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The determination of a function from its projections

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Let $H(t, \theta)$ be the hyperplane in R^n ($n \geqslant 2$) which is perpendicular to the unit vector θ and perpendicular distance t from the origin, that is $H(t, \theta) = \{\mathbf{x} \in R^n : \mathbf{x}.\theta = t\}$. (Note that $H(t, \theta)$ and $H(-t, -\theta)$ are the same hyperplane.) If $f(\mathbf{x}) \in L^1(R^n)$ we will denote by $F(t, \theta)$ the projection of f perpendicular to θ, that is the integral of $f(\mathbf{x})$ over $H(t, \theta)$ with respect to $(n-1)$-dimensional Lebesgue measure. By Fubini's Theorem, if $f(\mathbf{x}) \in L^1(R^n)$, $F(t, \theta)$ exists for almost all t for every θ.

A number of results of the form that if f is a function such that $F(t, \theta) = 0$ for a certain collection of hyperplanes then $f(\mathbf{x}) = 0$ for almost all $\mathbf{x}$ in some set have been obtained, a summary of such results being given in the latter part of the article by Smith, Solmon and Wagner (4). The basic theorem of this type is the well-known Uniqueness Theorem, that if $f \in L^1(R^n)$ has $F(t, \theta) = 0$ for all t and θ then $f(\mathbf{x}) = 0$ almost everywhere. A similar but more sophisticated result is that if X is a proper compact convex subset of R^n, and $f(\mathbf{x})$ is an integrable function on R^n with compact support such that $F(t, \theta) = 0$ for all hyperplanes $H(t, \theta)$ that do not cut X, then $f(\mathbf{x}) = 0$ almost everywhere outside X. In what follows we prove that if X is a compact convex subset of R^n, and f is an integrable function on R^n such that $F(t, \theta)$ is constant for all hyperplanes $H(t, \theta)$ that intersect X, then $f(\mathbf{x}) = 0$ almost everywhere on X providing that the dimension n is odd. We show that this result does not hold if n is even. The method of proof is related to that used by the author (2) to show that a compact set is convex if all its Steiner symmetrands are convex. We require the following Lemma:

Lemma. *Let n be an odd integer at least 3, and let $f_0(r)$ be a continuous function defined for $0 \leqslant r < \infty$ with $\int_0^\infty |f_0(r)|\, r^{n-1} dr < \infty$.* Let

$$F_0(t) = \int_t^\infty f_0(r)\, r(r^2 - t^2)^{\frac{1}{2}(n-3)} dr.$$

Then if $F_0(t) = M$ (a constant) for $t < \epsilon$, we have that $f_0(r) = 0$ for $r < \epsilon$.

Proof. Let k be an integer such that $0 \leqslant k \leqslant \frac{1}{2}(n-3)$ and define

$$F_k(t) = \int_t^\infty f_0(r)\, r^{(2k+1)}(r^2 - t^2)^{\frac{1}{2}(n-3-2k)} dr. \tag{1}$$

Suppose that $F_k(t) = M$ for $t < \epsilon$ and $k < \frac{1}{2}(n-3)$. We show that $F_{k+1}(t) = M$ for $t < \epsilon$ also. From (1) we have that

$$\frac{dF_k(t)}{dt} = -(n-3-2k)\, t \int_t^\infty f_0(r)\, r^{(2k+1)}(r^2 - t^2)^{\frac{1}{2}(n-5-2k)} dr. \tag{2}$$

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differentiation under the integral sign being valid as the integrand obtained is uniformly convergent over the relevant range of t. Thus (1) and (2) give

$$F_k(t) - \frac{t}{(n-3-2k)}\frac{dF_k(t)}{dt} = \int_t^\infty f_0(r)\, r^{(2k+3)}(r^2-t^2)^{\frac{1}{2}(n-5-2k)}dr$$
$$= F_{k+1}(t).$$

Thus if $F_k(t)$ is constant for $t < \epsilon$, this will also be true of $F_{k+1}(t)$. But by the conditions of the Lemma $F_0(t)$ is constant for $t < \epsilon$, so we have by induction that $F_{\frac{1}{2}(n-3)}(t)$ is constant for $t < \epsilon$. But this is just that $\int_t^\infty f_0(r)\, r^{(n-2)}dr$ is constant $t < \epsilon$ so as $f_0(r)$ is continuous, $f_0(r) = 0$ for $r < \epsilon$.

