{(2)}), \ldots, \delta_k^{(k)}(s^{(k)}), 0, 0, \ldots).$$

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These maps δ_k will play the same role as the corresponding maps did in earlier sections; the point of making them mappings into S itself is that we can now compose them: $\delta_h \circ \delta_k = \delta_k \circ \delta_h = \delta_h$ if $h \leqslant k$. In fact the image $\delta_k(S)$ is finite, so that there is a unique continuous extension $\hat{\delta}_k : \beta S \to S$ (with $\hat{\delta}_k(\beta S) \subseteq \delta_k(S)$). As in Section 2 we write $S_k = \delta^{-1}(0)$ and find $\mathrm{cl}_{\beta S} S_k = \hat{\delta}_k^{-1}(0)$.

The continuity properties of the mappings involved and the definition of δ_k show us that

$$\hat{\delta}_k(s) = (\hat{\delta}_k^{(1)}(\mathrm{pr}^{(1)}(s)), \ldots, \hat{\delta}_k^{(k)}(\mathrm{pr}^{(k)}(s)), 0, 0, \ldots).$$

The proof of the lemma which corresponds to Conjecture 2·2 is now immediate.

6·1. Lemma. *Let $e \in \beta S$ be idempotent and let the notation be as just described. Then $\hat{\delta}_k(e) = 0$ for every $k \in \mathbb{N}$.* ∎

The second lemma will give us little more trouble.

6·2. Lemma. *Again with the above notation, there is a map $\nu : S \to \mathbb{N}$ such that for each $s \in S$ with $s \neq 0$,*

$$\min\{k : \delta_k(u+s) \neq \delta_k(u)\} \leqslant \nu(s)$$

for every $u \in S$.

Proof. The only real point here is to define ν. We choose $\nu(0)$ arbitrarily. If $s \neq 0$, then there is i for which $\mathrm{pr}^{(i)}(s) \neq 0$, and then $\nu^{(i)}(\mathrm{pr}^{(i)}(s))$ is properly defined by the assumption that Conjecture 2·3 is satisfied for $S^{(i)}$. We define

$$\nu(s) = \min\{k : \text{for some } i \leqslant k \text{ both } \mathrm{pr}^{(i)}(s) \neq 0$$

and $\nu^{(i)}(\mathrm{pr}^{(i)}(s)) \leqslant k\}$.

We check that ν has the required property. Take $s \neq 0$ in S. We can find $i \leqslant \nu(s)$ such that $\mathrm{pr}_i(s) \neq 0$ and $\nu^{(i)}(\mathrm{pr}^{(i)}(s)) \leqslant \nu(s)$. The determining property of $\nu^{(i)}$ allows us to find $h \leqslant \nu^{(i)}(\mathrm{pr}^{(i)}(s))$ such that, for every $u \in S$,

$$\delta_h^{(i)}(\mathrm{pr}^{(i)}(u) + \mathrm{pr}^{(i)}(s)) \neq \delta_h^{(i)}(\mathrm{pr}^{(i)}(u)).$$

We put $k = \nu(s) \geqslant h$. Then also $k \geqslant i$. Since $i \leqslant k$ we have $\mathrm{pr}^{(i)} \circ \delta_k = \delta_k^{(i)}$ and since $h \leqslant k$ we have $\delta_h \circ \delta_k = \delta_k$, so from the fact that $\mathrm{pr}^{(i)}$ is a homomorphism we obtain

$$\delta_k(u+s) \neq \delta_k(u),$$

as required. ∎

The basic requirement of a digitization process was that if s and t were elements of S with the property that the digits of either were non-zero only where the digits of the other were zero, then the digits of $s+t$ consisted of the non-zero digits of both (in the same places as before) with zeros elsewhere. This property also holds for our present situation. In particular, each $s \in S$ has only a finite number of non-zero coordinates in $\bigoplus_i S^{(i)}$ and each coordinate $\mathrm{pr}^{(i)}(s)$ has only a finite number of non-zero digits. Therefore we can find k_0 such that $\delta_k(s) = s$ for all $k \geqslant k_0$. If then t is such that $\delta_k(t) = 0$ for some $k \geqslant k_0$, the digits $d_h^{(i)}(s)$ and $d_h^{(i)}(t)$ cannot both be non-zero for any pair i, h, and therefore $d_h^{(i)}(s+t) = d_h^{(i)}(s) + d_h^{(i)}(t)$. This property is exploited in the following lemma.

