J. Aust. Math. Soc. **120** (2026), 107–140
doi:10.1017/S1446788725101171

QUANTITATIVE GROWTH OF LINEAR RECURRENCES

ARMAND NOUBISSIE

(Received 17 May 2024; accepted 24 March 2025; first published online 6 October 2025)

Communicated by Dmitry Badziahin

Abstract

Let $\{u_n\}_n$ be a nondegenerate linear recurrence sequence of integers with Binet's formula given by $u_n = \sum_{i=1}^{m} P_i(n)\alpha_i^n$. Assume $\max_i |\alpha_i| > 1$. In 1977, Loxton and Van der Poorten conjectured that for any $\epsilon > 0$, there is an effectively computable constant $C(\epsilon)$ such that if $|u_n| < (\max_i\{|\alpha_i|\})^{n(1-\epsilon)}$, then $n < C(\epsilon)$. Using results of Schmidt and Evertse, a complete noneffective (qualitative) proof of this conjecture was given by Fuchs and Heintze ['On the growth of linear recurrences in function fields', *Bull. Aust. Math. Soc.* **104**(1) (2021), 11–20] and, independently, by Karimov *et al.* ['The power of positivity', *Proc. LICS* **23** (2023), 1–11]. In this paper, we give an effective upper bound for the number of solutions of the inequality $|u_n| < (\max_i\{|\alpha_i|\})^{n(1-\epsilon)}$, thus extending several earlier results by Schmidt, Schlickewei and Van der Poorten.

2020 *Mathematics subject classification*: primary 11B37; secondary 11J25, 11H06, 11D61.

Keywords and phrases: recurrence, Diophantine equations, geometry of numbers.

1. Introduction

A sequence of integer numbers $\{u_n\}$ is a linear recurrence sequence of order l if it is defined by the recurrence relation

$$u_{n+l} = a_1 u_{n+l-1} + a_2 u_{n+l-2} + \cdots + a_l u_n, \quad (n \in \mathbb{N}), \tag{1-1}$$

where the coefficients $a_1, \ldots, a_l$ are complex numbers and l is minimal. It is well known that we can write

$$u_n = \sum_{i=1}^{m} P_i(n)\alpha_i^n,$$

where α_i are the roots of the characteristic polynomial P of $\{u_n\}$ and the coefficients of polynomials P_i belong to $K := \mathbb{Q}(\alpha_1, \ldots, \alpha_m)$ and d is the degree of the number field K over $\mathbb{Q}$. One says that such a sequence is nondegenerate if none of the ratios

This research was funded in whole or in part by Austrian Science Fund (FWF): I4406.

© The Author(s), 2025. Published by Cambridge University Press on behalf of Australian Mathematical Publishing Association Inc.

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

$\alpha_i/\alpha_j\,(i \neq j)$ is a root of unity. Let α_1 be a dominant root of P. We denote by s the cardinality of the set containing all the prime ideals above α_i and all the Archimedean places over K. Here, we are interested in the study of the exponential Diophantine equation $u_n = 0$. Since any linear recurrence sequence can be decomposed as the interleaving of nondegenerate sequences, we study the previous equation only for the nondegenerate sequences. This problem was first studied by Skolem, Mahler and Lech, who used p-adic analysis to prove that this equation has finitely many solutions. Their proof is ineffective in the sense that it does not allow one to bound the number of solutions. Van der Poorten and Schlickewei [12] gave an upper bound for the number of solutions depending only on d, l and s. This result was extended by Schlickewei and Schmidt [11, Theorem 2.1], who provided an upper bound depending only on d and l, and by Schmidt [14], who provided an upper bound depending only on l.

We now turn to a more general question concerning the rate of growth of the recurrence. It is not difficult to see there is an effectively computable constant C such that, for all $n \geq 1, |u_n| < Cn^a|\alpha_1|^n$, where α_1 is a dominant root of the characteristic polynomial of the sequence $\{u_n\}$ with a, the maximum multiplicity of the roots α_i. In 1977, Loxton and Van der Poorten [8] conjectured that any nondegenerate LRS (linear recurrence sequence) has, essentially, a maximal possible growth rate, that is, for any $\epsilon > 0$, there is an effectively computable constant $C(\epsilon)$ such that if $|u_n| < (\max_i\{|\alpha_i|\})^{n(1-\epsilon)}$, then $n < C(\epsilon)$. Using results of Schmidt [14] and Evertse [4], a complete noneffective (qualitative) proof of this conjecture was given by Fuchs and Heintze [6] and, independently, by Karimov *et al.* [7].

This question about the rate of growth of LRS is connected to the Positivity and Ultimate Positivity Problems (in theoretical computer science). In fact, Ouaknine and Worrell [10] have proven that for a simple linear recurrence sequence, the Ultimate Positivity Problem is decidable for all orders. Their algorithm arises from model theory and the qualitative version of the Loxton–Van der Poorten conjecture above. The noneffectiveness of this result makes that algorithm nonconstructive in the sense that, for a given linear recurrence sequence $\{u_n\}$, the procedure does not produce a threshold N such that $u_n \geq 0$ for all $n \geq N$. This is why the positivity problem for simple linear recurrence sequences is still open.

In this article, we make progress in that direction by giving an explicit upper bound for the number of solutions.

Let $\{u_n\}$ be the nondegenerate linear recurrence sequence of integers of order $l \geq 1$ defined by the Recurrence Relation (1-1) with Binet's formula given by

$$u_n = \sum_{i=1}^{m} P_i(n)\alpha_i^n,$$

where α_i are the roots of the characteristic polynomial P of $\{u_n\}$ with multiplicity at most a; the coefficients a_{ij} of the polynomials P_i belong to $K := \mathbb{Q}(\alpha_1, \ldots, \alpha_m)$ and d is the degree of the number field K over $\mathbb{Q}$. Assume α_1 is a dominant root of P and $|\alpha_1| > 1$. We denote by S the set containing all the prime ideals above α_i, and all the

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

Archimedean places over K and by s its cardinality. Let n_{ij} be the smallest positive integer such that $n_{ij}a_{ij} \in O_K$ and we denote by $\rho := \mathrm{lcm}(n_{ij})$. Put

$$A' := \min_{\sigma,i,j}\{1, |\sigma(\rho a_{ij})|, \sigma(a_{ij}) \neq 0\}, \quad A := \max_{\sigma,i,j}\{1, |\sigma(\rho a_{ij})|\}, \tag{1-2}$$

where the minimum and the maximum are taken respectively over all elements of $\mathrm{Gal}(K/\mathbb{Q})$ and coefficients of ρP_i. Let Δ_K be the discriminant of K. For $x \in \mathbb{R}^*$, we define the functions $g(x) = 2^{34} \cdot 2^{11l} s^2 l^{12} x^{-5}$ and

$$\tau(x) = \max\left\{\frac{20a^l A^l}{A'}, \frac{-\log(A')}{\epsilon \log(|\alpha_1|)}, \frac{11mA + e^{g(x)} \log(l!\,) \log(2d|\Delta_K|)}{x \log|\alpha_1|}\right\}.$$

Notice that $\rho = 1$ is equivalent to all the coefficients of the polynomials P_i being algebraic integers. Hence, our main result can be stated as the following theorem.

THEOREM 1.1. *For each $0 < \epsilon < 1/12$, the number of solutions of the inequality*

$$|u_n| < |\alpha_1|^{n(1-\epsilon)} \tag{1-3}$$

does not exceed:

- $\tau(\epsilon/2d) + 2 \cdot s^2(d^2 l^7 2^{4l+19} \epsilon^{-2})^{ls+l} \exp((7l^a)^{8l^a})$ *if $\rho = 1$; and*
- $(2\epsilon^{-1} \log\rho/\log|\alpha_1|) + \tau(\epsilon/4d) + 2 \cdot s^2(d^2 l^7 2^{4l+21} \epsilon^{-2})^{ls+l} \exp((7l^a)^{8l^a})$ *if $\rho \neq 1$.*

The following result owing to Schmidt [14] plays a crucial role in the proof of Theorem 1.1.

LEMMA 1.2. *Suppose that $\{u_n\}_{n\in\mathbb{Z}}$ is a nondegenerate linear recurrence sequence of complex numbers, whose characteristic polynomial has k distinct roots of multiplicity at most a. Then, the number of solutions $n \in Z$ of the equation $u_n = 0$ can be bounded above by*

$$c(k, a) = \exp((7k^a)^{8k^a}).$$

REMARK 1.3. An effective upper bound for the number of solutions of Inequality (1-3) can be obtained by 'carefully' combining a variant of the main theorem in [5] and Lemma 1.2. However, our Theorem 1.1 provides a better upper bound.

If the characteristic polynomial of $\{u_n\}_n$ has just one root, that is, $m = 1$, then Inequality (1-3) becomes

$$|P_1(n)\alpha_1^n| < (|\alpha_1|)^{n(1-\epsilon)},$$

which implies $|P_1(n)\alpha_1^{n\epsilon}| < 1$. If $n > (2(aA + 1)/A')$, where A and A' are defined in (1-2), then one gets $|P_i(n)| > A'$. So $A'|\alpha_1^{n\epsilon}| < 1$. Hence, we conclude that any nonnegative integer solution n of the inequality (1-3) satisfies the relation

$$n < \max\left(\frac{2(aA + 1)}{A'}, \frac{-\log(A')}{\epsilon \log(|\alpha_1|)}\right).$$

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

In the remainder of the article, we assume $m > 1$. Let M_K be the collection of places over K. For $v \in M_K$, $x \in K$, we define $|x|_v$ as follows:

- $|x|_v = |\sigma(x)|^{1/d}$ if v corresponds to the embedding $\sigma : K \to \mathbb{R}$;
- $|x|_v = |\sigma(x)|^{2/d}$ if v corresponds to the embedding $\sigma : K \to \mathbb{C}$;
- $|x|_v = (N_{K/\mathbb{Q}}(\mathcal{P}))^{-\mathrm{ord}_{\mathcal{P}}(x)/d}$ if v corresponds to the prime ideal $\mathcal{P}$ of O_K and $\mathrm{ord}_{\mathcal{P}}(x)$ to the exponent of $\mathcal{P}$ in the decomposition of the ideal generated by x.

With this norm, we have the product formula

$$\prod_{v \in M_K} |x|_v = 1 \quad \text{for } x \in K.$$

For $X = (x_1, \ldots, x_m) \in K^m$, $X \neq 0$ and $|X|_v = \max(|x_1|_v, \ldots, |x_m|_v)$, we define the height of X as

$$H(X) = \prod_{v \in M_K} |X|_v.$$

Note that $H(X)$ depends only on X and not on the choice of the number field K (see [1, Lemma 1.5.2]). Let K be the splitting field of the characteristic polynomial of $\{u_n\}$ and S be the set containing all the prime ideals above α_i and all the Archimedean places over K, and s be its cardinality. The ring of S-integers and its set of units are respectively defined as

$$O_S = \{x \in K : |x|_v \leq 1 \text{ for all } v \notin S\}$$

and

$$O_S^* = \{x \in K : |x|_v = 1 \text{ for all } v \notin S\}.$$

We denote by id the embedding over K corresponding to the identity and for $v \in M_K$, we define the quantity $s(v)$ by

$$s(v) = \begin{cases} \dfrac{1}{d} & \text{if } v \text{ is real infinite,} \\ \dfrac{2}{d} & \text{if } v \text{ is complex infinite,} \\ 0 & \text{if } v \text{ is finite,} \end{cases}$$

where d is the degree of K over $\mathbb{Q}$. By the definition of $s(v)$, we get

$$\sum_{v \in M_K} s(v) = 1. \tag{1-4}$$

For $v \in S \setminus \{\mathrm{id}\}$, we define m linear forms $l_{1v}, \ldots, l_{mv}$ with m variables $x := (x_1, \ldots, x_m)$ over K as $l_{iv}(x) = x_i$. For $v = \mathrm{id}$, we define $l_{1v}(x) = x_1 + \cdots + x_m$ and $l_{iv}(x) = x_i$ for $i = 2, \ldots, m$.

Notice that to prove Theorem 1.1, it suffices to show only the first part (that is, the case $\rho = 1$, meaning all the coefficients of the polynomials P_i are algebraic integers)

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

since the second part follows easily from the first one. Indeed, assume $\rho \neq 1$ and fix $\epsilon > 0$. It is clear that all the solutions of Inequality (1-3) are solutions of the inequality

$$|\rho u_n| < \frac{\rho}{|\alpha_1|^{n\epsilon/2}} |\alpha_1|^{n(1-\epsilon/2)}.$$

If $\rho > |\alpha_1|^{n\epsilon/2}$, then we obtain

$$n < \frac{2\epsilon^{-1} \log \rho}{\log |\alpha_1|}.$$

Otherwise, one obtains $|\rho u_n| < (|\alpha_1|)^{n(1-\epsilon/2)}$. Since the coefficients of the polynomials ρP_i in Binet's formula for the nondegenerate sequence $v_n := \rho u_n$ are all algebraic integers, the desired result follows by applying the first case of Theorem 1.1 with $\epsilon/2$ instead of ϵ.

Thus, in the rest of the paper, we assume $\rho = 1$, meaning that all the coefficients of the polynomials P_i are algebraic integers. Put

$$\mathcal{N} := \{(P_1(n)\alpha_1^n, \ldots, P_m(n)\alpha_m^n), n \in \mathbb{N}\}$$

and $X_n = (P_1(n)\alpha_1^n, \ldots, P_m(n)\alpha_m^n)$. Our main result is a consequence of the following theorem.

THEOREM 1.4. *For each $0 < \epsilon < 1/12$, the set of $X_n \in \mathcal{N}$ with $n > \tau(\epsilon)$ that are solutions of the inequality*

$$\prod_{v \in S} \prod_{i=1}^{m} |l_{iv}(X_n)|_v < |\alpha_1|^{-n\epsilon} \tag{1-5}$$

is contained in at most

$$2 \cdot s^2 (m^7 2^{4m+17} \epsilon^{-2})^{ms+m}$$

proper subspaces in K^m.