THEOREM 1. *Let $f(\mathbf{x}) \in L^1(R^n)$ where n is odd and at least 3, and let X be a proper convex subset of R^n such that $F(t, \boldsymbol{\theta})$ is constant for every hyperplane $H(t, \boldsymbol{\theta})$ that cuts X. Then $f(\mathbf{x}) = 0$ for almost every $\mathbf{x} \in X$.*

Proof. Suppose first that $f(\mathbf{x})$ is a continuous function and assume that $F(t, \boldsymbol{\theta}) = M$ for all $|t| < \epsilon$ for all $\boldsymbol{\theta}$. Defining

$$f_0(r) = \frac{1}{A_n}\int f(r, \boldsymbol{\theta})\, d\boldsymbol{\theta},$$

where $f(r, \boldsymbol{\theta})$ represents f in polar coordinates with respect to the origin and A_n is the surface measure of the n-dimensional unit ball, we have that $f_0(r, \boldsymbol{\theta}) = f_0(r)$ defines a radially symmetric function in $L^1(R^n)$. An application of Fubini's Theorem gives that the projections of f_0 are given by $F_0(t, \boldsymbol{\theta}) = F_0(t)$ where

$$F_0(t) = \frac{1}{A_n}\int F(t, \boldsymbol{\theta})\, d\boldsymbol{\theta}, \tag{3}$$

and a calculation of the projections of a radially symmetric function gives that

$$F_0(t) = A_{n-1}\int_t^\infty f_0(r)\, r(r^2-t^2)^{\frac{1}{2}(n-3)}dr. \tag{4}$$

Thus from (3) $F_0(t) = M$ for all $t < \epsilon$ and so applying the Lemma to (4) gives that $f_0(r) = 0$ for $r < \epsilon$. Thus, as $f(\mathbf{x})$ is a continuous function, we have in particular that $f(\mathbf{0}) = f_0(\mathbf{0}) = 0$. But if we take any point in the interior of X as origin we will have that $F(t, \boldsymbol{\theta}) = M$ for all t sufficiently small (with respect to that origin). Thus $f(\mathbf{x}) = 0$ for all $\mathbf{x}$ in the interior of X, and therefore for all $\mathbf{x}$ in X in the case of $f(\mathbf{x})$ a continuous function.

Now let $f(\mathbf{x})$ be any function in $L^1(R^n)$ and let $f_\delta(\mathbf{x})$ be the continuous averaged function defined by

$$f_\delta(\mathbf{x}) = \frac{1}{V(\delta)}\int_{\mathbf{y} \in B(\mathbf{x}, \delta)} f(\mathbf{y})\, d\mathbf{y}$$

where $B(\mathbf{x}, \delta)$ is the ball with centre $\mathbf{x}$ and radius δ, and $V(\delta)$ is the n-dimensional measure of $B(\mathbf{x}, \delta)$. By Fubini's Theorem we have that $f_\delta \in L^1(R^n)$ and that the projections of f_δ, $F_\delta(t, \boldsymbol{\theta})$, are constant for all hyperplanes $H(t, \boldsymbol{\theta})$ that intersect $X_{-\delta}$, where

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$X_{-\delta}$ is the (convex) set of points that are distance at least δ from the complement of X. Thus by the result for continuous functions $f_\delta(\mathbf{x}) = 0$ for all $\mathbf{x} \in X_{-\delta}$ for all $\delta > 0$. But by Lebesgue's Theorem on the differentiation of the integral (see, for example, De Guzmán (1) p. 39), if $f \in L^1(R^n)$, then $f_\delta(\mathbf{x}) \to f(\mathbf{x})$ as $\delta \to 0$ for almost all $\mathbf{x}$, so that as $X_{-\delta}$ increases to X as $\delta \to 0$ we must have that $f(\mathbf{x}) = 0$ for almost all $\mathbf{x} \in X$.

Note that the same result will hold, by an identical proof to the above, if we merely require that $F(t, \boldsymbol{\theta}) = M(\boldsymbol{\theta})$ for all hyperplanes $H(t, \boldsymbol{\theta})$ that cut X, where $M(\boldsymbol{\theta})$ may depend on $\boldsymbol{\theta}$.

We now show that Theorem 1 does not hold for even dimensions.