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6·3. LEMMA. *If $u \in \beta S$ and $s \in \bigcap_{k=1}^{\infty} \mathrm{cl}_{\beta S} S_k$ then $u+s \notin S$.*

Proof. Suppose $u+s = w \in S$. We first point out we may assume $u \in S$. Indeed, take a net (u_l) in S with $u_l \to u$. Then $u_l + s \to u+s = w$. But $w \in S$ and so is isolated in βS, whence $u_l + s = w \in S$ for sufficiently large l.

We take (s_j) in S with $s_j \to s$. Since now $u \in S \subseteq \Lambda(\beta S)$ there is j_0 such that $u+s_j = w$ for all $j \geqslant j_0$. For each k, $s \in \mathrm{cl}_{\beta S} S_k$, and the latter is a clopen set in βS; hence for sufficiently large j, $s_j \in \mathrm{cl}_{\beta S} S_k$, and as $s_j \in S$, in fact $s_j \in S_k$. We choose k so large that both $\delta_k(u) = u$ and $\delta_k(w) = w$, and then take j so that $s_j \in S_k$ and $s_j \neq 0$. We can find $h > k$ and i such that $d_h^{(i)}(s_j) \neq 0$, and then

$$0 = d_k^{(i)}(w) = d_h^{(i)}(u+s_j) = d_h^{(i)}(u) + d_h^{(i)}(s_j) = d_h^{(i)}(s_j)$$

gives a contradiction. ∎

The proof we have given for Lemma 6·3 is valid for any direct sum of digitizable semigroups of the kind described at the beginning of this section. However, the only semigroups for which we have constructed digitizations are groups, and therefore cancellative. Now if S is a discrete, cancellative semigroup its one-point compactification $S \cup \{\infty\}$ is a semigroup when ∞ acts as a semigroup zero ($s+\infty = \infty+s = \infty$ for $s \in S$). The identity map $\phi: S \to S$ extends to a continuous homomorphism $\hat{\phi}: \beta S \to S \cup \{\infty\}$, so that $S^* = \beta S \backslash S = \hat{\phi}^{-1}(\infty)$ is a closed ideal in βS. We state this as a further lemma for reference. Obviously if S also satisfies our digitization conditions, $\bigcap_{k=1}^{\infty} \mathrm{cl}_{\beta S} S_k \subseteq S^*$.

6·4. LEMMA. *If S is discrete and cancellative, S^* is an ideal in βS.* ∎

7. *The last basic lemma*

In this section, S is a countable discrete semigroup of the type discussed in Section 6, and so a countable direct sum. The key points for us are that Lemmas 6·1 and 6·2 hold. These lemmas do involve digitizations of the summands being carried out in such a way that their conclusions are valid and so depend on the idempotent e which is in question. Below we shall fix such an idempotent. Then suitable maps δ_k, subsets $S_k = \delta_k^{-1}(0) \backslash \{0\} \subseteq S$, etc., can be found as in the proofs of these lemmas.

We are (at last) going to discuss directly subsemigroups of βS. We remind the reader that a Λ-semigroup is a compact right topological semigroup T in which $\Lambda(T)$ is dense. Our objective is to prove this result.