2. Deduction of Theorem 1.1 from Theorem 1.4

Fix $\epsilon > 0$. Let n be a solution of the inequality (1-3) with $n > \tau(\epsilon/2d)$ and $X_n \in \mathcal{N}$. For $v \in S$,

$$|P_i(n)|_v \leq ((d_i + 1)n^{d_i})^{s(v)} A^{s(v)} \leq (an^a)^{s(v)} A^{s(v)}, \tag{2-1}$$

where a is the maximum multiplicity of the roots of the characteristic polynomial of $\{u_n\}$; d_i the degree of P_i and A is defined in (1-2). By using the relation (2-1), (1-4), the product formula and the fact that $\alpha_i \in O_S^*$,

$$\prod_{v \in S} \prod_{i=1}^{m} |P_i(n)\alpha_i^n|_v \leq (an^a A)^m. \tag{2-2}$$

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

Hence,

$$\begin{aligned}\prod_{v\in S}\prod_{i=1}^{m}|l_{iv}(X_n)|_v &= \frac{|u_n|_{\mathrm{id}}}{|P_1(n)\alpha_1^n|_{\mathrm{id}}}\cdot\prod_{v\in S}\prod_{i=1}^{m}|P_i(n)\alpha_i^n|_v\\ &\le \frac{(an^aA)^l}{|P_1(n)\alpha_1^n|_{id}}\cdot|\alpha_1|^{ns(id)(1-\epsilon)}\\ &\le n^{l^2}\cdot|\alpha_1|^{-n\epsilon/d}\\ &\le |\alpha_1|^{-n\epsilon/2d},\end{aligned}$$

where for the first inequality, we use Relation (2-2), the fact that n is a solution of Inequality (1-3) and $m \le l$, and for the last one, [9, Lemma 2.4] and the fact that $n > \tau(\epsilon/2d)$. So X_n is a solution of Inequality (1-5) (replacing ϵ by $\epsilon/2d$). Applying Theorem 1.4 and the fact that $X_n \in \mathcal{N}$, we deduce that there are at most

$$2\cdot s^2(d^2l^7 2^{4l+19}\epsilon^{-2})^{ls+l}$$

exponential Diophantine equations of the shape

$$\sum_{i\in I} c_i P_i(n)\alpha_i^n = 0 \tag{2-3}$$

with c_i non-zero elements of K and $I \subseteq \{1, 2, \dots, m\}$ such that for all solutions X_n of Inequality (1-5), n is a solution of at least one of them. From Lemma 2, each Diophantine equation given by the relation (2-3) has at most $\exp((7l^a)^{8l^a})$ solutions with $n \in \mathbb{Z}$. Hence, we conclude that the cardinality of the set of solutions n of the inequality (1-3) with $n > \tau(\epsilon/2d)$ does not exceed

$$2\cdot s^2(d^2l^7 2^{4l+19}\epsilon^{-2})^{ls+l}\exp((7l^a)^{8l^a}),$$

which gives us the desired result.

3. Preliminaries

Before starting the proof of Theorem 1.4, we need some notation and facts about exterior algebra, inequalities related to heights, and results from the geometry of numbers over number fields.

Let V be a vector space over the splitting field K of the characteristic polynomial P of $\{u_n\}$ of dimension m (number of distinct roots of P), with canonical basis e_i $(i = 1, \dots, m)$. For $1 \le p \le m$, we equip the exterior power $\wedge^p V$ with the standard basis

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p} \quad \text{with } i_1 < i_2 < \cdots < i_p.$$

Put $N = \binom{m}{p}$. Let $\sigma_1, \dots, \sigma_N$ be the subsets of $\{1, \dots, m\}$ given by

$$\sigma_1 = \{1, \dots, p\}, \sigma_2 = \{1, \dots, p-1, p+1\}, \dots,$$

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

and

$$\sigma_{N-1} = \{m-p, m-p+2, \dots, n\}, \quad \sigma_N = \{m-p+1, \dots, m\}$$

ordered lexicographically. For the vectors $x_t = (x_{t1}, \dots, x_{tm}) \in K^m$, $1 \le t \le m$, we define

$$\Delta_{\sigma_i} = \Delta_i(x_1, \dots, x_p) = \begin{vmatrix} x_{1,i_1} & x_{1,i_2} & \cdots & x_{1,i_p} \\ x_{2,i_1} & x_{2,i_2} & \cdots & x_{2,i_p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{p,i_1} & x_{p,i_2} & \cdots & x_{p,i_p} \end{vmatrix},$$

where $\sigma_i = \{i_1, \dots, i_p\}$ and also

$$x_1 \wedge x_2 \wedge \cdots \wedge x_p = (\Delta_{\sigma_1}, \Delta_{\sigma_2}, \dots, \Delta_{\sigma_N}).$$

Note that $x_1 \wedge x_2 \wedge \cdots \wedge x_p = 0$ if and only if $\{x_1, x_2, \dots, x_p\}$ is linearly dependent (see, for example, [13, Lemma 6B, Ch. IV]). For $x = (x_1, x_2, \dots, x_m)$, we define $x^* = (x_m, -x_{m-1}, \dots, (-1)^{m-1}x_1)$ and extend the standard scalar product $x \cdot y = \sum_{i=1}^m x_i y_i$, where $x, y \in K^m$, to $\wedge^p V$, using the Laplace identity,

$$(x_1 \wedge x_2 \wedge \cdots \wedge x_p) \cdot (y_1 \wedge y_2 \wedge \cdots \wedge y_p) = \det(x_i \cdot y_j)_{i,j=1,\dots,p} \tag{3-1}$$

and the Laplace extension

$$x_1 \cdot (x_2 \wedge x_3 \wedge \cdots \wedge x_m)^* = \det(x_1, \dots, x_m) \quad \text{with } x_1, \dots, x_m \in K^m. \tag{3-2}$$

A similar notation is used with regards to linear forms. If $l(X) = a \cdot X = \sum_{i=1}^m a_i X_i$, where $a = (a_1, \dots, a_m)$, we put $l^*(X) = a^* \cdot X$. Further, for linear forms $l_t(X) = a^{(t)} \cdot X$, with $1 \le t \le p$, in m variables, we define the linear forms in N variables by

$$(l_1 \wedge l_2 \wedge \cdots \wedge l_p)(x) = (l_1 \wedge l_2 \wedge \cdots \wedge l_p) \cdot x.$$

Let $\{a_1, \dots, a_m\}$ and $\{b_1, \dots, b_m\}$ be two bases of K^m. Then $\{A_1, \dots, A_N\}$ and $\{B_1, \dots, B_N\}$ with $A_i = a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{i_{m-p}}$ and $B_i = b_{i_1} \wedge b_{i_2} \wedge \cdots \wedge b_{i_{m-p}}$ are two bases of K^N.

The following lemma, established in [3, Lemma 3.3.1, Ch. IV], is used at several places in this paper.

LEMMA 3.1. *Let $k \in \{1, \dots, m\}$. If $\{a_1, \dots, a_m\}$ and $\{b_1, \dots, b_m\}$ are two bases of K^m, then $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_k\}$ generate the same vector space if and only if $\{A_1, \dots, A_{N-1}\}$ and $\{B_1, \dots, B_{N-1}\}$ generate the same vector space with $N = \binom{m}{k}$.*

This lemma implies the existence of an injective map f_k from the collection of k-dimensional subspaces of K^m to the collection of $(N-1)$-dimensional subspaces of K^N such that if V has basis $\{a_1, \dots, a_k\}$ and $a_{k+1}, \dots, a_n$ are any vectors such that $\{a_1, \dots, a_n\}$ is a basis of K^m, then $\{A_1, \dots, A_{N-1}\}$ is a basis of $f_k(V)$.

LEMMA 3.2. *Let $v \in M_K$. The following inequalities hold:*

$$|x \cdot y|_v \le m^{s(v)} |x|_v \cdot |y|_v \quad \text{for } x, y \in K^m \tag{3-3}$$

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

and

$$|x_1 \wedge x_2 \wedge \cdots \wedge x_k|_v \leq (k!)^{s(v)}|x_1|_v \cdots |x_k|_v \quad \text{for } x_1, \ldots, x_k \in K^m, k \leq m. \tag{3-4}$$

If $x_1, \ldots, x_m$ are linearly independent vectors in K^m, then also

$$|\det(x_1, \ldots, x_m)|_v \leq (m!)^{s(v)}|x_1|_v \cdots |x_m|_v \tag{3-5}$$

and

$$H(x_1 \wedge x_2 \wedge \cdots \wedge x_k) \leq k! \cdot H(x_1) \cdot H(x_2) \cdots H(x_k). \tag{3-6}$$

PROOF. Let x, y be two elements of K^m. By the above definition of the scalar product on the vector space K^m, we infer that

$$|x \cdot y|_v = \left| \sum_{i=1}^{m} x_i y_i \right|_v \leq m^{s(v)}|x|_v \cdot |y|_v,$$

where for the inequality, we use the fact that $|\cdot|_v^{1/s(v)}$ is the usual absolute value when v is an infinite place over K and the strong triangle inequality property of $|\cdot|_v$, whereas v is a finite place over K. Therefore, the relation (3-3) holds. By the above definition of the exterior power over the vector space K^m,

$$x_1 \wedge x_2 \wedge \cdots \wedge x_k = (\Delta_{\sigma_1}, \Delta_{\sigma_2}, \ldots, \Delta_{\sigma_N}),$$

with $N = \binom{m}{k}$ and

$$\Delta_{\sigma_i} = \Delta_i(x_1, \ldots, x_k) = \begin{vmatrix} x_{1,i_1} & x_{1,i_2} & \cdots & x_{1,i_k} \\ x_{2,i_1} & x_{2,i_2} & \cdots & x_{2,i_k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{k,i_1} & x_{k,i_2} & \cdots & x_{k,i_k} \end{vmatrix},$$

where $\sigma_i = \{i_1 < \cdots < i_k\}$. From the Leibniz formula for the determinant Δ_{σ_i}, it follows that

$$|\Delta_{\sigma_i}|_v \leq (k!)^{s(v)}|x_1|_v \cdots |x_k|_v$$

for all $i \in \{1, \ldots, N\}$. So

$$|x_1 \wedge x_2 \wedge \cdots \wedge x_k|_v \leq (k!)^{s(v)}|x_1|_v \cdots |x_k|_v.$$

If $x_1, \ldots, x_m$ are linearly independent vectors in K^m, then

$$|\det(x_1, \ldots, x_m)|_v = |x_1 \wedge x_2 \wedge \cdots \wedge x_m|_v$$

and so the relation (3-5) follows immediately from (3-4). By taking the product over all the places of M_K on the relation (3-4) and using the definition of the height function $H(\cdot)$, we obtain the relation (3-6). □

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

LEMMA 3.3. *Let $v \in M_K$, H be positive real numbers and $x_1, \ldots, x_p$ linearly independent vectors in K^m, with $H(x_i) \le H$ for $i = 1, \ldots, p$. Then,*

$$\frac{H^{-p}}{p!} \le \frac{|x_1 \wedge x_2 \wedge \cdots \wedge x_p|_v}{|x_1|_v \cdots |x_p|_v} \le (p!)^{s(v)}. \tag{3-7}$$

In particular, if $p = m$, then

$$\frac{H^{-m}}{m!} \le \frac{|\det(x_1, \ldots, x_m)|_v}{|x_1|_v \cdots |x_m|_v} \le (m!)^{s(v)}. \tag{3-8}$$

PROOF. We prove this result by using the same argument as in the proof of [5, Lemma 2]. Since $x_1, \ldots, x_p$ are linearly independent vectors, it follows from [13, Lemma 6B, Ch. IV] that $x_1 \wedge x_2 \wedge \cdots \wedge x_p \neq 0$; therefore, $H(x_1 \wedge x_2 \wedge \cdots \wedge x_p) \ge 1$. Hence,

$$\begin{aligned}
\frac{|x_1 \wedge x_2 \wedge \cdots \wedge x_p|_v}{|x_1|_v \cdots |x_p|_v} &= \frac{(\prod_{w \in M_K \setminus \{v\}} |x_1 \wedge x_2 \wedge \cdots \wedge x_p|_v)^{-1} \cdot H(x_1 \wedge x_2 \wedge \cdots \wedge x_p)}{|x_1|_v \cdots |x_p|_v} \\
&\ge \prod_{w \in M_K \setminus \{v\}} (p!)^{-s(w)} \cdot \prod_{w \in M_K} (|x_1|_w \cdots |x_p|_w)^{-1} \\
&\ge (p!)^{-1} H(x_1)^{-1} \cdots H(x_p)^{-1} \\
&\ge \frac{H^{-p}}{p!},
\end{aligned}$$

where, for the first inequality, we use Relation (3-4) of Lemma 3.2, the second inequality follows from the fact that

$$\sum_{w \in M_K} s(w) = 1,$$

whereas the last inequality holds because $H(x_i) \le H$ for $i = 1, \ldots, p$. So we proved the first inequality of Relation (3-7). The second inequality follows immediately from Relation (3-4). Therefore, the proof of the relation (3-7) is complete. By the definition above of the exterior power of the vector space K^m,

$$|\det(x_1, \ldots, x_m)|_v = |x_1 \wedge x_2 \wedge \cdots \wedge x_m|_v.$$

Hence, the relation (3-8) follows from the relation (3-7), which completes the proof of our lemma. □

We now define $\operatorname{rank}(x_1, \ldots, x_r)$, the maximum number of vectors in $M = \{x_1, \ldots, x_r\}$ that are linearly independent over K^m. We need the following lemma, which is a slight modification of [5, Lemma 3].

LEMMA 3.4. *Let H be a positive real number, $a_1, \ldots, a_r \in \overline{\mathbb{Q}}^m$ with $H(a_i) \le H$ for $i = 1, \ldots, r$ and $x \in K^m$ be nonzero such that*

$$a_i \cdot x = 0 \quad \text{for } i = \{1, \ldots, r\}.$$

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

Suppose that $\operatorname{rank}(a_1, \dots, a_r) \le m-1$. *Then, there exists* $y \in K^m$ *that only depends on* a_i *such that* $y \neq 0, a_i \cdot y = 0$ *for* $i = 1, \dots, r$ *and* $H(y) \le (m-1)!\, H^{m-1}$.