THEOREM 2. *Let n be even and at least 2. Then there exists a function $f(\mathbf{x}) \in L^1(R^n)$ not almost everywhere zero, and with the compact support, such that $F(t, \boldsymbol{\theta}) = 0$ for all $|t| < 1$ for all $\boldsymbol{\theta}$.*

Proof. If b is a positive real number define $f_b(\mathbf{x})$ by

$$\begin{aligned} f_b(\mathbf{x}) &= (b^2 - |\mathbf{x}|^2)^{-\frac{1}{2}} && (|\mathbf{x}| < b) \\ &= 0 && (|\mathbf{x}| \geqslant b). \end{aligned}$$

Then $f_b \in L^1(R^n)$ and has projections given by

$$\begin{aligned} F_b(t, \boldsymbol{\theta}) &= A_{n-1} \int_t^b (b^2 - r^2)^{-\frac{1}{2}} r (r^2 - t^2)^{\frac{1}{2}(n-3)} \, dr && (|t| < b) \\ &= 0 && (|t| \geqslant b). \end{aligned}$$

By substituting $r^2 = a^2 \sin^2 u + b^2 \cos^2 u$, if k is a non-negative integer, we have that

$$\int_a^b (r^2 - a^2)^{k - \frac{1}{2}} (b^2 - r^2)^{-\frac{1}{2}} r \, dr = \gamma (b^2 - a^2)^k,$$

where

$$\gamma = \frac{\pi}{2} \frac{(2k)!}{(k! \, 2^k)^2}.$$

Thus letting $k = \frac{1}{2}n - 1$ for even n,

$$\left.\begin{aligned} F_b(t, \boldsymbol{\theta}) &= A_{n-1} \gamma (b^2 - t^2)^{\frac{1}{2}n - 1} && (|t| < b) \\ &= 0 && (|t| \geqslant b). \end{aligned}\right\} \tag{5}$$

Let $b(1), \ldots, b(\frac{1}{2}n + 1)$ be $\frac{1}{2}n + 1$ distinct numbers all at least 1. Then as the polynomials $(b(i)^2 - t^2)^{\frac{1}{2}n - 1}$ all lie in a vector space of polynomials of dimension $\frac{1}{2}n$, we may find scalars λ_i $(1 \leqslant i \leqslant \frac{1}{2}n + 1)$ not all zero such that

$$\sum_{i=1}^{\frac{1}{2}n+1} \lambda_i (b(i)^2 - t^2)^{\frac{1}{2}n - 1} = 0.$$

Thus if we define

$$f(\mathbf{x}) = \sum_{i=1}^{\frac{1}{2}n+1} \lambda_i f_{b(i)}(\mathbf{x})$$

then $f \in L^1(R^n)$ and from (5), $F(t, \boldsymbol{\theta}) = 0$ for all $|t| < 1$. $f(\mathbf{x})$ does not vanish almost everywhere, indeed $f(\mathbf{x})$ is unbounded near $|\mathbf{x}| = b(i)$. By taking an averaged function $f_\delta(\mathbf{x})$ it is easy to see that we may even find a continuous function with the required properties.

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We now consider the case where the support of f lies in the set X of Theorem 1. We prove the following generalization of the result of Green (3) using a new method.

THEOREM 3. *Let X be a proper convex subset of R^n and let $f(\mathbf{x}) \in L^1(X)$ be a function (not equal to zero almost everywhere) such that $F(t, \boldsymbol{\theta}) = M(\boldsymbol{\theta})$ for every hyperplane $H(t, \boldsymbol{\theta})$ that cuts X, where $M(\boldsymbol{\theta})$ depends only on $\boldsymbol{\theta}$. Then*

(1) *X is an n-ellipsoid*

(2) *$n = 2$*

(3) *If T is an affine transformation that maps X onto the unit disc, then f is of the form*

$$f(\mathbf{x}) = \gamma(1 - |T\mathbf{x}|^2)^{-\frac{1}{2}}$$

for almost all $\mathbf{x} \in X$, for a suitable constant γ.

Proof. If the integral of $f(\mathbf{x})$ over X were zero, then we would require that $M(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta}$ giving that $f(\mathbf{x}) = 0$ almost everywhere by the Uniqueness Theorem. Thus $\int_X f(\mathbf{x})\, d\mathbf{x} \neq 0$ so that X weighted by f has a centroid (that is a point $\mathbf{c}$ such that

$$\int_X (\mathbf{x} - \mathbf{c}) f(\mathbf{x})\, d\mathbf{x} = \mathbf{0})$$

which we may take to be the origin. Let $h(\boldsymbol{\theta})$ be the support function of X with respect to this origin. Then for each unit vector $\boldsymbol{\theta}$ we must have (taking moments)