7·1. LEMMA. *Let T be a Λ-subsemigroup of βS. Fix any idempotent e in T and take S_k so that Lemmas 6·1 and 6·2 hold. Let there be $u \in \Lambda(T)$ with $u \notin S$. Then, for some $k \in \mathbb{N}$,*

$$(S_k + e) \cap T = \varnothing.$$

In this lemma we do allow e to be the identity 0 of S (provided it is in T). However, if S is cancellable (as in the cases we can treat) then S^* is a semigroup, so that $S^* \cap T$ is a compact semigroup and therefore contains an idempotent not in S.

Our presentation of the proof of this lemma is via a few auxiliary results, but we first call attention to a slightly irritating point. We fix $u \in \Lambda(T) \backslash S$. Then $u \in \beta S$ and for each k we have from the finiteness of the range of δ_k that $\hat{\delta}_k(u) \in S$. We would therefore not expect $\hat{\delta}_k(u)$ to be independent of k, but it is conceivable that this

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happens – indeed, $\hat{\delta}_k(e) = 0$ for every k. If there does exist an element $w \in S$ such that, for some $k_0 \in \mathbb{N}$, $\hat{\delta}_k(u) = w$ for all $k \geqslant k_0$, we shall refer to w as the exceptional point of S.

We begin our proof of Lemma 7·1 by defining a function whose rôle in [**22**] was played by a cardinal function associated with the usual digitization of $\mathbb{N}$ (the number of non-zero digits in the expression for an integer). We believe our function is technically a little easier to handle. We have already noted that because $\delta_k : S \to S$ has finite image it extends to a continuous map $\hat{\delta}_k : \beta S \to S$. With u fixed as above, we define $\mu_u : S \to \mathbb{N}$ by

$$\mu_u(s) = \min\{k : \delta_k(s) \neq \hat{\delta}_k(u)\}$$

provided s is not the exceptional point w (if the latter exists); it is of no importance what $\mu_u(w)$ is taken to be, so we may as well put $\mu_u(w) = 1$. We extend μ_u to a unique continuous map $\hat{\mu}_u : \beta S \to \beta\mathbb{N}$.

7·2. Lemma. $\hat{\mu}_u(u) \in \mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$.

Proof. Take (u_i) in S with $u_i \to u$ in βS. Since $u \notin S$, eventually $u_i \neq w$, the exceptional point (if it exists). As the image $\hat{\delta}_k(\beta S) = \delta_k(S)$ is finite, for each k we have $\delta_k(u_i) = \hat{\delta}_k(u)$ eventually in i, and this means that $\mu_u(u_i) > k$. Thus $\mu_u(u_i) \to \infty$, which is equivalent to our assertion. ∎

Here is another simple result about μ_u.

7·3. Lemma. *Let* $s \in \bigcap_{k=1}^{\infty} \mathrm{cl}_{\beta S} S_k$. *Then if* $v \in \beta S$ *and* $v \neq w$ (*the exceptional point*), $\hat{\mu}_u(v+s) = \hat{\mu}_u(v)$.

Proof. We first consider the case in which $v \in S$. Then, if we choose k large enough, $\delta_k(v) = v$; we fix such a k. We take (s_j) in S with $s_j \to s$. Then we can find j_0 such that when $j \geqslant j_0$, $\delta_k(s_j) = \hat{\delta}_k(s_j) = \hat{\delta}_k(s) = 0$, since $s \in \mathrm{cl}_{\beta S} S_k$. Thus for $j \geqslant j_0$,

$$\delta_k(v + s_j) = v.$$

Now $\delta_h \circ \delta_k = \delta_h$ when $1 \leqslant h \leqslant k$, so we find $\delta_h(v+s_j) = \delta_h(v)$ for such h.