Here, we provide the result of Bombieri and Vaaler about a generalisation of Minkowski's result on geometry of numbers to an Adele ring of a number field (see [1, Appendix C] for more details). Throughout this section, K is the splitting field of the characteristic polynomial of $\{u_n\}$, and S the set of places consisting of all Archimedean places and prime ideals above $\alpha_1, \dots, \alpha_m$. Let O_v be a local ring $\{x \in K_v : |x|_v \le 1\}$. A set C_v of K_v^m (with respect to the v-adic topology) is called a symmetric convex body in K_v^m if it satisfies the following conditions:

- 0 is an interior point of C_v and C_v compact;
- if $x \in C_v$ and $\alpha \in K_v, |\alpha|_v = 1$, then $\alpha x \in C_v$;
- if $v \mid \infty$ and $x, y \in C_v$, then $\lambda x + (1-\lambda)y \in C_v$ for all $\lambda \in \mathbb{R}, 0 \le \lambda \le 1$; otherwise if $x, y \in C_v$, then $x + y \in C_v$.

The ring of K-Adele V_K is the set of infinite tuples $(x_v, v \in M_K)$ with $x_v \in K_v$ and $|x_v|_v \le 1$ for all but finitely many v, endowed with component-wise addition and multiplication. The set V_K^m may be identified with a set of infinite tuples of vectors (x_v) with $x_v \in K_v^m$ for all $v \in M_K$ and $x_v \in O_v^m$ for all but finitely many v. By the diagonal embedding, we mean the map

$$\psi : K^m \to V_K^m : x \mapsto (x_v),$$

with $x_v = x$ for $v \in M_K$. A convex body in V_K^m is a Cartesian product

$$C = \prod_{v \in M_K} C_v = \{(x_v) \in V_K^m : x_v \in C_v \text{ for all } v \in M_K\},$$

where for every $v \in M_K, C_v$ is a symmetric convex body in V_K and $C_v = O_v^m$ for all but finitely many $v \in M_K$. For $\lambda \in \mathbb{R}$, we define

$$\lambda C := \prod_{v \mid \infty} \lambda C_v \times \prod_{v \nmid \infty} C_v.$$

The i th successive minima $\lambda_i := \lambda_i(C)$ is defined by

$$\lambda_i := \min\{\lambda \in \mathbb{R}_{>0} : \psi^{-1}(\lambda C) \text{ contains } i \text{ linearly independent points}\}.$$

This minimum exists (see [1, Appendix C]). We recall that we use the volume with respect to the following measure. If $v \nmid \infty$, then K_v is a commutative locally compact set so that the Haar measure exists and it is unique up to a constant. Further, we consider the measure β_v on K_v defined by $\beta_v(\Omega) = |\Delta_{K_v}|_p^{1/2} \mu_v(\Omega)$, with μ_v defined in [1, Example C.1.3] and Ω a μ_v-measurable set of K_v. If $v \mid \infty$ and is real, then $K_v = \mathbb{R}$ and we take β_v to be the usual Lebesgue measure on $\mathbb{R}$. If v is complex infinite, then $K_v = \mathbb{C}$ and we take β_v to be two times Lebesgue measure on the complex plane. The

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

corresponding product measure on K_v^m is denoted β_v^m. If ρ is a linear transformation of K_v^m into itself, then

$$\beta_v^m(\rho D) = |\det(\rho)|_v^d \beta_v^m(D)$$

for any β_v^m-measurable set D of K_v^m (see [5, page 239]). Now, let $\beta = \prod_v \beta_v$ be the product measure on V_K and β^m the n-fold product measure on V_K^m.

Bombieri and Vaaler [2, Theorems 3 and 6] proved the following generalisation of Minkowski's theorem (see also [5, Lemma 6]).

LEMMA 3.5. *Let Δ_K be the discriminant of K and r_2 the number of complex infinite places of K. Further, let C be a symmetric convex body in V_K^m and $\lambda_1, \dots, \lambda_m$ its successive minima. Then,*

$$\left(\frac{\pi^m m!}{2}\right)^{r_2/d} \frac{2^m}{m!} |\Delta_K|^{-m/2d} \le \lambda_1 \cdots \lambda_m \cdot \beta^m(C)^{1/d} \le 2^m.$$

By a tuple $(N, \overline{\gamma}, \mathcal{L}, Q)$ we mean a collection satisfying:

- $Q > 1; N \ge 2$; $\overline{\gamma} = (\gamma_{iv}, v \in S, i = 1, \dots, N)$;
- $\mathcal{L}$ is a system of linear forms $\mathcal{L} = \{\bar{l}_{iv} : v \in S, i = 1, \dots, N\}$ in N variables such that $\{\bar{l}_{1v}, \dots, \bar{l}_{Nv}\}$ is linearly independent for each $v \in S$.

We define the set

$$\Pi(N, \overline{\gamma}, \mathcal{L}, Q) := \{y \in \mathcal{O}_S^N : |\bar{l}_{iv}(y)|_v \le Q^{\gamma_{iv}} \text{ for } v \in S, i = 1, \dots, N\}$$

and

$$V(N, \overline{\gamma}, \mathcal{L}, Q) := \text{the } K\text{-vector space generated by } \Pi(N, \overline{\gamma}, \mathcal{L}, Q).$$

Notice that the map f_k is defined previously. We deduce our Theorem 1.4 from the two following results.

THEOREM 3.6. *Let $0 < \epsilon < 1/12$, and $X_n \in \mathcal{N}$ with $n > \tau(\epsilon)$ be a solution of Inequality (1-5) such that $l_{iv}(X_n) \ne 0$ for $v \in S, i = 1, \dots, m$. There are a proper linear subspace W of K^m and a tuple $(N, \overline{\gamma}, \mathcal{L}, Q)$ with $N = \binom{m}{k}$, where $k = \dim_K W$, such that the three conditions below are satisfied:*

$$X_n \in W, \quad f_k(W) = V(N, \overline{\gamma}, \mathcal{L}, Q), \quad \dim_K(V(N, \overline{\gamma}, \mathcal{L}, Q)) = N - 1; \tag{3-9}$$

$$\overline{\gamma} = (\gamma_{iv}, v \in S, i = 1, \dots, N) \quad \text{with } \gamma_{iv} \le s(v) \text{ and } \sum_{v \in S} \sum_{i=1}^{N} \gamma_{iv} \le -\frac{\epsilon}{9m^3}; \tag{3-10}$$

$$Q \ge (|\alpha_1|)^{3mn}, \tag{3-11}$$

and such that $(N, \overline{\gamma}, \mathcal{L}) \in C$, where C is a fixed set independent of X_n of cardinality at most $C' := (64em^4 2^m \epsilon^{-1})^{ms+m}$.

The next result gives us an effective upper bound for the number of vector spaces $V(N, \overline{\gamma}, \mathcal{L}, Q)$ when Q is large enough.

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

THEOREM 3.7. *Fix a tuple $(N, \overline{\gamma}, \mathcal{L})$ as in Theorem 3.6. Then there exists a collection of $(N-1)$-dimensional linear subspaces of K^N of cardinality at most*

$$C'' = 2^{11m+34} s^2 \epsilon^{-4} m^{12}$$

such that for every Q with

$$\dim_K(V(N, \overline{\gamma}, \mathcal{L}, Q)) = N-1; \quad Q > (9m!)^{e^{2C''}}, \tag{3-12}$$

the vector spaces $V(N, \overline{\gamma}, \mathcal{L}, Q)$ belong to this collection.

By combining Lemma 3.1 and Theorems 3.6 and 3.7, we easily deduce Theorem 1.4. Indeed, we notice that Lemma 3.1 implies the map f_k from the collection of k-dimensional subspaces of K^m to the collection of $(N-1)$-dimensional subspaces of K^N such that, for any k-dimensional subspace V of basis $\{a_1, \dots, a_k\}$ and any vectors $a_{k+1}, \dots, a_n$ such that $\{a_1, \dots, a_n\}$ is a basis of K^m, then $f_k(V)$ is the subspace generated by $\{A_1, \dots, A_{N-1}\}$, with $N = \binom{m}{m-k}$ with $A_i = a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{i_{m-k}}$ is injective. Let $0 < \epsilon < 1/12$ and $X_n \in \mathcal{N}$ with $n > \tau(\epsilon)$ solutions of Inequality (1-5) such that $l_{iv}(X_n) \neq 0$ for $v \in S, i = 1, \dots, m$. Since $n > \tau(\epsilon)$, we deduce

$$|\alpha_1|^{3mn} \geq (9m!)^{e^{2C''}} \quad \text{with } C'' = 2^{11m+34} s^2 \epsilon^{-4} m^{12}.$$

It follows from Theorem 3.6 that a proper subspace W of K^m and a tuple $(N, \overline{\gamma}, \mathcal{L})$ independent of n such that

$$X_n \in W, \quad f_k(W) = V(N, \overline{\gamma}, \mathcal{L}, Q), \quad \dim_K(V(N, \overline{\gamma}, \mathcal{L}, Q)) = N-1, \quad Q \geq (9m!)^{e^{2C''}}$$

exist. The cardinality of the collection of such tuples $(N, \overline{\gamma}, \mathcal{L})$ does not exceed $C' := (64em^4 2^m \epsilon^{-1})^{ms+m}$. For each tuple $(N, \overline{\gamma}, \mathcal{L})$, we deduce from Theorem 3.7 that the collection of vector spaces $V(N, \overline{\gamma}, \mathcal{L}, Q)$ satisfying the properties

$$\dim_K(V(N, \overline{\gamma}, \mathcal{L}, Q)) = N-1 \quad Q \geq (9m!)^{e^{2C''}}$$

does not exceed C''. Since f_k is injective, we obtain the set of proper subspaces W of K^m for which $X_n \in W$ does not exceed

$$C'C'' = (64em^4 2^m \epsilon^{-1})^{ms+m} \cdot 2^{11m+34} s^2 \epsilon^{-4} m^{12} \leq s^2 (m^7 2^{4m+17} \epsilon^{-2})^{ms+m}.$$

By adding the collection of ms proper subspaces associated to the collection $\{l_{iv}, v \in S, i = 1, \dots, m\}$, we conclude that the set of $X_n \in \mathcal{N}$ with $n > \tau(\epsilon)$ solutions of Inequality (1-5) is contained in at most

$$ms + s^2 (m^7 2^{4m+17} \epsilon^{-2})^{ms+m} \leq 2 \cdot s^2 (m^7 2^{4m+17} \epsilon^{-2})^{ms+m}$$

proper subspaces in K^m.

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

4. Proof of Theorem 3.6

We prove this result using the same machinery as in the proof of [5, Theorem B], with the difference that the norm of vectors in K^m that we are working with is not the Euclidean norm that Evertse considered in his paper. Let $0 < \epsilon < 1/12$ and $X_n \in \mathcal{N}$ with $n > \tau(\epsilon)$ solutions of Inequality (1-5) such that $l_{iv}(X_n) \neq 0$ for $v \in S, i = 1, \dots, m$. Fixing $v \in S$, we define $\mathcal{O}_v = \{x \in K_v : |x|_v \leq 1\}$ and the v-adic approximation domain $\Pi_v(n)$ of X_n to be the paralellepiped in K_v^m given by

$$\Pi_v(n) = \{y \in K_v^m : |l_{iv}(y)|_v \leq |l_{iv}(X_n)|_v \text{ for } i = 1, \dots, m\}.$$

This is a compact convex body of K_v^m. The approximate domain is defined in V_K^m by

$$\begin{aligned}\Pi(n) &= \prod_{v \in S} \Pi_v(n) \times \prod_{v \notin S} \mathcal{O}_v^m \\ &= \{y_v \in V_K^m : |l_{iv}(y_v)|_v \leq |l_{iv}(X_n)|_v \text{ for } v \in S \text{ and } |y_v|_v \leq 1 \text{ for } v \notin S\}.\end{aligned}$$

We denote by $\lambda_1(n), \dots, \lambda_m(n)$ the successive minima of $\Pi(n)$;

$$R_v(n) = \left(\prod_{i=1}^{m} |l_{iv}(X_n)|_v \right)^{-1} \quad \text{and} \quad R(n) = \prod_{v \in S} R_v(n).$$

LEMMA 4.1. *Let $v \in M_K$ and D_{K_v} be the discriminant over K_v. When v is a finite place over K, we denote by p the rational prime above v.*

(1) *If $v \notin S$, then $\beta_v^m(n) := \beta_v^m(\Pi_v(n)) = \beta_v^m(\mathcal{O}_v^m) = |D_{K_v}|_p^{m/2}$.*
(2) *If $v \in S$ is not Archimedean, then $\beta_v^m(n) = |D_{K_v}|_p^{m/2} \cdot R_v(n)^{-d}$.*
(3) *If $K_v = \mathbb{R}$, then $\beta_v^m(n) = 2^m \cdot R_v(n)^{-d}$.*
(4) *If $K_v = \mathbb{C}$, then $\beta_v^m(n) = (2\pi)^m \cdot R_v(n)^{-d}$.*
(5) *We also have*

$$\frac{1}{m!} R(n) \leq \lambda_1(n) \cdots \lambda_m(n) \leq \Delta_K^{m/2d} R(n).$$

PROOF. It is easy to see that, for $v \in S$,

$$\Pi_v(n) = \rho \mathcal{O}_v^m \quad \text{with } \rho = \begin{pmatrix} l_{1v}(X_n) & l_{2v}(X_n) \cdot \{0, -1\} & \cdots & l_{mv}(X_n) \cdot \{0, -1\} \\ 0 & l_{2v}(X_n) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & l_{mv}(X_n) \end{pmatrix},$$

where $l_{iv}(X_n) \cdot \{0, -1\} = -l_{iv}(X_n)$ if $v = \mathrm{id}$ and 0 otherwise. Hence, the first four statements follow from the definition of β_v and the fact that

$$\beta_v^m(n) = \beta_v^m(\rho \mathcal{O}_v^m) = |\det(\rho)|_v^d \beta_v^m(\mathcal{O}_v^m).$$

