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On invariant distributions and mixing

YVES COUDENE

IRMAR, Université Rennes 1, campus beaulieu, bâtiment 23, 35042 Rennes cedex, France
(e-mail: yves.coudene@univ-rennes1.fr)

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Abstract. We show that any probability preserving transformation of a metric space is mixing as soon as there are no non-constant L^2-functions which are invariant under both the stable and unstable distributions. This generalizes a result of Babillot.

1. *Introduction*

The original proof of mixing of Anosov diffeomorphisms and flows, given by Anosov and Sinai [**A69**], relied on the ergodicity of the strong stable foliation of the system. This ergodicity, together with the existence of a measurable partition subordinated to the foliation, allowed them to show that Anosov systems are in fact K-mixing, as soon as they are topologically mixing.

The joint ergodicity of the stable and unstable distributions of any C^2 diffeomorphism with non-zero exponents is, in fact, sufficient to obtain K-mixing. This surprisingly general result, due to Ledrappier and Young [**LY85**] relies on Pesin theory to bypass the lack of hyperbolicity.

It has been remarked recently that, if we are only interested in strong mixing, then a short proof can be obtained, which does not make use of entropy and Pesin theory. The elegant argument of Babillot [**B02**] relies on a trick from spectral theory, which connects weak accumulation points of the positive iterates of a function with accumulation points for its negative iterates.

We propose an elementary proof of a generalization of a result of Babillot, which does not use spectral theory. The proof is modelled on the famous Hopf argument [**H71**], which originally dealt with geodesic flows on negatively curved manifolds.

In order to make the proof of the mixing property more transparent, we first recall the Hopf argument, in a generality which may not have been known before. Here are some motivations for such a generality: billiards maps are examples of transformations which are not smooth; still they retain sufficient hyperbolicity to carry out the argument of Babillot, as was pointed out by Pène. Another example is given by hyperbolic systems on infinite-dimensional manifolds. Here, the phase space is not locally compact, still the Hopf argument can be applied. The simplest example of such a system is the geodesic flow on the constant negatively curved Hilbert ball.

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2. *The Hopf argument*

We consider a metric space X, endowed with a Borel probability measure μ. Let $T : X \to X$ be a measure-preserving measurable transformation.

Definition 1. The stable distribution of T is defined by

$$W^{ss}(x) = \{y \in X \mid d(T^n x, T^n y) \to 0 \text{ as } n \to +\infty\}.$$

A measurable function $g : X \to \mathbb{R}$ is W^{ss}-invariant if there is a set $\Omega \subset X$ of full measure, such that for all $x, y \in \Omega$, $y \in W^{ss}(x)$ implies that $g(y) = g(x)$.

If T is invertible, we may also define the *unstable distribution* $W^{su}(x)$ of a point $x \in X$. This is just the stable distribution of x with respect to the transformation T^{-1}.

Finally, recall first that bounded Lipschitz functions are dense in $L^2(X, \mu)$; this follows from the exterior regularity of μ. Second, that any L^2 convergent sequence admits a subsequence that converges almost everywhere.

THEOREM 1. *Let X be a metric space, μ a Borel probability measure on X, and $T : X \to X$ a measure-preserving transformation of X. Then any T-invariant L^2-function is W^{ss}-invariant.*

Proof. We denote by $\mathcal{I}$ the σ-algebra of T-invariant measurable sets; let us apply the L^2 ergodic theorem: for each bounded Lipschitz function $f : X \to \mathbb{R}$, with Lipschitz constant C, there exists a set $\Omega_f \subset X$ of full measure, and integers $n_i \to \infty$ such that

$$\frac{1}{n_i} \sum_{k=0}^{n_i - 1} f(T^k(x)) \xrightarrow[i \to +\infty]{} E(f \mid \mathcal{I})(x) \quad \text{for all } x \in \Omega_f.$$

Let $y \in \Omega_f \cap W^{ss}(x)$. We have

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) - \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(y)) \right| \leq \frac{1}{n} C \sum_{k=0}^{n-1} d(T^k x, T^k y) \xrightarrow[n \to +\infty]{} 0.$$

This implies that $E(f \mid \mathcal{I})(x) = E(f \mid \mathcal{I})(y)$.

Let f be any T-invariant L^2-function. There exists a sequence f_n of bounded Lipschitz functions such that $f_n \to f$ in L^2-norm. This implies that $E(f_n \mid \mathcal{I}) \to E(f \mid \mathcal{I}) = f$ in L^2-norm and, passing to a subsequence n_ℓ, for all x in a set Ω_0 of full measure. So, for all points $x, y \in \Omega_0 \cap \Omega_{f_{n_\ell}}$, and $y \in W^{ss}(x)$,

$$f(x) = \lim E(f_{n_\ell} \mid \mathcal{I})(x) = \lim E(f_{n_\ell} \mid \mathcal{I})(y) = f(y).$$

This ends the proof of the theorem. □

COROLLARY 1. *Let X be a metric space, μ a Borel probability measure on X and $T : X \to X$ an invertible measure-preserving transformation of X. Assume that any function which is W^{ss}-invariant, W^{su}-invariant and T-invariant is constant almost everywhere. Then T is ergodic.*

The proof of the corollary is straightforward: Theorem 1, applied to T and T^{-1}, shows that any T-invariant function is W^{ss} and W^{su}-invariant.