Since v is not the exceptional point, if we take k so large that $\delta_k(v) = v$ and also $k \geqslant \mu_u(v)$, we find, for $j \geqslant j_0$,

$$\mu_u(v + s_j) = \mu_u(v).$$

Now $v \in S \subseteq \Lambda(\beta S)$ and $s_j \to s$, so that $v + s_j \to v + s$ and hence $\hat{\mu}_u(v+s) = \mu_u(v)$. This is the result for $v \in S$. But to extend it to $v \in \beta S$ is a simple matter using continuity again. ∎

We now assume that the conclusion of Lemma 7·1 is false, so that $(S_k + e) \cap T \neq \varnothing$ for every $k \in \mathbb{N}$. Then for each k we can choose $s_k \in S_k$ with $s_k + e \in T$. We shall derive a contradiction by considering the sequence $(\hat{\mu}_u(u + s_k + e))_{k=1}^{\infty}$ in $\beta\mathbb{N}$.

7·4. Lemma. *The only cluster point of* $(\hat{\mu}_u(u + s_k + e))$ *is* $\hat{\mu}_u(u)$.

Proof. Any cluster point of the sequence can be written as $\lim_j \hat{\mu}_u(u + s_{k_j} + e)$ for some subnet (s_{k_j}) of the sequence (s_k). Since βS is compact, we may (by passing to a further subnet) assume that (s_{k_j}) converges, say $s_{k_j} \to s \in \beta S$. Since $s_h \in S_h \subseteq S_k$ when $h \geqslant k$, we find that $s \in \mathrm{cl}_{\beta S} S_k$ for every k, so that $s \in \bigcap_k \mathrm{cl}_{\beta S} S_k$.

Because βS is right topological we deduce from $s_{k_j} \to s$ that $s_{k_j} + e \to s + e$. Now each

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$s_{k_j}+e$ is in T and since T is compact, $s+e$ is also in T. From $u\in\Lambda(T)$ and $s_{k_j}+e\to s+e$ in T we deduce that $u+s_{k_j}+e\to u+s+e$. The continuity of $\hat{\mu}_u$ yields

$$\lim_j \hat{\mu}_u(u+s_{k_j}+e)=\hat{\mu}_u(u+s+e).$$

The cluster points of our sequence are thus all of the form $\hat{\mu}_u(u+s+e)$. If $e\neq 0$, we apply Lemma 7·3 twice (noting that $u+s\neq w$ from Lemma 6·3 and $u\neq w$) first using $e\in\bigcap_k \mathrm{cl}_{\beta S}S_k$ from Lemma 6·1 and then $s\in\bigcap_k \mathrm{cl}_{\beta S}S_k$, to find

$$\hat{\mu}_u(u+s+e)=\hat{\mu}_u(u+s)=\hat{\mu}_u(u)$$

as claimed. If $e=0$, the first of the last two equalities is obvious and the second follows from Lemma 7·3 as before. ∎

The proof of Lemma 7·1 is completed by the next result which flatly contradicts Lemma 7·4, showing that the assumption that $(S_k+e)\cap T\neq 0$ for every $k\in\mathbb{N}$ was invalid.

7·5. LEMMA. *$\hat{\mu}_u(u)$ is not the only cluster point of $(\hat{\mu}_u(u+s_k+e))$.*

Proof. If $e=0$, it is trivial to see that, for each k,

$$\hat{\mu}_u(u+s_k+e)=\hat{\mu}_u(u+s_k),$$

while if $e\neq 0$ this follows from Lemma 7·3.

We next take a net (u_i) in S with $u_i\to u$. We know that, for each $r\in\mathbb{N}$, $\delta_r(u_i)\to\hat{\delta}_r(u)$. We fix k for the moment, and take $r=\nu(s_k)$. Then, since $\hat{\delta}_r(\beta S)$ is finite, we can find i_0 such that $\delta_r(u_i)=\hat{\delta}_r(u)$ for all $i>i_0$. Therefore, for each such i,

$$\{h\leqslant r=\nu(s_k)\colon \delta_h(u_i+s_k)\neq\hat{\delta}_h(u)\}=\{h\leqslant r\colon \delta_h(u_i+s_k)\neq\delta_h(u_i)\}.$$

From the definition of μ_u and Lemma 6·2 we conclude that $\mu_u(u_i+s_k)\leqslant\nu(s_k)$. Taking the limit using the continuity of $\hat{\mu}_u$ gives $\hat{\mu}_u(u+s_k)\leqslant\nu(s_k)$. Thus, for each k, $\mu_u(u+s_k)$ is in $\mathbb{N}$.