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

From [1, B.1.20 and B.1.21, page 606]),

$$|\Delta_K|_p = \prod_{v|p} |\Delta_{K_v}|_p.$$

Together with the product formula,

$$|\Delta_K|^{-1} = \prod_p \prod_{v|p} |\Delta_{K_v}|_p. \tag{4-1}$$

By the definition of β^m,

$$\begin{aligned}\beta^m(n) &= 2^{mr_1} \cdot (2\pi)^{mr_2} \prod_p \prod_{v|p} |\Delta_{K_v}|_p^{m/2} \cdot \Big(\prod_{v\in S} R_v(n) \Big)^{-d} \\ &= 2^{mr_1} \cdot (2\pi)^{mr_2} |\Delta_K|^{-m/2} \cdot R(n)^{-d},\end{aligned}$$

where, for the last equality, we use Relation (4-1) and r_1, r_2 correspond respectively to the numbers of real and complex places. Together with Lemma 3.5,

$$\begin{aligned}\lambda_1(n)\cdots\lambda_m(n) &\le 2^m \beta^m(n)^{-1/d} \\ &\le 2^m \cdot 2^{-mr_1/d} (2\pi)^{-mr_2/d} |\Delta_K|^{m/2d} \cdot R(n) \\ &\le \left(\frac{2}{\pi}\right)^{mr_2/d} |\Delta_K|^{m/2d} \cdot R(n) \\ &\le |\Delta_K|^{m/2d} \cdot R(n),\end{aligned}$$

and

$$\begin{aligned}\lambda_1(n)\cdots\lambda_m(n) &\ge \left(\frac{\pi^m m!}{2}\right)^{r_2/d} \frac{2^m}{m!} \cdot 2^{-mr_1/d} (2\pi)^{-mr_2/d} \cdot R(n) \\ &\ge \left(\frac{m!}{2}\right)^{r_2/d} \frac{(2^m)^{r_2/d}}{m!} \cdot R(n) \\ &\ge \frac{R(n)}{m!}.\end{aligned}$$

Hence, the proof is completed. □

We also need the following lemma.

LEMMA 4.2. *Let $\lambda > 0$. For each $y \in \psi^{-1}(\lambda\Pi(n))$:*

- $|y|_v \le (m^2(m-1)!\, R(n)^{11/10\epsilon}\lambda)^{s(v)}$ *for* $v \in S$;
- $\lambda_1(n) \ge m^{-2}(m-1)!^{-1}\, R(n)^{-11/10\epsilon}$.

PROOF. Let $y \in \psi^{-1}(\lambda\Pi(n))$. Fix $v \in S$ and set $\Delta_v = \det(l_{1v}, \dots, l_{mv})$. Clearly, $\Delta_v = 1$. Let a_j be the vector $(l_{1v} \wedge \cdots \wedge l_{j-1v} \wedge l_{j+1v} \wedge \cdots \wedge l_{mv})^*$. The fact that $\{a_1, \dots, a_m\}$ is a basis of K^m implies

$$y = \sum_{j=1}^m \Delta_v^{-1} l_{jv}(y) a_j = \sum_{j=1}^m l_{jv}(y) a_j. \tag{4-2}$$

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

Using the fact $n > \tau(\epsilon)$, one gets $|X_n|_v \le (|\alpha_1|)^{11ns(v)/10}$. Namely, if v is a finite place over K, then $|X_n|_v \le 1$, as all the coordinates $P_i(n)\alpha_i^n$ of X_n are algebraic integers, so we are done. Otherwise, by using the fact that $|\cdot|_v^{1/s(v)}$ is the usual absolute value,

$$\begin{aligned}|X_n|_v &= |(P_i(n)\alpha_1^n, \dots, P_m(n)\alpha_m^n)|_v \\ &\le \max_i |(P_i(n)\alpha_i^n|_v \\ &\le \left(\frac{(a+1)An^a}{|\alpha_1|^{n/10}}\right)^{s(v)} \cdot (|\alpha_1|)^{11ns(v)/10} \\ &\le (|\alpha_1|)^{11ns(v)/10},\end{aligned}$$

where, for the second inequality, we use the triangle inequality for the absolute value and the definition of A in (1-2), while for the last inequality, we use the fact that $n > \tau(\epsilon)$. Therefore,

$$\begin{aligned}|l_{jv}(y)|_v &\le \lambda^{s(v)}|l_{jv}(X_n)|_v \\ &\le (\lambda m(|\alpha_1|)^{11n/10})^{s(v)} \\ &\le (\lambda m(R(n))^{11/(10\epsilon)})^{s(v)}. \end{aligned} \tag{4-3}$$

By Relation (3-4),

$$|a_j|_v = |l_{1v} \wedge \cdots \wedge l_{j-1v} \wedge l_{j+1v} \wedge \cdots \wedge l_{mv}|_v \le (m-1)!^{s(v)}.$$

Together with (4-2) and (4-3),

$$\begin{aligned}|y|_v &\le m^{s(v)} \max_j\{|l_{jv}(y)a_j|_v\} \\ &\le (\lambda m^2(m-1)!\,(R(n))^{11/(10\epsilon)})^{s(v)}.\end{aligned}$$

Hence, the proof of the first statement is completed. Let $y \in \psi^{-1}(\lambda_1(n)\Pi(n))$ with $y \neq 0$. Then, $|y|_v \le 1$ for $v \notin S$ and since $H(y) \ge 1$, it follows that

$$1 \le H(y) \le \prod_{v\in S} |y|_v \le m^2(m-1)!\,\lambda_1(n)(R(n))^{11/(10\epsilon)},$$

which implies the second statement of the lemma. □

The following lemma is a particular case of one due to Evertse [5, Lemma 15].

LEMMA 4.3. *Let $b_1, \dots, b_m$ be linearly independent vectors in K^m. For each infinite place v on K, let $\mu_{1v}, \dots, \mu_{mv}$ be real numbers with*

$$0 < \mu_{1v} \le \mu_{2v} \le \cdots \le \mu_{mv}.$$

Suppose $|l_{iv}(b_j)|_v \le \mu_{jv}$ for $1 \le i, j \le m, v \mid \infty$. Then there are permutations k_v of $\{1, \dots, m\}$ for each infinite place v on K and vectors

$$v_1 = b_1, \quad v_i = \sum_{j=1}^{i-1} \epsilon_{ij} v_j + b_i \quad \text{with } \epsilon_{ij} \in O_K$$

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

such that

$$|l_{k_v(i),v}(v_j)|_v \le (2d|\Delta_K|^{1/2})^{s(v)(i+j)} \min\{\mu_{iv}, \mu_{jv}\} \quad \textit{for } 1 \le i,j \le m, v \mid \infty.$$

Choose linearly independent vectors $g_1, \dots, g_m$ with $g_j \in \psi^{-1}(\lambda_j \Pi(n))$ for any $j = 1, \dots, m$. By Lemma 4.2, we have $g_j \in O_K$ and

$$\begin{cases} |l_{iv}(g_j)|_v \le \lambda_j^{s(v)} |l_{iv}(X_n)|_v & \text{for } v \mid \infty, \\ |l_{iv}(g_j)|_v \le |l_{iv}(X_n)|_v & \text{for } v \in Sv \nmid \infty. \end{cases}$$

By applying Lemma 4.3 with the vectors $g_1, \dots, g_m$ and the linear forms $l_{iv}(X_n)^{-1} l_{iv}$ $(v \mid \infty, i = 1, \dots, m)$, and the real numbers $\mu_{jv} = \lambda_j^{s(v)} (v \mid \infty, j = 1, \dots, m)$, we infer that there are vectors $v_1, \dots, v_m$ with

$$v_1 = g_1, \quad v_i = \sum_{j=1}^{i-1} \epsilon_{ij} v_j + g_i \quad \text{with } \epsilon_{ij} \in O_K$$

and permutations k_v of $\{1, \dots, m\}$ for $v \mid \infty$ such that

$$|l_{k_v(i),v}(v_j)|_v \le ((2d|\Delta_K|^{1/2})^{2m} \lambda_{\min\{i,j\}})^{s(v)} |l_{k_v(i),v}(X_n)|_v \quad \text{for } 1 \le i,j \le m, v \mid \infty.$$

Fix $v \in S$, $v \nmid \infty$. Since $\epsilon_{ij} \in O_K$ and $g_j \in O_K^m$,

$$v_j \in O_K^m, \quad |l_{iv}(v_j)|_v \le |l_{iv}(X_n)|_v.$$

Hence, we have proven the following lemma.

LEMMA 4.4. *There are linearly independent vectors $v_1, \dots, v_m \in O_K^m$ and permutations k_v of $\{1, \dots, m\}$ for $v \mid \infty$ all depending of X_n, with the following properties:*

- *for $j = 1, \dots, m$, the vectors $v_1, \dots, v_j$ belong in the K-vector space generated by $\psi^{-1}(\lambda_j(n)\Pi(n))$;*
- *we also have*

$$\begin{cases} |l_{k_v(i),v}(v_j)|_v \le ((2d|\Delta_K|^{1/2})^{2m} \lambda_{\min\{i,j\}}(n))^{s(v)} |l_{k_v(i),v}(X_n)|_v & \textit{for } v \mid \infty \\ |l_{iv}(v_j)|_v \le |l_{iv}(X_n)|_v & \textit{for } v \in S, v \nmid \infty. \end{cases}$$

In the next lemma, we make a first step towards the proof of Theorem 1.4.

LEMMA 4.5. *There is a set Γ of cardinality*

$$|\Gamma| \le (m!)^{-s} (64em^4 2^m \epsilon^{-1})^{ms+m}$$

consisting of the tuples of real numbers

$$(c; d) = (c_{iv} : v \in S, i = 1, \dots, m; d_i : i = 1, \dots, m)$$

with:

- $c_{iv} \le (12s(v)/10\epsilon)$ *for* $v \in S, i = 1, \dots, m$;
- $d_1 \le 0, -12/10\epsilon \le d_1 \le \cdots \le d_m$

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

such that for every solution X_n of Inequality (1-5), with $n > \tau(\epsilon)$ and $l_{iv}(X_n) \neq 0$ for $v \in S, i = 1, \ldots, m$, there is a tuple $(c; d) \in \Gamma$ with

$$R(n)^{c_{iv}-\{1/sc(m)\}} < |l_{iv}(X_n)|_v \leq R(n)^{c_{iv}}$$

and

$$R(n)^{d_i} \leq \lambda_i(n) \leq R(n)^{d_i+\{1/c(m)\}}$$

for $v \in S, i = 1, \ldots, m$ with $c(m) = 4m^3 2^m$.

PROOF. Put $u_{iv} := |l_{iv}(X_n)|_v (R(n)^{-12/10\epsilon})^{s(v)}$ for $v \in S, i = 1, \ldots, m$. Since $|X_n|_v \leq (|\alpha_1|)^{11ns(v)/10}$, it follows that

$$u_{iv} \leq |X_n|_v (mR(n)^{-12/10\epsilon})^{s(v)} \leq (mR(n)^{-1/10\epsilon})^{s(v)} \leq 1,$$

where for the second inequality, we use the fact that X_n is a solution of Inequality (1-5) and for the last inequality, $n > \tau(\epsilon)$. However,

$$\begin{aligned}\prod_{v\in S}\prod_{i=1}^{m} u_{iv} &= \Big(\prod_{v\in S}\prod_{i=1}^{m} |l_{iv}(X_n)|_v\Big) R(n)^{-12m/10\epsilon} \\ &\geq (R(n))^{-2m/\epsilon}.\end{aligned}$$

By applying [5, Lemma 9] with $\theta = 0.5\epsilon c(m)^{-1}$, there is a set Γ_1 of nonnegative reals $\beta = (\beta_{iv}, v \in S, i = 1, \ldots, m)$ of cardinality

$$|\Gamma_1| \leq (2e + 2ec(m)\epsilon^{-1})^{ms}$$

such that for some $\beta \in \Gamma_1$,

$$R(n)^{c_{iv}-\{1/sc(m)\}} < |l_{iv}(X_n)|_v \leq R(n)^{c_{iv}},$$

with

$$c_{iv} = \frac{12}{10\epsilon} s(v) - \frac{2m}{\epsilon}\beta_{iv} \quad \text{for } v \in S, i = 1, \ldots, m.$$

Therefore, $c_{iv} \leq (12/10\epsilon)s(v)$. Finally, $c = (c_{iv}, v \in S, i = 1, \ldots, m)$ depends only on $\gamma \in \Gamma_1$; hence, for c, we have at most $|\Gamma_1|$ possibilities.