As a final remark, we note that we did not need any continuity assumption on T, regularity property on W^{ss} or separability hypothesis on X.

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3. *Mixing*

The proof of the mixing property is based on the following two results concerning Hilbert spaces, cf. Riesz and Sz.-Nagy [**RN90**] for short elementary proofs.

- *Banach–Alaoglu*. The unit ball of a Hilbert space is weakly sequentially compact.
- *Banach–Saks*. If $f_n \to f$ weakly, then there is a subsequence n_k such that we have the convergence $(1/n)\sum_{k=0}^{n-1} f_{n_k} \to f$ in L^2-norm.

THEOREM 2. *Let X be a metric space, μ a Borel probability measure on X and $T : X \to X$ a measure preserving transformation of X and $f \in L^2(X, \mu)$. Then any weak limit of a subsequence of $f \circ T^n$ is W^{ss}-invariant.*

Proof. Let g be a weak limit of $f \circ T^{n_i}$. First assume that f is bounded and Lipschitz. The Banach–Saks lemma gives subsequences m_ℓ, n_{i_k} such that

$$\Psi_\ell(x) := \frac{1}{m_\ell}\sum_{k=1}^{m_\ell} f \circ T^{n_{i_k}} \underset{\text{a.e.}}{\longrightarrow} g.$$

If $y \in W^{ss}(x)$,

$$|\Psi_\ell(x) - \Psi_\ell(y)| \le C\frac{1}{m_\ell}\sum_{k=1}^{m_\ell} d(T^{n_{i_k}}(x), T^{n_{i_k}}(y)) \xrightarrow[\ell\to\infty]{} 0.$$

Hence, g is W^{ss}-invariant.

Let $f \in L^2$. For all $\varepsilon > 0$, there exists f_0 bounded and Lipschitz such that $\|f - f_0\| < \varepsilon$. On passing to a subsequence, we may assume that the sequence $f_0 \circ T^{n_i}$ weakly converges to some function g_0, which is W^{ss}-invariant. This gives $(f - f_0) \circ T^{n_i} \to g - g_0$ weakly, which, in turn, implies that

$$\|g - g_0\| \le \underline{\lim}\|(f - f_0) \circ T^{n_i}\| \le \|f - f_0\| < \varepsilon.$$

So there exists a sequence of W^{ss}-invariant functions that converges to g in L^2-norm and, taking a subsequence, almost everywhere. This shows that g is W^{ss}-invariant. □

Recall that the stable distribution W^{ss} is *ergodic* if all W^{ss}-invariant functions are constant almost everywhere. On the other hand, T is *strongly mixing* if the sequence $f \circ T^n$ converges weakly to $\int f\, d\mu$. So we get the following corollary.

COROLLARY 2. *If W^{ss} is ergodic, then T is strongly mixing.*

We now come to the main result. In contrast to the Hopf argument, the invertible case is not a straightforward consequence of the non-invertible case.

THEOREM 3. *Let X be a metric space, μ a Borel probability measure on X and $T : X \to X$ an invertible measure-preserving transformation of X and $f \in L^2(X, \mu)$. Then any weak limit of a subsequence of $f \circ T^n$, $n \in \mathbb{N}$, is W^{ss}-invariant and W^{su}-invariant.*

Proof. Let g be a weak limit of $f \circ T^{n_i}$. We first show that there is a function g' which is both W^{ss}- and W^{su}-invariant, and that satisfies $\langle f, g'\rangle = \langle g, g\rangle$.

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We apply the preceding lemma to T^{-1} and to the sequence $g \circ T^{-n_i}$. There is a subsequence n_{i_k} and a W^{su}-invariant function g' such that $g \circ T^{-n_{i_k}} \to g'$ weakly. The function g is W^{ss}-invariant, so for all $n \in \mathbb{Z}$, the functions $g \circ T^{-n}$ are W^{ss}-invariant. Hence, g' is W^{ss}-invariant. Now we get

$$\langle f, g' \rangle = \lim_{k \to +\infty} \langle f, g \circ T^{-n_{i_k}} \rangle = \lim_{k \to +\infty} \langle f \circ T^{n_{i_k}}, g \rangle = \langle g, g \rangle.$$

We denote by I the closed subspace of L^2 containing all functions which are both W^{ss}- and W^{su}-invariant. This subspace is T-invariant. Now write f as a sum $f = f_1 + f_2$, with $f_1 \in I$ and $f_2 \in I^{\perp}$. Extracting subsequences, we can find two functions g_1 and g_2 and a sequence n_i' such that $f_1 \circ T^{n_i'} \to g_1$ weakly and $f_2 \circ T^{n_i'} \to g_2$ weakly. This shows that $g_1 \in I$, $g_2 \in I^{\perp}$ and $g_1 + g_2 = g$. However, there is a function $g_2' \in I$ such that $\langle f_2, g_2' \rangle = \langle g_2, g_2 \rangle$. Hence, $g_2 = 0$ and we are done.

Remarks.

- Any T-invariant function g is a weak limit of a sequence $f \circ T^n$ (just take $f = g$). Hence, this gives a proof of the Hopf argument which does not use the ergodic theorem. Note that the L^2 ergodic theorem can be deduced from the Banach–Alaoglu and Banach–Saks theorems.
- Any eigenvalue f is also a weak limit of a subsequence of $f \circ T^n$. Hence, it is both W^{ss}- and W^{su}-invariant.
- The results of this paper extend to flows without difficulty.

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