Now either $(\mu_u(u+s_k))$ is a bounded sequence, in which case it cannot converge to $\hat{\mu}_u(u)\in\mathbb{N}^*$ (Lemma 7·2), or it is an unbounded sequence in $\mathbb{N}$ and so has 2^c distinct cluster points in $\mathbb{N}^*$. ∎

We close this section by applying Lemma 7·1 to determine $\Lambda(T)$ for certain subsemigroups T of βS when S is cancellative. The idea is at root the same as in the proof in [**22**] that the centre of $\mathbb{N}^*$ is empty, and the parallels with Strauss's work indicate that our Lemma 7·1 is a 'non-commutative' analogue of her Lemma 3.

For each infinite subsemigroup S_0 of S, $\beta S_0=\mathrm{cl}_{\beta S}S_0$ is a compact subsemigroup of βS. For cancellative S_0, S_0^* is an ideal in βS_0 (Lemma 6·4); thus it contains an idempotent e and $S_0+e\subseteq S_0^*$. We use e to produce a digitization of S, and since $e\in\mathrm{cl}_{\beta S}S_k$ for each k, we have $S_k\cap S_0\neq\varnothing$. Therefore, if either $T=\beta S_0$ or $T=S_0^*$, we find

$$(S_k+e)\cap T\supseteq((S_k\cap S_0)+e)\cap S_0^*=(S_k\cap S_0)+e\neq\varnothing.$$

We conclude from Lemma 7·1 that, in each case, $\Lambda(T)\backslash S=\varnothing$. This yields the following result.

7·6. THEOREM. *Let S_0 be an infinite subsemigroup of a cancellative, digitizable semigroup S. Then*

(i) $\Lambda(\beta S_0)=S_0$, (ii) $\Lambda(S_0^*)=\varnothing$. ∎

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The first part of this theorem is a special case of Theorem 3·6 of Ruppert's paper [**20**] (and a more complete result for all locally compact groups will appear in [**13**]), while the second part is Corollary 4 of [**22**] for the case of $\mathbb{N}^*$, and is also related to Theorem 4·6 of [**1**].

8. *Results*

Now at last we can present the consequences for the structure of compact subsemigroups of Stone–Čech compactifications of the preliminary work which has taken up the bulk of this paper. First we need a result about subgroups of Stone–Čech compactifications.

8·1. Theorem. *A compact subgroup G of βS, where S is a countable discrete semigroup, must be finite.*

Proof. Since S is discrete, we might as well assume $G \subseteq S^*$, for G is homogeneous (that is, given two points x, y of G, there is a homeomorphism – namely right translation by $x^{-1}y$ – which sends x to y). Then, as Strauss points out in [**22**], any compact subset of S^* is an F-space ([**12**], chapter 11, lemma 1·2·2) and homogeneous F-spaces are finite ([**12**], chapter 11, corollary 3·4·2, or [**5**], theorem 16·21). ∎

We come now to our conclusions. The simplest general class of semigroups to which they apply is described in our next lemma.

8·2. Lemma. *The commutative cancellative countable semigroups are precisely those which can be embedded in a countable direct sum of groups each of which is isomorphic either to $\mathbb{Q}$ or to $\mathbb{Z}(p^\infty)$ for some prime p.*

Proof. Cancellative commutative semigroups are precisely those which can be embedded in commutative groups ([**4**], §1·10). Commutative groups are embeddable in direct sums of copies of $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$s ([**9**], theorems 19·1 and 20·1, [**10**], theorems A 14 and A 15). That a countable subgroup only involves a countable number of the coordinates in a direct sum is easy to see. ∎

The purpose of our earlier sections is now clear – they enable us to treat the semigroups characterized in Lemma 8·2. Indeed, they do not permit us to deal with cases which are much more general, for all we can do is augment our direct sums by a countable number of non-commutative finite groups.