We define the numbers $v_j(n) := R(n)^{-12/10\epsilon}\lambda_j(n)^{-1}$ for $v \in S, i = 1, \ldots, m$. By Lemma 4.2, we have $\lambda_1(n) \geq m^{-2}(m-1)!^{-1} R(n)^{-11/10\epsilon}$. Together with $n > \tau(\epsilon)$,

$$1 \geq v_1(n) \geq v_2(n) \geq \cdots \geq v_m(n).$$

Then, it follows from Lemma 4.1 that

$$v_1(n) \cdots v_m(n) = \frac{R(n)^{-12m/10\epsilon}}{\lambda_1(n) \cdots \lambda_m(n)} \geq R(n)^{-2m/\epsilon}.$$

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

By applying [5, Lemma 9] with $\theta = 0.5\epsilon c(m)^{-1}$, there is a set Γ_2 of m-tuples of nonnegative reals $\delta = (\delta_i, i = 1, \dots, m)$ independent of X_n of cardinality

$$|\Gamma_2| \le (2e + 2ec(m)\epsilon^{-1})^m$$

such that for some $\delta \in \Gamma_2$,

$$R(n)^{-2m/\epsilon(\delta_j+\epsilon/2mc(m))} < v_j(n) \le R(n)^{-2m\delta_j/\epsilon}, \tag{4-4}$$

and the relation (4-4) remains valid after replacing $\delta_1, \dots, \delta_m$ by

$$\delta'_1 = \min\{\delta_1, \dots, \delta_m\}, \quad \delta'_2 = \min\{\delta_2, \dots, \delta_m\}, \dots, \delta'_m = \delta_m.$$

Putting

$$d_j = \frac{-12}{10\epsilon} + \frac{2m\delta'_j}{\epsilon} \quad \text{for } j = 1, \dots, m,$$

finally,

$$R(n)^{d_i} \le \lambda_i(n) \le R(n)^{d_i + \{1/c(m)\}};$$

therefore, $d = (d_1, \dots, d_m)$ depends only on Γ_2. From the definition d_i, we have the desired inequality. Since $X_n \in \psi^{-1}(\Pi(n))$, it follows by definition of first successive minima, we have $\lambda_1(n) \le 1$; therefore, $d_1 \le 0$. For such a tuple d, we have at most $|\Gamma_2|$ possibilities. It follows that the number of possibilities for $(c; d)$ is at most

$$\begin{aligned} |\Gamma_1||\Gamma_2| &\le (2e + 2ec(m)\epsilon^{-1})^{ms+m} \\ &\le (16e^2m^3 2^m \epsilon^{-1})^{ms+m} \\ &\le (m!)^{-s}(64em^4 2^m \epsilon^{-1})^{ms+m}, \end{aligned}$$

where we use Stirling's approximation to get the last inequality. □

Let X_n be a solution of Inequality (1-5) with $n > \tau(\epsilon)$ and $(c; d)$ the corresponding tuple from Lemma 4.5. For any finite place $v \in S$, choose k_v to be an identity map from $\{1, \dots, m\}$ to $\{1, \dots, m\}$. We define

$$l'_{iv}(X) = l_{k_v(i),v}(X) \quad \text{for } v \in S, i = 1, \dots, m$$

and the numbers

$$e_{iv} = c_{k_v(i),v} \quad \text{for } v \in S, i = 1, \dots, m.$$

We have constructed a tuple

$$\mathcal{T} = (l'_{iv} : v \in S, i = 1, \dots, m; e_{iv} : v \in S, i = 1, \dots, m; d_i : i = 1, \dots, m).$$

By Lemma 4.5,

$$R(n)^{e_{iv} - \{1/sc(m)\}} < |l'_{iv}(X_n)|_v \le R(n)^{e_{iv}} \tag{4-5}$$

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

and

$$R(n)^{d_i} \le \lambda_i(n) \le R(n)^{d_i+\{1/c(m)\}} \tag{4-6}$$

for $v \in S, i = 1, \dots, m$.

LEMMA 4.6. *$\mathcal{T}$ belongs to a set independent of X_n of cardinality at most*

$$C' = (64em^4 2^m \epsilon^{-1})^{ms+m},$$

and we also have:

- $e_{iv} \le (12s(v)/10\epsilon)$ *for* $v \in S, i = 1, \dots, m$;
- $\sum_{v\in S} \sum_{i=1}^{m} e_{iv} \le -1 + (4m^2 2^m)^{-1}$;
- $1 - (3m^2 2^m)^{-1} \le \sum_{i=1}^{m} d_i \le 1 + (3m^2 2^m)^{-1}$.

PROOF. For infinite places $v \in S$, the linear forms l'_{iv} are uniquely determined by the permutation of $\{1, \dots, m\}$ from Lemma 4.3. Therefore, $\mathcal{T}$ is uniquely determined by $k_v(v \mid \infty)$ and $(c; d)$. It follows that for $\mathcal{T}$, we have at most

$$(m!)^{r_1+r_2}(m!)^{-s}(64em^4 2^m \epsilon^{-1})^{ms+m} \le C'$$

possibilities. The second point follows from the definition of $\mathcal{T}$. Namely, from Lemma 4.5,

$$e_{iv} \le \frac{12s(v)}{10\epsilon} \quad \text{for } v \in S, i = 1, \dots, m.$$

Moreover,

$$R_v(n)^{-1} = \prod_{i=1}^{m} |l_{iv}(X_n)|_v \ge R(n)^{\sum_{i=1}^{m} e_{iv} - m/c(m)s}$$

for $v \in S$. Hence,

$$R(n)^{-1} = \prod_{v\in S} R_v(n)^{-1} \ge R(n)^{\sum_{i,v} e_{iv} - m/c(m)},$$

which gives us the desired result. By (4-6) and Lemma 4.1,

$$R(n)^{d_1+\cdots+d_m} \le \lambda_1(n) \cdots \lambda_m(n) < \Delta_K^{m/2d} R(n) \le R(n)^{1+1/3m^2 2^m},$$

where for the last inequality, we use the fact $n > \tau(\epsilon)$. Moreover,

$$R(n)^{d_1+\cdots+d_m} \ge \lambda_1(n) \cdots \lambda_m(n) R(n)^{-1/4m^2 2^m} > R(n)^{1-1/3m^2 2^m},$$

where for the last inequality, we also use that $n > \tau(\epsilon)$. Combining the two relations above completes the proof of the lemma. □

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

Now, we consider the vectors $v_1, \dots, v_m$ from Lemma 4.5. We recall that $l'_{iv} = l_{k_v(i),v}$ for infinite places v and $l'_{iv} = l_{iv}$ if v is a finite place in S. Lemma 4.3, (4-5) and (4-6) thus yield that

$$|l'_{iv}(v_j)|_v \le ((2d|\Delta_K|^{1/2})^{2m})^{s(v)} R(n)^{e_{iv}+s(v)(d_{\min(i,j)}+1/c(m))}. \tag{4-7}$$

Let t be the largest integer such that $\lambda_t \le 1$ and V, the K-vector space generated by $\psi^{-1}(\lambda_t \Pi(n))$. We have $X_n \in V$ since $\lambda_{t+1} > 1$ and, moreover, $\{v_1, \dots, v_t\}$ is a basis of V. We also have $R(n)^{d_t} \le \lambda_t \le 1$, which implies $d_t \le 0$. Hence, it follows that X_n lies in the K-vector space generated by $v_1, \dots, v_r$, where r is the largest integer such that $d_r \le 0$.

Next, we construct a tuple $(N, \overline{\gamma}, \mathcal{L})$ depending only on $\mathcal{T}$. Since the number of possibilities for $\mathcal{T}$ is at most C', it follows that the number of such tuples is at most

$$(64em^4 2^m \epsilon^{-1})^{ms+m}.$$

Let $(l'_{iv}; e_{iv}; d_i) \in \mathcal{T}$. There is an integer k with

$$1 \le k \le m-1, \quad d_{k+1} > 0, \quad d_{k+1} - d_k \ge \frac{1}{m^2}. \tag{4-8}$$

Namely, let r be the largest integer such that $d_r \le 0$. By taking k as the smallest index of the set $\{d_r, d_{r+1}, \dots, d_m\}$ such that $d_{k+1} - d_k$ is maximal, we obtain the relation (4-8) as follows. By Lemma 4.5, it is immediate that

$$d_m \ge \frac{1}{m} - \frac{1}{3m^3 2^m}. \tag{4-9}$$

Therefore,

$$d_{k+1} - d_k \ge \frac{1}{m-r}(d_m - d_r) \ge \frac{1}{m-r}\left(\frac{1}{m} - \frac{1}{3m^3 2^m}\right) \ge \frac{1}{m^2},$$

where for the second inequality, we use (4-9). Put $N = \binom{m}{k}$. As before, let $\sigma_1, \dots, \sigma_N$ be the sequence of subsets of $\{1, \dots, m\}$ of cardinality $m-k$, ordered lexicographically. We define the set of linear forms $\mathcal{L} = \{\bar{l}_{iv} : v \in S, i = 1, \dots, N\}$ with

$$\bar{l}_{iv} := \alpha_{iv} l'_{i_1 v} \wedge \cdots \wedge l'_{i_{m-k} v} \quad \text{for } v \in S, i = 1, \dots, N$$

such that $|\bar{l}_{iv}|_v = 1$. We also define

$$\overline{e}_{iv} = e_{i_1 v} + \cdots + e_{i_{m-k} v}, \quad \overline{d}_i = d_{i_1} + \cdots + d_{i_{m-k}}$$

and the tuple $\overline{\gamma} = (\gamma_{iv}, v \in S, i = 1, \dots, N)$ with

$$\gamma_{iv} = \frac{\epsilon}{3m}\left\{\frac{1}{c(m)s} + \overline{e}_{iv} + s(v)\left(\overline{d}_i + \frac{m}{c(m)}\right)\right\} \quad \text{for } v \mid \infty, i = 1, \dots, N-1$$

and

$$\gamma_{Nv} = \frac{\epsilon}{3m}\left\{\frac{1}{c(m)s} + \overline{e}_{Nv} + s(v)\left(\overline{d}_{N-1} + \frac{m}{c(m)}\right)\right\} \quad \text{for } v \mid \infty$$

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

and

$$\gamma_{iv} = \frac{\epsilon}{3m}\left\{\min\left(0, \frac{1}{c(m)s} + \overline{e}_{iv}\right)\right\} \quad \text{for } v \nmid \infty, i = 1, \dots, N,$$

where $c(m) = 4m^3 2^m$.

LEMMA 4.7. *We have $\gamma_{iv} \le s(v)$ for $v \in S, i = 1, \dots, N$ and*

$$\sum_{v \in S} \sum_{i=1}^{m} \gamma_{iv} \le -\frac{\epsilon}{9m^3}.$$

PROOF. For any finite place $v \in S$, we have $\gamma_{iv} \le 0 \le s(v)$. Let v be an infinite place over K. By Lemma 4.6, it follows that

$$\overline{e}_{iv} = e_{i_1 v} + \cdots + e_{i_{m-k} v} \le (m-k)\frac{12s(v)}{10\epsilon} \le \frac{12ms(v)}{10\epsilon}$$

and

$$\overline{d}_i = d_{i_1} + \cdots + d_{i_{m-k}} \le d_1 + \cdots + d_m - kd_1 \le 1 + \frac{1}{3m^2 2^m} + \frac{12m}{10\epsilon}.$$

By inserting these relations in the expression of γ_{iv},

$$\begin{aligned}\gamma_{iv} &\le \frac{\epsilon}{3m}\left\{\frac{2+m}{c(m)} + \frac{24m}{10\epsilon} + 1 + \frac{1}{3m^2 2^m}\right\} s(v) \\ &\le \frac{\epsilon}{3}\{3\epsilon + 2.5\} s(v) \\ &\le s(v),\end{aligned}$$

where we use the fact that $\epsilon < 1/12$. We know that $\overline{d}_N - \overline{d}_{N-1} = d_{k+1} - d_k$, so we obtain

$$\begin{aligned}\sum_{v \in S} \sum_{i=1}^{N} \gamma_{iv} &\le \frac{\epsilon}{3m}\left\{\sum_{v \in S} \sum_{i=1}^{N} \left(\frac{1}{c(m)s} + \overline{e}_{iv} + s(v)\left(\overline{d}_i + \frac{m}{c(m)}\right)\right) - (\overline{d}_N - \overline{d}_{N-1})\right\} \\ &\le \frac{\epsilon}{3m}\left\{\frac{N}{c(m)} + \sum_{v \in S} \sum_{i=1}^{N} \overline{e}_{iv} + \sum_{i=1}^{N} \overline{d}_i + \frac{Nm}{c(m)} - (d_{k+1} - d_k)\right\} \\ &\le \frac{\epsilon}{3m}\left\{\frac{(m+1)2^m}{4m^3 2^m} + \frac{7 \cdot 2^{m-1}}{12m^3 2^m} - \frac{1}{m^2}\right\} \\ &\le -\frac{\epsilon}{9m^3},\end{aligned}$$

where, for the third inequality, we have applied Lemma 4.6 and the equality

$$\sum_{v \in S} \sum_{i=1}^{N} \overline{e}_{iv} = \binom{m-1}{m-k-1} \sum_{v \in S} \sum_{i=1}^{N} e_{iv} \quad \text{and} \quad \sum_{i=1}^{N} \overline{d}_i = \binom{m-1}{m-k-1} \sum_{i=1}^{N} d_i.$$

Hence, the proof is complete. □

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

Put $Q = (R(n)^{1/\epsilon})^{3m}$. We have proved $(N, \overline{\gamma}, \mathcal{L}, Q)$ satisfies the relations (3-10) and (3-11) of Theorem 3.6. Hence, we complete the proof of that result by showing there is a vector space W for which (3-9) holds.