$$\int_{-h(-\boldsymbol{\theta})}^{h(\boldsymbol{\theta})} tM(\boldsymbol{\theta})\, dt = 0,$$

giving that $h(\boldsymbol{\theta}) = h(-\boldsymbol{\theta})$ for all $\boldsymbol{\theta}$, so that X is centro-symmetric about the origin. By taking coordinates parallel to and perpendicular to $\boldsymbol{\theta}$ and using Fubini's Theorem we have that

$$\int_X f(\mathbf{x})\, (\mathbf{x}.\boldsymbol{\theta})^2 d\mathbf{x} = \int F(t, \boldsymbol{\theta})\, t^2\, dt$$

for all $\boldsymbol{\theta}$, and integrating and using that $F(t, \boldsymbol{\theta}) = M(\boldsymbol{\theta}) = M/(2h(\boldsymbol{\theta}))$ where M is a constant this becomes

$$\int_X f(\mathbf{x})\, (\mathbf{x}.\boldsymbol{\theta})^2 d\mathbf{x} = \tfrac{1}{3} M h(\boldsymbol{\theta})^2.$$

Thus $h(\boldsymbol{\theta})^2$ may be expressed as a second degree homogeneous polynomial in $\boldsymbol{\theta}$, that is if $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ with respect to some coordinate basis then

$$h(\boldsymbol{\theta})^2 = \sum_{i,j} a_{i,j} \theta_i \theta_j$$

for constants $a_{i,j}$. But this is the support function of a central conic, so that as the conditions of the Theorem imply that X is bounded, X must be an n-ellipsoid.

Now let T be an affine transformation that maps X onto the n-dimensional unit ball, and define the function $g(\mathbf{y})$ on B by

$$g(\mathbf{y}) = f(T^{-1}\mathbf{y}).$$

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g must have the property that the projections $G(t, \boldsymbol{\theta}) = M_1$ for some constant M_1 for all hyperplanes $H(t, \boldsymbol{\theta})$ that cut B. As $g(\mathbf{y}) \in L^1(B)$, g will have a continuous n-dimensional Fourier transform defined by

$$\hat{g}(\mathbf{d}) = \int g(\mathbf{y})\, e^{i\mathbf{y}.\mathbf{d}} d\mathbf{y} \quad (\mathbf{d} \in R^n).$$

Taking coordinates parallel to and perpendicular to $\boldsymbol{\theta}$ and applying Fubini's Theorem this may be written as

$$\hat{g}(\mathbf{d}) = \int G(t, \boldsymbol{\theta})\, e^{it|\mathbf{d}|} dt,$$

where $\boldsymbol{\theta} = \mathbf{d}/|\mathbf{d}|$ so that in this case

$$\begin{aligned} \hat{g}(\mathbf{d}) &= (2M_1 \sin|\mathbf{d}|)/|\mathbf{d}| \quad (\mathbf{d} \neq 0) \\ \hat{g}(\mathbf{0}) &= 2M_1. \end{aligned} \tag{6}$$

If $n = 2$ this is the Fourier transform of the function

$$\begin{aligned} g(\mathbf{y}) &= \frac{M_1}{\pi}(1 - |\mathbf{y}|^2)^{-\frac{1}{2}} \quad (|\mathbf{y}| < 1) \\ &= 0 \qquad\qquad\qquad\quad (|\mathbf{y}| \geqslant 1). \end{aligned}$$

Otherwise, as $\hat{g}(\mathbf{d})$ is radially symmetric it follows from the uniqueness of the Fourier transform that $g(\mathbf{y})$ must be a radially symmetric function that we will denote by $g(r)$. Thus as in (4)

$$G(t, \boldsymbol{\theta}) = A_{n-1} \int_t^1 g(r)\, r(r^2 - t^2)^{\frac{1}{2}(n-3)} dr. \tag{7}$$

But if $g \in L^1(X)$,

$$\int_0^1 |g(r)|\, r^{n-1} dr < \infty$$

so that estimating (7) in terms of this integral we must have that $G(t, \boldsymbol{\theta}) \to 0$ as $t \to 1$ if $n \geqslant 3$, contradicting the requirement that $G(t, \boldsymbol{\theta}) = M_1$ for all $t < 1$. Thus, if $n \geqslant 3$ the Fourier transform given in (6) is the transform of a distribution that is not an integrable function. (It is easy to see that in the case $n = 3$ (6) gives the transform of the uniform mass distribution on the unit sphere.)

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