Here, then, is our main result. It includes Strauss's Theorem 1 about $\beta\mathbb{N}$ [**22**], for we can interpret that as saying that singly generated compact semigroups with dense centres embedded in $\beta\mathbb{N}$ must be finite.

8·3. Theorem. *Let S be a discrete countable semigroup which is algebraically isomorphic with a subsemigroup of a countable direct sum of copies of $\mathbb{Q}, \mathbb{Z}(p^\infty)$s and finite groups. Let T be a Λ-subsemigroup of βS for which $\Lambda(T)\backslash S \neq \varnothing$ and for which there is a countable subsemigroup T_0 of $\Lambda(T)$ which is dense in T.*

(i) *T_0 contains an idempotent (and indeed an idempotent minimal in T).*

(ii) *Each minimal right ideal in T is finite (and, a fortiori, each maximal group in the minimal ideal of T is finite).*

(iii) *If T_0 is a group (commutative or not) then T is a finite group.*

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(iv) *If T_0 is commutative, the minimal ideal of T is a finite commutative group.*

(v) *If T_0 is singly generated, T is finite.*

Proof. If S' is a discrete direct sum of the kind described in the statement and $S \subseteq S'$, then βS is a subsemigroup of $\beta S'$. Thus subsemigroups of βS are subsemigroups of $\beta S'$ and we lose nothing by supposing $S = S'$.

Now T, being a compact right topological semigroup, has a minimal ideal K. Let e be an idempotent in K, that is, a minimal idempotent. Then e is idempotent in βS, so Lemma 6·1 tells us $e \in \mathrm{cl}_{\beta S} S_k$ for each $k \in \mathbb{N}$. Idempotence and right continuity give

$$e = e + e \in (\mathrm{cl}_{\beta S} S_k) + e = \mathrm{cl}_{\beta S}(S_k + e),$$

and we have also $e \in T = \mathrm{cl}_{\beta S} T_0$. Thus $\mathrm{cl}_{\beta S}(S_k + e) \cap \mathrm{cl}_{\beta S} T_0 \neq \varnothing$. Lemma 2·1 tells us that, for each k, either $(S_k + e) \cap \mathrm{cl}_{\beta S} T_0 \neq \varnothing$ or $(\mathrm{cl}_{\beta S}(S_k + e)) \cap T_0 \neq \varnothing$. Lemma 7·1 says the former is impossible for at least one k, and therefore, for this k, $((\mathrm{cl}_{\beta S} S_k) + e) \cap T_0 = (\mathrm{cl}_{\beta S}(S_k + e)) \cap T_0 \neq \varnothing$. In particular there is some $s \in T_0$ with $s \in (\mathrm{cl}_{\beta S} S_k) + e$, so that $s = s + e$. We have proved that, for every idempotent e in the minimal ideal of T, $T_0 \cap (T_0 + e) \neq \varnothing$.

We now begin to prove (ii) by showing that the maximal groups in the minimal ideal of T are finite. Because they are all algebraically isomorphic, we need only prove that one of them is finite. Take $s \in T_0 \cap (T_0 + e)$ as above. Then s is in the minimal ideal of T, so $s + T + s$ is one of the groups we are considering ([**2**], theorem 2·16). By right continuity, $T + s$ is compact, and since $s \in T_0 \subseteq \Lambda(T)$, $s + (T + s)$ is compact, so that we have a right topological compact group. Theorem 8·1 tells us it is finite.

We can now dispose of (i). Continuing with the same notation, since $s + T_0 + s$ is dense in the finite set $s + T + s$ it must be equal to it. Therefore the identity of $s + T + s$ belongs to $s + T_0 + s \subseteq T_0$.