LEMMA 4.8. *Let $v_1, \dots, v_m$ be linearly independent vectors from Lemma 4.4 and W the K-vector space generated by $\{v_1, \dots, v_k\}$. Then,*

$$\dim_K W = k, \quad X_n \in W, \quad f_k(W) = V(N, \overline{\gamma}, \mathcal{L}, Q).$$

PROOF. It is clear that $\dim_K W = k$. We have proved previously that $X_n \in V$, where V is the vector space generated by $\{v_1, v_2, \dots, v_r\}$ with r the largest integer such that $d_r \leq 0$. By the construction of the integer k above, it is immediate that $k \geq r$, which implies $X_n \in W$. We now define the vector

$$\widehat{v_i} := \widehat{v}_{\sigma_i} = v_{i_1} \wedge \cdots \wedge v_{i_{m-k}} \quad \text{where } \sigma_i = \{i_1, \dots, i_{m-k}.\}$$

By definition, $f_k(W)$ has basis $\{\widehat{v_1}, \dots, \widehat{v_{N-1}}\}$. Furthermore, $f_k(W)$ is the K- vector space generated by $\Pi := \Pi(N, \overline{\gamma}, \mathcal{L}, Q)$. Namely, we first show that $\widehat{v_1}, \dots, \widehat{v_{N-1}} \in \Pi$ and that any vector $\widehat{v_0} \in \Pi$ is linearly dependent on $\widehat{v_1}, \dots, \widehat{v_{N-1}}$. This gives us what we want. Take $v \in S$ and $i, j \in \{1, \dots, N\}$. Suppose σ_i, σ_j given as above. By the Laplace rule,

$$|\bar{l}_{iv}(\widehat{v_j})|_v = |\alpha_{iv}|_v |\det(l'_{pv}(v_q))_{p \in \sigma_i, q \in \sigma_j}|_v.$$

From the definition of $\bar{l}_{iv}$ and Lemma 3.3 (apply with $H = 1$), Relation (3-7), we have $|\alpha_{iv}|_v = |l'_{i_1 v} \wedge \cdots \wedge l'_{i_{m-k} v}|_v^{-1} \leq m!$ and by taking the maximum over all permutations κ of σ_j, one gets from (4-7) and Leibniz's formula for the determinant that

$$|\bar{l}_{iv}(\widehat{v_j})|_v \leq m! \, (m!)^{s(v)} \max_{\kappa} \prod_{t=1}^{m-k} (2d\Delta_K^{1/2})^{2ms(v)} R(n)^{e_{i_t v} + s(v)(d_{\min(i_t, \kappa(j_t))} + 1/c(m))}.$$

This expression thus yields that

$$|\bar{l}_{iv}(\widehat{v_j})|_v \leq m! \, (m! \, (2d\Delta_K^{1/2})^{2m(m-k)})^{s(v)} R(n)^{\overline{e}_{iv} + s(v)(d_0 + m/c(m))},$$

where

$$d_0 = \max_{\kappa} \sum_{i=1}^{m-k} d_{\min(i_t, \kappa(j_t))} \leq \min\{\overline{d}_i, \overline{d}_j\}.$$

Using the fact that $n > \tau(\epsilon)$,

$$(m!^2 \, (2d\Delta_K^{1/2})^{2m(m-1)}) \leq |\alpha_1|^{n\epsilon/sc(m)} \leq R(n)^{1/sc(m)}.$$

Hence,

$$|\bar{l}_{iv}(\widehat{v_j})|_v \leq R(n)^{\{1/c(m)s + \overline{e}_{iv} + s(v)(\overline{d}_{\min(i,j)} + m/c(m))\}} \tag{4-10}$$

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

for $v \in S$; $(i,j) \in \{1,\dots,N\} \times \{1,\dots,N-1\}$. Together with the fact that $Q = (R(n)^{1/\epsilon})^{3m}$ and the definition of γ_{iv}, we deduce that for infinite place v and $i = 1,\dots,N$; $j = 1,\dots,N-1$, one gets

$$|\bar{l}_{iv}(\widehat{v_j})|_v \le Q^{\gamma_{iv}}.$$

We know that $v_1,\dots,v_m \in O_K^m$ and so $\widehat{v_j} \in O_K^N$. For every finite place v and $j = 1,\dots,N-1$, one deduces from Lemma 3.2, Relation (3-3) that

$$|\bar{l}_{iv}(\widehat{v_j})|_v \le |\bar{l}_{iv}|_v |\widehat{v_j}|_v \le 1.$$

Together with (4-10),

$$|\bar{l}_{iv}(\widehat{v_j})|_v \le R(n)^{\min\{0,1/c(m)s+\bar{e}_{Nv}\}} \le Q^{\gamma_{iv}}.$$

Therefore, $\widehat{v_1},\dots,\widehat{v_{N-1}} \in \Pi$. Take $\widehat{v_0} \in \Pi$. We show that $\widehat{v_0}$ is linearly dependent of $\widehat{v_1},\dots,\widehat{v_{N-1}}$. Fix $v \in S$. Then,

$$\det(\widehat{v_0},\dots,\widehat{v_{N-1}}) = \det(\bar{l}_{1v},\dots,\bar{l}_{Nv})^{-1} \det(\bar{l}_{iv}(\widehat{v_j}))_{i,j}. \tag{4-11}$$

Put $f_v = \det(\bar{l}_{iv}(\widehat{v_j}))_{i,j}$. By the relations (3-4), (3-6) and (3-8),

$$|\det(\bar{l}_{1v},\dots,\bar{l}_{Nv})^{-1}|_v \le N!\,(m-1)!^N\,(m-1)!^{Ns(v)}. \tag{4-12}$$

Since $v_0,\dots,v_{N-1} \in \Pi$, it follows that

$$\begin{aligned} |f_v|_v &\le (N!\,)^{s(v)} \max_{\kappa} |\bar{l}_{1v}(\widehat{v}_{\kappa(0)}) \cdots \bar{l}_{Nv}(\widehat{v}_{\kappa(N-1)})|_v \\ &\le (N!\,)^{s(v)} \cdot Q^{\gamma_{1v}+\cdots+\gamma_{Nv}}. \end{aligned} \tag{4-13}$$

By combining (4-11), (4-12) and (4-13),

$$|\det(\widehat{v_0},\dots,\widehat{v_{N-1}})|_v \le N!\,(m-1)!^{N(1+s(v))}\,(N!\,)^{s(v)} \cdot Q^{\gamma_{1v}+\cdots+\gamma_{Nv}}.$$

By taking the product over $v \in S$ and using Lemma 3.5,

$$\begin{aligned} \prod_{v\in S} |\det(\widehat{v_0},\dots,\widehat{v_{N-1}})|_v &\le (N!\,(m!\,))^{N)^{(s+1)}} \cdot Q^{\sum_{i,v}\gamma_{iv}} \\ &\le ((2^m)!\,(m!\,))^{2^m})^{(s+1)} \cdot Q^{-\epsilon/9m^3} \\ &< 1, \end{aligned}$$

where for the last inequality, we use the relation (3-11) and the fact that $n > \tau(\epsilon)$. Since $\widehat{v_j} \in O_S^N$ for $j = 0,\dots,N-1$, $\det(\widehat{v_0},\dots,\widehat{v_{N-1}})$ is an S-integer. By the product formula, we deduce that

$$\det(\widehat{v_0},\dots,\widehat{v_{N-1}}) = 0,$$

and hence the proof is completed. □

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

5. Proof of Theorem 3.7

Let l, N be the integers ≥ 2. For $h = 1, \ldots, l$ denote by x_h the block of N variables $(x_{h1}, \ldots, x_{hN})$ and $\overline{\mathbb{Q}}[x_1, \ldots, x_l]$, the ring of polynomials in the lN variables $x_{11}, \ldots, x_{lN}$ with coefficients from $\overline{\mathbb{Q}}$. We use $\mathbf{i}$ to denote a tuple of nonnegative integers $i_{hj} : h = 1, \ldots, l; j = 1, \ldots, N$. For such a tuple $\mathbf{i}$, we define the partial derivative of $F \in \overline{\mathbb{Q}}[x_1, \ldots, x_l]$ by

$$F_{\mathbf{i}} = \prod_{h=1}^{l} \prod_{j=1}^{N} \left(\frac{1}{i_{hj}!} \frac{\partial^{i_{hj}}}{\partial x_{hj}^{i_{hj}}} F \right).$$

Let $d := (d_1, \ldots, d_l)$ be a tuple of positive integers and for a tuple $\mathbf{i}$ as above, put

$$(\mathbf{i}/d) = \sum_{h=1}^{l} \frac{1}{d_h} \{i_{h1} + \cdots + i_{hN}\}.$$

Assume $F \neq 0$. Then, the index of F at x with respect to d, denoted by $\mathrm{Ind}_{x,d}(F)$ is defined as the largest number σ such that $F_{\mathbf{i}}(x) = 0$ for all $\mathbf{i}$ with $(\mathbf{i}/d) \leq \sigma$. It is well known (see for example, [13, Lemma 4A, Ch. VI]) that

$$\mathrm{Ind}_{x,d}(FG) = \mathrm{Ind}_{x,d}(F) + \mathrm{Ind}_{x,d}(G) \quad \text{for } F, G \in \overline{\mathbb{Q}}[X_1, \ldots, X_l].$$

We say that F is homogeneous of degree d_h in x_h for $h = 1, \ldots, l$ if F is a linear combination of monomials

$$\prod_{h=1}^{l} \prod_{j=1}^{N} X_{hj}^{i_{hj}} \quad \text{with} \quad \sum_{j=1}^{l} i_{hj} = d_h, h = 1, \ldots, l.$$

Let $\Gamma(d)$ be the set of homogeneous polynomials of degree d_h in x_h for $h = 1, \ldots, l$ with coefficients in $\mathbb{Z}$ whose gcd is 1. We define the height $H(F) := H(a_F)$, where a_F is the vector of coefficients of F. Let L be a number field, v a place on L and $F \in L[x_1, \ldots, x_l]$. Put $|F|_v = |a_F|_v$ and

$$H(F) = \prod_{v \in M_L} |F|_v.$$

From the inequality $\binom{a}{b} \leq 2^a$ with $a \geq b$ integers, it is easy to see that

$$|F_{\mathbf{i}}|_v \leq 2^{(d_1 + \cdots + d_l)s(v)} |F|_v. \tag{5-1}$$

By taking the product over all places,

$$H(F_{\mathbf{i}}) \leq 2^{(d_1 + \cdots + d_l)} H(F).$$

The height of an $(N-1)$-dimensional linear subspace of L^N defined by

$$V = \{x \in L^N : a_1 x_1 + \cdots + a_N x_N = 0\},$$

with $a = (a_1, \ldots, a_N) \in L^N \setminus \{0\}$, is given by $H(V) = H(a)$.

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

We need the next lemma, which is a generalisation of Roth's lemma owing to Evertse [5, Lemma 24] and based on Schmidt's result [13, Theorem 10B, Ch. VI].

LEMMA 5.1. *Let $d = (d_1, \dots, d_l)$ be a tuple of positive integers and $0 < \theta \le 1$. Suppose that*

$$\frac{d_h}{d_{h+1}} \ge \frac{2l^2}{\theta} \quad \text{with } h = 1, \dots, l-1.$$

Further, let F be a nonzero polynomial in $\Gamma(d)$ and $V_1, \dots, V_l$ be $(N-1)$-dimensional linear subspaces of L^N with

$$H(V_h)^{d_h} \ge \{e^{d_1+\cdots+d_l} H(F)\}^{(N-1)(3l^2/\theta)^l} \quad \text{with } h = 1, \dots, l.$$

Then, there is $x_h \in V_h$ for $h = 1, \dots, l$ such that for $x = (x_1, \dots, x_l)$, we have $\mathrm{Ind}_{x,d}(F) < l\theta$.

Let V be a $(N-1)$-dimensional linear subspace of L^N. A grid of size A in V is a set

$$\Gamma = \{g_1 x_1 + \cdots + g_{N-1} x_{N-1}; x_i \in \mathbb{Z}; |x_i| \le A\},$$

where $\{g_1, \dots, g_{N-1}\}$ is a basis of V.

The following lemma is proven in [5] (see Lemma 26) using Schmidt's argument in the proof of [13, Lemmas 8A and 8B, Ch. VI].

LEMMA 5.2. *Let $d = (d_1, \dots, d_l), F, V_1, \dots, V_l, \theta$ be defined as in Lemma 5.1 and satisfy the condition of Lemma 5.1. For $h = 1, \dots, l$, let Γ_h be any grid in V_h of size N/θ. Then, there are $x_1 \in \Gamma_1, \dots, x_l \in \Gamma_l$ such that for $x = (x_1, \dots, x_l)$, we have $\mathrm{Ind}_{x,d}(F) < 2l\theta$.*

In what follows, we fix $(N, \overline{\gamma}, \mathcal{L})$ as defined in the proof of Theorem 3.6 meaning that $N = \binom{m}{t}$ with some integer t and m is the number of distinct roots of the characteristic polynomial P of $\{u_n\}$. Here, K is the splitting field of P and S is the set containing all the prime ideals above α_i, and all the Archimedean places over K and s is its cardinality, $\overline{\gamma} = (\gamma_{iv} : v \in S; i = 1, \dots N)$ with $\gamma_{iv} \le s(v)$, by setting $\delta = \epsilon/9m^3$,

$$\sum_{i,v} \gamma_{iv} \le -\delta \quad \text{and} \quad \mathcal{L} = \{\bar{l}_{iv} : v \in S; i = 1, \dots N\}$$

with $|\bar{l}_{iv}|_v = 1$. We keep fixed $\Pi(Q)$, $V(Q)$ for $\Pi(N, \overline{\gamma}, \mathcal{L}, Q)$, $V(N, \overline{\gamma}, \mathcal{L}, Q)$, respectively. We assume that Q satisfies Relation (3-12). The next auxiliary result gives the relation between Q and the height $H(V(Q))$ of $V(Q)$.

LEMMA 5.3. *There is an $(N-1)$-dimensional linear subspace of K^N with the following property. For every Q, with (3-12),*

$$V(Q) = V \quad \text{or} \quad H(V(Q)) \ge Q^{\delta/3s},$$

where V is a fixed $(N-1)$-dimensional linear subspace of K^N.

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

PROOF. Fix Q as in the relation (3-12). The first expression in Relation (3-12) implies there are linearly independent vectors $g_1, \dots, g_{N-1} \in \Pi(Q)$. Now put $g^* = (g_1 \wedge \cdots \wedge g_{N-1})^*$, where x^* is defined previously. It is clear from (3-2) that

$$V(Q) = \{x \in K^N : g^* \cdot x = 0\}.$$

Define the linear forms

$$l^*_{kv} = (\bar{l}_{1v} \wedge \cdots \wedge \bar{l}_{k-1v} \wedge \bar{l}_{k+1v} \wedge \cdots \wedge \bar{l}_{Nv})^*$$

and put

$$D_{kv} = l^*_{kv}(g^*) \quad \text{for } v \in S; i = 1, \dots N.$$

By Laplace Identity (3-1),

$$D_{kv} = \det(\bar{l}_{iv}(g_j)_{i \neq k, j})$$

and using the fact that $g_1, \dots, g_{N-1} \in \Pi(Q)$, one deduces $|l_{iv}(g_j)|_v \leq Q^{\gamma_{iv}}$ for $v \in S$, $i = 1, \dots, N; j = 1, \dots, N-1$. Therefore, it follows from the Leibniz formula for the determinant that

$$|D_{kv}|_v \leq (N-1)!^{s(v)} Q^{\gamma_{1v} + \cdots + \gamma_{Nv} - \gamma_{kv}}.$$

Suppose, for the moment, there is a tuple $(i_v, v \in S)$ with $i_v \in \{1, \dots, N\}$ such that

$$D_{i_v, v} \neq 0 \tag{5-2}$$

and

$$\sum_v \gamma_{i_v, v} \geq -\frac{\delta}{2}. \tag{5-3}$$

Then,

$$\prod_{v \in S} |D_{i_v, v}|_v \leq (N-1)!\, Q^{-\delta/2}. \tag{5-4}$$

Fix $v \in S$ and $k := i_v$. The product formula applied to D_{kv} and the relation (3-3) give

$$\begin{aligned}
1 &= \prod_{w \in M_K} |D_{kv}|_w \\
&\leq |D_{kv}|_v \Big(\prod_{w \neq v} N^{s(w)} |l^*_{kv}|_w |g^*|_w \Big) \\
&\leq \frac{|D_{kv}|_v}{N^{s(v)} |l^*_{kv}|_v |g^*|_v} \Big(\prod_{w \in M_K} N^{s(w)} |l^*_{kv}|_w |g^*|_w \Big) \\
&\leq \frac{|D_{kv}|_v}{N^{s(v)-1} |l^*_{kv}|_v |g^*|_v} H(l^*_{kv}) H(g^*).
\end{aligned}$$

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

This implies

$$|D_{kv}|_v \geq N^{s(v)-1}|l^*_{kv}|_v|g^*|_v H(l^*_{kv})^{-1}H(g^*)^{-1}.$$

Together with Lemma 3.3 (Relation (3-7) with $H = m!$),

$$|l^*_{kv}|_v \geq \frac{(m!)^{1-N}}{(N-1)!} \quad \text{and} \quad H(l^*_{kv}) \leq (N-1)!\,(m!)^{N-1}.$$

Hence,

$$\prod_{v\in S}|D_{kv}|_v \geq N(m!)^{-2s(N-1)}(N)!^{-2s}\,H(V(Q))^{-s}\cdot\prod_{v\in S}|g^*|_v.$$

By combining this with (5-4),

$$H(V(Q))^s \geq N(m!)^{-2s(N-1)}(N)!^{-2s-1}\,Q^{\delta/2} \geq Q^{\delta/3},$$

where, for the last inequality, we use the fact that $N = \binom{m}{t} \leq 2^m$ and

$$Q^{-\delta/6} < (9m!)^{-\delta e^{2C''}/6} < N(m!)^{-2s(N-1)}(N)!^{-1}$$

with C'' defined as in Theorem 3.7. Now, assume there is no tuple $(i_v, v \in S)$ with $i_v \in \{1,\dots,N\}$ satisfying (5-2) and (5-3). We show there is a fixed $(N-1)$-dimensional linear subspace V of K^N, independent of Q such that $V(Q) = V$. Fix $v \in S$. Put

$$I_v = \{i \in \{1,\dots,N\} : D_{iv} \neq 0\}.$$

We have

$$l^*_{kv}(g^*) = 0 \quad \text{for } v \in S, k \in \{1,\dots,N\}\setminus I_v.$$

If $I_v = \{1,\dots,N\}$ for each $v \in S$, take $h = (1,0,\dots,0)$ and we define

$$V := \{x \in K^N : x\cdot h = 0\}.$$

Otherwise, by the relation (3-6) and the definition of $\bar{l}_{iv}$ in the proof of Theorem 3.6,

$$H(l^*_{kv}) \leq (N-1)!\,(m!)^{N-1} \quad \text{for } v \in S, i = 1,\dots,N.$$

Since the rank of the set of such l^*_{kv} is less than $N-1$ (by the definition of g^*), it follows from Lemma 3.4 (with $H = (N-1)!\,(m!)^{N-1}$) that there exists a nonzero vector $h \in K^N$ such that

$$l^*_{kv}(h) = 0 \quad \text{for } v \in S, i \in \{1,\dots,N\}\setminus I_v \tag{5-5}$$

with

$$H(h) \leq (N-1)!^N\,(m!)^{(N-1)^2}.$$

Therefore,

$$V := \{x \in K^N : x\cdot h = 0\}.$$

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

We now want to show that $V(Q) = V$. Since $V(Q)$ is the vector space generated by $\Pi(Q)$, and $V(Q)$ and V have dimension $N-1$, it suffices to show that $\Pi(Q) \subset V$. Let $x \in \Pi(Q)$, we prove that $x \cdot h = 0$. Put $\Delta_v = \det(\bar{l}_{1v}, \ldots, \bar{l}_{Nv})$. Using the fact that $\{\bar{l}_{1v}, \ldots, \bar{l}_{Nv}\}$ is a basis of K^N,

$$x \cdot h = \Delta_v^{-1} \sum_{i=1}^{N} \bar{l}_{iv}(x) l_{iv}^*(h).$$

In view of the relation (5-5),

$$x \cdot h = \Delta_v^{-1} \sum_{i \in I_v} \bar{l}_{iv}(x) l_{iv}^*(h).$$

From Lemma 3.4 (the relation (3-8) with $H = m!$),

$$|\Delta_v|_v^{-1} \leq N!\,(m!)^N \quad \text{for } v \in S.$$

Further, by applying the relations (3-3) and (3-4),

$$|l_{iv}^*(h)|_v \leq |\bar{l}_{1v} \wedge \cdots \wedge \bar{l}_{i-1v} \wedge \bar{l}_{i+1v} \wedge \cdots \wedge \bar{l}_{Nv}|_v \cdot |h|_v \leq (N!\,(m!)^{N-1})^{s(v)} |h|_v. \tag{5-6}$$

If there exists $v \in S$ such that $I_v = \emptyset$ then, using the above equality and (5-5), one deduces $x \cdots h = 0$; therefore, the desired result follows. Otherwise, for $v \in S$, choose $j_v \in I_v$ such that $\gamma_{j_v,v} = \max_{i \in I_v} \gamma_{iv}$. Since $x \in \Pi(Q)$, it follows that

$$|\bar{l}_{iv}(x)|_v \leq Q^{\gamma_{j_v,v}} \quad \text{for } v \in S, i \in I_v.$$

Combining this inequality with (5-5) and (5-6),

$$\begin{aligned} |x \cdot h|_v &= |\Delta_v|_v^{-1} \left| \sum_{i \in I_v} \bar{l}_{iv}(x) l_{iv}^*(h) \right|_v \\ &\leq N!\,(m!)^N \cdot N^{s(v)} \cdot (N!\,(m!)^{N-1})^{s(v)} |h|_v Q^{\gamma_{j_v,v}}. \end{aligned} \tag{5-7}$$

Further, $j_v \in I_v$ for $v \in S$, the tuple $(j_v, v \in S)$ satisfies (5-2), so by assumption, it does not satisfy (5-3). Together with (5-7) and the product formula, if $|x \cdot h|_v \neq 0$, then

$$\begin{aligned} 1 &= \prod_{v \in S} |x \cdot h|_v \cdot \prod_{v \notin S} |x \cdot h|_v \\ &\leq (m!)^{Ns} \cdot N!^s N \cdot (N!\,(m!)^{N-1}) Q^{-\delta/2} H(h) \\ &\leq (m!)^{Ns+(N-1)^2+(N-1)} \cdot (N!)^{N+s+1} Q^{-\delta/2}, \end{aligned}$$

which contradicts the fact that $Q > (9m!)^{e^{2C''}}$ by using the fact that $N \leq 2^m$. □

We need the following gap principle.

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

LEMMA 5.4. *Let a, b be reals with $b > a > (9m!)^{e^{2C''}}$. There is a collection of $(N-1)$-dimensional linear subspaces of K^N of cardinality at most*

$$C(a,b) = 1 + 4\delta^{-1} \log\left(\frac{\log b}{\log a}\right)$$

such that for every Q satisfying (3-12) and $a \le Q \le b$, the vector space $V(Q)$ belongs to this collection.

PROOF. The proof is similar to [5, Lemma 28] so it is left to the reader. □

Let us consider $F \in \Gamma(d)$. For each $v \in S$,

$$U_{hiv} = \bar{l}_{iv}(x_h) \quad h = 1, \dots, l; i = 1, \dots, N$$

and

$$F_{\mathbf{i}} = \sum_{\mathbf{j}} C(\mathbf{i}, \mathbf{j}, v) U_{11v}^{j_{11}} \cdots U_{lNv}^{j_{lN}},$$

where the sum is taken over tuples $\mathbf{j}$ with

$$\sum_{k=1}^{N} j_{hk} = d_h - \sum_{k=1}^{N} i_{hk} \quad \text{for } h = 1, \dots l.$$

LEMMA 5.5. *Let θ be real with $\theta < 1/N$. Assume $l > 4\theta^{-2} \log(2Ns)$ and $d = (d_1, \dots, d_l)$ for any l-tuple of positive integers. Then, there is a polynomial $F \in \Gamma(d)$ with the following properties:*

- *$H(F) \le (8^N m!)^{d_1 + \cdots + d_l}$;*
- *for all tuples $v \in S$ and all $\boldsymbol{i}, \boldsymbol{j}$ with $(\boldsymbol{i}/d) \le 2l\theta$,*

$$C(\boldsymbol{i}, \boldsymbol{j}, v) \ne 0 \quad \textit{implies} \ \max_k \left| \sum_{h=1}^{l} \frac{j_{hk}}{d_h} - \frac{l}{N} \right| < 3lN\theta;$$

- *for all tuples $\boldsymbol{i}$,*

$$\prod_{v \in S} \max_{\boldsymbol{j}} |C(\boldsymbol{i}, \boldsymbol{j}, v)|_v \le (2 \cdot 16^N m! N!^{s+1})^{d_1 + \cdots + d_l}.$$

PROOF. By definition of $\bar{l}_{iv}$, its coefficients are rational numbers. Let t be the maximal number of pairwise nonproportional linear forms among $\bar{l}_{iv} (v \in Si = 1, \dots, N)$. Then, $t \le Ns$ and we have $l > 4\theta^{-2} \log(2t)$. This is precisely the condition of [13, Theorem 6A, Ch. VI]. Then, there exists a polynomial $F \in \Gamma(d)$ whose coefficients have gcd 1. For $v \in S$ infinite, we have $|\bar{l}_{iv}|_v \le m!$. Together with [13, Theorem 6A, Ch. VI], the first point of our lemma is proven and by the polynomial theorem

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

[13, Theorem 7A, Ch. VI], we have our second point. Fix $v \in S$. Since the coefficients of F have gcd 1, it follows that

$$|F|_v = H(F)^{s(v)} \leq (8^N m!)^{s(v)(d_1+\cdots+d_l)}.$$

Together with (5-1),

$$|F_{\mathbf{i}}|_v \leq (2 \cdot 8^N m!)^{s(v)(d_1+\cdots+d_l)}. \tag{5-8}$$

Fix $v \in S$ and consider the matrix $A_v = (\bar{l}_{1v}, \ldots, \bar{l}_{Nv})$. By Cramer's rule,

$$x_{hi} = \sum_{k=1}^{N} \eta_{ik} U_{hkv} \quad \text{for } h = 1, \ldots, l; i = 1, \ldots, N, \tag{5-9}$$

where $\eta_{ik} = \pm\Delta_{ik}\Delta_v^{-1}$, with Δ_{ik} the determinant of the matrix obtained by removing the i th row and k th column of A_v and $\Delta_v = \det(A_v)$. Using the Leibniz formula for the determinant and the fact that $|\bar{l}_{iv}|_v = 1$, we obtain $|\Delta_{ij}|_v \leq (N-1)!^{s(v)}$ for $i, j = 1, \ldots, N$. Together with (3-8) and the fact that the coordinates of $\bar{l}_{iv}$ are 0 or 1,

$$|\eta_{ij}|_v \leq (N-1)!^{s(v)} \cdot N!\,.$$

Write

$$F_{\mathbf{i}}(x_1, \ldots, x_n) = \sum_{\mathbf{j}} P(\mathbf{i}, \mathbf{j}) x_{11}^{j_{11}} \cdots x_{lN}^{j_{lN}},$$

where $x_i = (x_{i1}, \ldots, x_{iN})$. By inserting (5-9),

$$F_{\mathbf{i}}(x_1, \ldots, x_n) = \sum_{\mathbf{j}} P(\mathbf{i}, \mathbf{j}) \prod_{h=1}^{l} \prod_{k=1}^{N} \Big(\sum_{j=1}^{N} \eta_{kj} U_{hjv} \Big)^{j_{hk}}.$$

Put

$$|F_{\mathbf{i}}|_v = \max_{\mathbf{j}} |P(\mathbf{i}, \mathbf{j})|_v.$$

We finally obtain

$$F_{\mathbf{i}}(x_1, \ldots, x_l) = \sum_{\mathbf{p}} C(\mathbf{i}, \mathbf{p}, v) U_{11v}^{p_{11}} \cdots U_{lNv}^{p_{lN}},$$

where, for v infinite places over K, $|\cdot|_v^{1/s(v)}$ is the usual absolute value and so

$$\begin{aligned} |C(\mathbf{i}, \mathbf{p}, v)|_v^{1/s(v)} &\leq \Big| \sum_{\mathbf{j}} P(\mathbf{i}, \mathbf{j}) \prod_{h=1}^{l} \prod_{k=1}^{N} \Big(\sum_{j=1}^{N} \eta_{kj} \Big)^{j_{hk}} \Big|_v^{1/s(v)} \\ &\leq \sum_{\mathbf{j}} |F_{\mathbf{i}}|_v^{1/s(v)} \Big| \prod_{h=1}^{l} \prod_{k=1}^{N} \Big(\sum_{j=1}^{N} \eta_{kj} \Big)^{j_{hk}} \Big|_v^{1/s(v)} \\ &\leq \sum_{\mathbf{j}} |F_{\mathbf{i}}|_v^{1/s(v)} \prod_{h=1}^{l} \prod_{k=1}^{N} (N!\, N!^{1/s(v)})^{j_{hk}}. \end{aligned}$$