The minimal right ideals of T are algebraically isomorphic, so again we need only prove one is finite. With s as above, we consider $R = s + T$, which is compact since $s \in \Lambda(T)$. Put $G = s + T + s$, a maximal group in R (and so finite as we have seen), and let F be the set of idempotents in R. Then F is right-zero ($e + f = f$ for $e, f \in F$) and algebraically R is isomorphic to the direct product $G \times F$ ([**2**], theorem 1·2·16(iv)). If we fix $f \in F$, then that direct product representation yields easily $F = \{x \in R : x + f = f\}$, so that F is a closed subset of R. Thus (ii) follows from our next theorem.

8·4. THEOREM. *Let S be as in Theorem* 8·3 *and let F be a compact right-zero subsemigroup of βS. Then F is finite.*

Proof. The map $(e, f) \mapsto e + f = f$ is obviously jointly continuous on F, showing $\Lambda(F) = F$. Suppose F is infinite. Let T_0 be a countably infinite subset of F and put $T = \mathrm{cl}_{\beta S} T_0 \subseteq F$. Then $\Lambda(T) = T$ contains a point not in S, for S has at most one idempotent. The minimal ideal of T is T itself, so from the beginning of the proof of Theorem 8·3 we see that for each $e \in T$ there is $s_e \in T_0$ such that $s_e = s_e + e$. Since T is right-zero this means $s_e = e$, or that $T_0 = T$. Thus T_0 is a countably infinite compact subspace of an extremally disconnected space, which is not possible. We conclude that F has no infinite subset, so must be finite. ∎

We have now completed the proof of Theorem 8·3(ii) and we turn to (iii). In the proof of (ii) we showed that for some $s \in T_0$, $s + T + s$ is a maximal subgroup of the

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minimal ideal of T and is finite. Now $s+T_0+s \subseteq s+T+s$, and in this case $s+T_0+s = T_0$ because T_0 is a group. Since $\mathrm{cl}_{\beta S} T_0 = T$, we conclude that $T = s+T+s$.

We next prove (iv). Because T_0 is now commutative, it is contained in the algebraic centre of T (this is easy to prove). Take s as above, so that also $s+s$ is an element of the minimal ideal K of T. Then $K = T+(s+s)+T = s+T+T+s \subseteq s+T+s$, a finite group. Since $s+T_0+s = s+T+s$, this group is commutative.

Finally we come to (v). Let $T_0 = \{t, t+t, t+t+t, \ldots\}$. From (iii) the minimal ideal of T is a finite group G and we know that for some $s \in T_0, s \in G$. If $s = t+t+\ldots+t = nt$ (say), this leads to $T = \{t, t+t, \ldots, (n-1)t\} \cup G$, so that T is finite. ∎

We close with some general comments. Theorem 8·1 says (a little more than) that every extremally disconnected right topological group is finite. Theorem 8·3(ii) implies that compact minimal right ideals (or compact right groups) with dense topological centres which are embedded in βS are finite. Could it be that extremally disconnected compact right topological right groups with dense topological centres are always finite? The answer is easily seen to be negative. Any extremally disconnected space F can be made into a right-zero semigroup by putting $e+f = f$ for all $e, f \in F$. There is something special about the structure of semigroups like βS which forces compact right-zero subsemigroups to be finite.

Right-zero semigroups of arbitrary finite size do exist in $\beta\mathbb{N}$, for each minimal right ideal in $\beta\mathbb{N}$ has 2^c idempotents [11] which form a right-zero semigroup, and any finite subset is a finite subsemigroup. In contrast, whether there is any non-trivial finite subgroup in $\beta\mathbb{N}$ is unknown, as also is whether a singly-generated Λ-subsemigroup of $\mathbb{N}^*$ can have more than one element. Making conjectures about the semigroup $\beta\mathbb{N}$ is a foolhardy business, but we would advise readers with plenty of spare time to try to prove that these subobjects were singletons rather than to try to find larger specimens.

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