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

Together with (5-8),

$$|C(\mathbf{i},\mathbf{p},v)|_v \le (2\cdot 16^N m!\,)^{s(v)(d_1+\cdots+d_l)}(N!^{s(v)}\,(N!\,))^{d_1+\cdots+d_l}$$

for any $v\in S$. By taking the product over $v\in S$, finally,

$$\begin{aligned}\prod_{v\in S}\max_{\mathbf{p}}|C(\mathbf{i},\mathbf{p},v)|_v &\le (2\cdot 16^N m!\,)^{d_1+\cdots+d_l}(N!^{s+1}\,)^{d_1+\cdots+d_l}\\ &\le (2\cdot 16^N m!\,N!^{s+1}\,)^{d_1+\cdots+d_l}.\end{aligned}$$

□

Now, we present the last step of the proof of Theorem 3.7. Let $(N,\overline{\gamma},\mathcal{L})$ be as constructed in the proof of Theorem 3.6 meaning that $N=\binom{m}{t}$ with some integer t and m is the number of distinct roots of the characteristic polynomial of $\{u_n\}$ and $\overline{\gamma}=(\gamma_{iv}: v\in S; i=1,\dots N)$ with $\gamma_{iv}\le s(v)$. By setting $\delta=\epsilon/9m^3$,

$$\sum_{i,v}\gamma_{iv}\le -\delta \quad\text{and}\quad \mathcal{L}=\{\bar{l}_{iv}: v\in S; i=1,\dots N\}$$

with $|\bar{l}_{iv}|_v=1$. Put $\theta=\delta/(30N^3)$ and we recall that $C''=2^{11m+34}s^2\epsilon^{-4}m^{12}$. Let l be the smallest integer such that $l>4\theta^{-2}\log(2Ns)$. So $l<5\theta^{-2}\log(2Ns)$. We assume that the number of subspaces $V(Q)$ satisfying (3-12) has cardinality $>C''$ and derive a contradiction from that. By denoting $\lfloor\cdot\rfloor$ as the floor function, this number of subspaces is more than

$$2+(l-1)t \quad\text{with } t=2+\lfloor 4\delta^{-1}\log(4l^2\theta^{-1})\rfloor$$

since, using the fact that $N<2^m$ and $l<5\theta^{-2}\log(2Ns)$, we get $2+(l-1)t<C''$. By Lemma 5.3, there are real $Q'_1,\dots,Q'_{1+(l-1)t}$ satisfying the relation (3-12) and

$$Q'_1<Q'_2<\cdots<Q'_{1+(l-1)t}$$

such that the spaces $V(Q'_1),\dots,V(Q'_{1+(l-1)t})$ are all distinct and different from V; therefore,

$$H(V(Q'_i))\ge Q_i'^{\delta/3s} \quad\text{for } i=1,\dots,1+t(l-1). \tag{5-10}$$

We set

$$Q_1=Q'_1,\quad Q_2=Q'_{1+t},\dots,Q_l=Q'_{1+t(l-1)}$$

and

$$V_h=V(Q_h)\quad\text{for } h=1,\dots,l.$$

There are t different spaces $V(Q)$ with $Q_h\le Q\le Q_{h+1}$. By applying Lemma 5.4 with $a=Q_{h+1}$ and $b=Q_h$,

$$Q_{h+1}\ge Q_h^{4l^2\theta^{-1}}\quad\text{for } h=1,\dots,l-1.$$

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

We define

$$d_1 = 1 + \left\lfloor \frac{\log Q_l}{\theta \log Q_1} \right\rfloor.$$

Let us define d_h as an integer satisfying the relation

$$d_1 \log Q_1 \le d_h \log Q_h < (1+\theta) d_1 \log Q_1 \quad \text{for } h = 2, \dots, l.$$

Furthermore, we may take

$$d_h = 1 + \left\lfloor \frac{d_1 \log Q_1}{\log Q_h} \right\rfloor \quad \text{for } h = 2, \dots, l.$$

In what follows, we denote by F the polynomial obtained from Lemma 5.5. In view of the definition of d_h,

$$\frac{d_h}{d_{h+1}} = \frac{d_h \log Q_h}{d_{h+1} \log Q_{h+1}} \cdot \frac{\log Q_{h+1}}{\log Q_h} \ge \frac{2l^2}{\theta}.$$

From (5-10),

$$H(V(Q_h))^{d_h} \ge Q_h^{d_h \delta/3s} > Q_1^{d_1 \delta/3s}.$$

However, $d_1 + \cdots + d_l \le l d_1$ and by Lemma 5.5,

$$\begin{aligned} \{e^{d_1+\cdots+d_l} H(F)\}^{(N-1)(3l^2/\theta)^l} &\le (e^{ld_1}(8^N m!)^{ld_1})^{(N-1)(3l^2/\theta)^l} \\ &\le (e 8^N m!)^{ld_1 (N-1)(3l^2/\theta)^l} \\ &\le (9m!)^{ld_1 N^2 (3l^2/\theta)^l} \\ &< Q_1^{d_1 \delta/3s}, \end{aligned}$$

where for the last inequality, we use the fact that $l < 5\theta^{-2}\log(2Ns)$ and $Q_1 > (9m!)^{e^{c''}}$. Therefore,

$$H(V_h)^{d_h} \ge \{e^{d_1+\cdots+d_l} H(F)\}^{(N-1)(3l^2/\theta)^l}$$

and so $l, N, \theta, d_1, \dots, d_l, F, V_1, \dots, V_l$ satisfy the conditions of Lemma 5.1. For $h = 1, \dots, l$, choose a linearly independent set of vectors

$$\{g_{h1}, \dots, g_{hN-1}\}$$

from $\Pi(Q_h)$ and let Γ_h be the grid of size $N\theta^{-1}$, given by

$$\Gamma_h = \{g_{h1}u_1 + \cdots + g_{hN-1}u_{N-1}, u_i \in \mathbb{Z}; |u_i| \le N\theta^{-1}\}.$$

By Lemma 5.2, there are $x_1 \in \Gamma_1, \dots, x_l \in \Gamma_l$ such that for $x = (x_1, \dots, x_l)$, we have $\mathrm{Ind}_{x,d}(F) < 2l\theta$ and so

$$f := F_{\mathbf{i}}(x_1, \dots, x_l) \ne 0$$

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

for some $\mathbf{i} = (i_{11}, \dots, i_{lN})$ with $(\mathbf{i}/d) = \mathrm{Ind}_{x,d}(F)$. Since $g_{hj} \in \Pi(Q_h)$, it follows that $x_h \in \mathcal{O}_S^N$ for $h = 1, \dots, N$. Further, the fact that $F_{\mathbf{i}}$ has coefficients in $\mathbb{Z}$, combined with the product formula yields $\prod_{v\in S} |f|_v \geq 1$. Now, we show that $\prod_{v\in S} |f|_v < 1$. Fix $v \in S$. Put $u_{hiv} = \bar{l}_{iv}(x_h)$ for $h = 1, \dots, l; i = 1, \dots, N$. Since $x_h \in \Gamma_h$ is of size $N\theta^{-1}$,

$$|u_{hiv}|_v \leq (N^2\theta^{-1})^{s(v)} Q_h^{\gamma_{iv}} \quad \text{for } h = 1, \dots, l; i = 1, \dots, N. \tag{5-11}$$

From Lemma 5.5, we may write

$$f = \sum_{\mathbf{j}} C(\mathbf{i}, \mathbf{j}, v) u_{11v}^{p_{11}} \cdots u_{lNv}^{j_{lN}} \tag{5-12}$$

with

$$\max_k \left| \sum_{h=1}^{l} \frac{j_{hk}}{d_h} - \frac{l}{N} \right| < 3lN\theta \quad \text{for } C(\mathbf{i}, \mathbf{j}, v) \neq 0. \tag{5-13}$$

The number of possibilities of $\mathbf{j}$ is at most $(2^N)^{(d_1+\cdots+d_l)}$ and we know that $d_1 + \cdots + d_l \leq ld_1$. Thus,

$$\begin{aligned} |f|_v &\leq (2^N)^{s(v)(d_1+\cdots+d_l)} \cdot \max_{\mathbf{j}} |C(\mathbf{i}, \mathbf{j}, v)|_v |u_{11v}|_v^{j_{11}} \cdots |u_{lNv}|_v^{j_{lN}} \\ &\leq (2^N N^2 \theta^{-1})^{s(v)(d_1+\cdots+d_l)} \max_{\mathbf{j}} |C(\mathbf{i}, \mathbf{j}, v)|_v Q_1^{\sum_k \gamma_{kv} j_{1k}} \cdots Q_l^{\sum_k \gamma_{kv} j_{lk}}, \end{aligned}$$

where for the last inequality, we use (5-11). However,

$$\begin{aligned} &\sum_{h,k} \gamma_{kv} \frac{j_{hk}}{d_h} \cdot \frac{d_h \log Q_h}{d_1 \log Q_1} \\ &\quad \leq \sum_{k=1}^{N} [(\gamma_{kv} - s(v)) + s(v)] \sum_{h=1}^{l} \frac{j_{hk}}{d_h} \cdot \frac{d_h \log Q_h}{d_1 \log Q_1} \\ &\quad \leq (1+\theta)\left(\sum_{k=1}^{N} l\gamma_{kv} \left(\frac{1}{N} - 3N\theta \right) + s(v)(3lN^2\theta - l) + s(v)(l + 3lN^2\theta) \right) \\ &\quad \leq l(1+\theta)\left(6N^2\theta s(v) + \left(\frac{1}{N} - 3N\theta \right) \sum_{k=1}^{N} \gamma_{kv} \right). \end{aligned}$$

From the equality

$$Q_h^{\sum_k \gamma_{kv} j_{hk}} = Q_1^{(\log Q_h / \log Q_1) \sum_k \gamma_{kv} j_{hk}}$$

together with (5-12) and (5-13), it follows that

$$\begin{aligned} \prod_{v\in S} |f|_v &\leq (2^N N^2 \theta^{-1})^{(ld_1)} \prod_{v\in S} \max_{\mathbf{j}} |C(i, \mathbf{j}, v)|_v \cdot Q_1^{ld_1(1+\theta)(6N^2\theta + (1/N - 3N\theta)\sum_{k,v} \gamma_{kv})} \\ &\leq (2^{5N+1} N^2 \theta^{-1} m! \, (N!)^{s+1})^{ld_1} \cdot Q_1^{1.25ld_1((\delta^2/10N^2) + (\delta/5N) - (\delta/N))} \end{aligned}$$

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press

$$\leq (2^{5N+1}N^2\theta^{-1}m!\,(N!\,)^{s+1})^{ld_1} \cdot Q_1^{-3ld_1\delta/5N}$$
$$\leq (2^{5N+1}N^2\theta^{-1}m!\,(N!\,)^{s+1} \cdot Q_1^{-3\delta/5N})^{ld_1}$$
$$< 1,$$

where for the last inequality, we use the fact that $N \leq 2^m$ and $Q_1 > (9m!\,)^{e^{2C''}}$, $\theta = \delta/30N^3$ and $\sum_{k,v} \gamma_{kv} \leq -\delta$, with $\delta = \epsilon/9m^3$ and $C'' = 2^{11m+34}s^2\epsilon^{-4}m^{12}$. Hence, the proof of Theorem 3.7 is complete.

Acknowledgements

The author would like to thank Prof. Pieter Moree for his valuable suggestions on the first version of this paper, as well as Prof. Yann Bugeaud for the clarification on how to refine the bounds of Theorems 1.1 and 1.4 in the previous draft.

References

[1] E. Bombieri and W. Gubler, *Heights in Diophantine Geometry* (Cambridge University Press, New York, 2006).
[2] E. Bombieri and J. Vaaler, 'On Siegel's lemma', *Invent. Math.* **73** (1983), 11–32.
[3] B. Edixhoven and J.-H. Evertse, *Diophantine Approximation and Abelian Varieties*, Lecture Notes in Mathematics, 1566 (Springer Verlag, Berlin, 1993).
[4] J.-H. Evertse, 'On sums of S-units and linear recurrences', *Compos. Math.* **53**(2) (1984), 225–244.
[5] J.-H. Evertse, 'An improvement of the quantitative subspace theorem', *Compos. Math.* **101**(3) (1996), 225–311.
[6] C. Fuchs and S. Heintze, 'On the growth of linear recurrences in function fields', *Bull. Aust. Math. Soc.* **104**(1) (2021), 11–20.
[7] T. Karimov, E. Kelmendi, J. Nieuwveld, J. Ouaknine and J. Worrell, 'The power of positivity', *Proc. LICS* **23** (2023), 1–11.
[8] B. H. Loxton and A. J. Van Poorten, 'On the growth of linear recurrences', *Math. Proc. Cambridge Philos. Soc.* **81** (1977), 369.
[9] F. Luca and A. Noubissie, 'Linear combinations of factorial and S-unit in a ternary recurrence sequence with a double root', *Period. Math. Hungar.* **86** (2023), 422–441.
[10] J. Ouaknine and J. Worrell, 'Ultimate positivity problem is decidable for simple linear recurrence sequences', in: *Proceedings of the ICALP*, Lecture Notes in Computer Science LNCS, 8573 (eds. J. Esparza, P. Fraigniaud, T. Husfeldt and E. Koutsoupias) (Springer Verlag, Berlin, 2014).
[11] H. P. Schlickewei and W. M. Schmidt, 'The number of solutions of polynomials-exponential equations', *Compos. Math.* **120** (2000), 193–225.
[12] H. P. Schlickewei and A. J. Van der Poorten, 'Zeros of linear recurrence sequences', *Bull. Aust. Math. Soc.* **44** (1991), 215–223.
[13] W. M. Schmidt, *Diophantine Approximation*, Lecture Notes in Mathematics, 785 (Springer Verlag, Berlin, 1980).
[14] W. M. Schmidt, 'The zero multiplicity of linear recurrence sequences', *Acta Math.* **182** (1999), 243–282.

ARMAND NOUBISSIE, Graz University of Technology,
Institute for Analysis and Number Theory, Münzgrabenstrasse 36/II,
8010 Graz, Austria
e-mail: armand.noubissie@tugraz.at

https://doi.org/10.1017/S1446788725101171 Published online by Cambridge University Press