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Gravitational induction

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Abstract. Near-field transfer of gravitational energy for quasi-static axisymmetric systems is investigated by a perturbation method; a relativistic analogue of Bondi's Newtonian Poynting vector is defined. It is concluded that a quasi-static axisymmetric system can lose energy only in the presence of a receiver.

1. *Introduction.* An important problem in general relativity is the elucidation of the mechanism for transfer of energy from one body to another through gravitational interactions. The two means by which energy may be transmitted across empty space are induction and radiation, both of which are well known from electromagnetic theory. The question of the existence of gravitational radiation, which is not an observed physical effect, remains one of the outstanding problems of gravitation theory. In contrast to radiation, gravitational induction is a well established physical phenomenon, an example of which is provided by the Earth–Moon system. Bondi and McCrea (1, 2) have discussed this question in some detail within Newtonian theory, and they have constructed a simple two-body model which quite clearly exhibits inductive energy transfer. In this paper, inductive energy transfer in general relativity is investigated. Because of the complexity of the two-body problem in relativity, it is impossible to discuss the simple model of Bondi and McCrea. Instead, gravitational induction is considered by a perturbation method for a class of axially symmetric systems. It is shown that the energy flux is described by means of a Poynting vector, which is similar to the one given by Bondi for Newtonian theory

2. *Energy transfer in Newtonian theory.* Before considering the relativistic theory of induction, it is useful to have an expression for the energy flux in Newtonian theory. The following derivation of such an expression is as given by Bondi (1).

Let Σ be a closed surface located in empty space which surrounds at least some of the moving matter present. The rate at which the gravitational forces are doing work within Σ is given by

$$W = -\int_{\Sigma} \psi \rho \mathbf{v} \,.\, d\Sigma + \int_{V} \psi \operatorname{div}(\rho \mathbf{v})\, dV$$
$$= -\int_{V} \psi \frac{\partial \rho}{\partial t} dV,$$

where the equation of continuity, and the fact that ρ vanishes on Σ, have been used. Next

$$W = -\int_{V} \psi \frac{\partial \rho}{\partial t} dV = -\frac{\partial}{\partial t} \frac{1}{2} \int_{V} \psi \rho \, dV + \frac{1}{2} \int_{V} (\psi \rho - \psi \dot{\rho})\, dV$$

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(where a dot denotes differentiation with respect to the time t). Therefore

$$W + \dot{U} = \frac{1}{8\pi} \int_V (\psi \nabla^2 \dot{\psi} - \dot{\psi} \nabla^2 \psi)\, dV, \tag{1}$$

where U is the total gravitational potential energy within Σ*. The integral on the right-hand side thus denotes the rate of flow of energy through Σ into V, and may immediately be transformed into

$$-\int_\Sigma \mathbf{p} . \mathbf{d}\Sigma,$$

where $\mathbf{p}$, which is interpreted as the Poynting vector of the gravitational field, is defined by

$$8\pi \mathbf{p} = (\psi \operatorname{grad} \dot{\psi} - \dot{\psi} \operatorname{grad} \psi). \tag{2}$$

It will be useful to have an expression for the flux F of the Poynting vector for the special case of an axisymmetric mass distribution. The potential ψ on the surface of a sphere Σ is

$$\psi = \sum_{n=0}^{\infty} (A_n R^{-(n+1)} + B_n R^n) P_n(\cos\theta),$$

where A_n, B_n are functions of t determined by the distribution of matter, and R, θ, ϕ are spherical polar coordinates. The flux is

$$F = \int_\Sigma \mathbf{p} . \mathbf{d}\boldsymbol{\Sigma} = \frac{1}{2} \sum_{n=0}^{\infty} (A_n \dot{B}_n - \dot{A}_n B_n),$$

so that the expression (1) may be put in the form

$$W + \dot{U} + \frac{1}{2} \sum_{n=0}^{\infty} (A_n \dot{B}_n - \dot{A}_n B_n) = 0. \tag{3}$$

Now Newtonian theory is a good guide to the behaviour of gravitational fields in relativistic theory, provided that all changes occur sufficiently slowly; the argument which follows will be confined to quasi-static fields.

3. *The line element.* The phenomenon of gravitational induction will now be investigated for two classes of systems, both with axial symmetry. First to be considered is the field outside a non-rotating body which slowly changes shape, while remaining always axisymmetric. Secondly, there is the case of two concentric spheroids, both of which are slowly changing shape in such a way that the field between them is always axisymmetric. The mathematical description for the two cases is the same, the difference arising only through the boundary conditions.

Accordingly it will be assumed that the system has a space-like hypersurface-orthogonal Killing field; the coordinate x^3 is adapted to this symmetry, so that the metric tensor satisfies $g_{\alpha 3} = 0$†, and the signature -2 (with x^0 assumed everywhere time-like) is adopted. Furthermore it will be assumed that initially the system was static, and that at time $x^0 = 0$, say, the dynamical phase began. Consequently, in an empty

* In relativistic units, with $c = G = 1$.

† Greek letters have the range 0, 1, 2.

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region of space-time for which $x^0 \leqslant 0$, one may use Weyl's canonical coordinates in order to simplify the metric to the form (3, 5).

$$ds^2 = \exp(2\overset{0}{u})\,dt^2 - \exp(2\overset{0}{k} - 2\overset{0}{u})\,(dr^2 + dz^2) - r^2 \exp(-2\overset{0}{u})\,d\phi^2, \tag{4}$$

where the notation $x^0 = t$, $x^1 = r$, $x^2 = z$, $x^3 = \phi$ has been adopted, and where $\overset{0}{u}$, $\overset{0}{k}$ are functions of r and z only which satisfy the vacuum field equations

$$\nabla^2 \overset{0}{u} = \overset{0}{u}_{,11} + r^{-1}\overset{0}{u}_{,1} + \overset{0}{u}_{,22} = 0, \tag{5}$$

$$\left.\begin{aligned} \overset{0}{k}_{,1} &= r(\overset{0}{u}_{,1})^2 - r(\overset{0}{u}_{,2})^2, \\ \overset{0}{k}_{,2} &= 2r\overset{0}{u}_{,1}\overset{0}{u}_{,2}. \end{aligned}\right\} \tag{6}$$

It will at this stage be convenient, because of the character of the approximation method to be employed, to define u and k to be functions of t, r and z which, for $t \leqslant 0$, coincide with $\overset{0}{u}$ and $\overset{0}{k}$ respectively, and which subsequently also satisfy the equations (5) and (6). The general solution for u is therefore of the form

$$u = \sum_{n=0}^{\infty} (A_n R^{-(n+1)} + B_n R^n)\, P_n(\cos\theta), \tag{7}$$

where A_n, B_n are functions of t only, and R, θ are 'spherical polar coordinates' defined by

$$R = (r^2 + z^2)^{\frac{1}{2}}, \quad \theta = \tan^{-1} r/z. \tag{8}$$

A matrix a_{ij}, which will be referred to as the 'pseudo Weyl metric', is defined by

$$a_{ij} = \begin{pmatrix} e^{2u} & 0 & 0 & 0 \\ 0 & -e^{2k-2u} & 0 & 0 \\ 0 & 0 & -e^{2k-2u} & 0 \\ 0 & 0 & 0 & -r^2 e^{-2u} \end{pmatrix},$$

and the metric g_{ij} may then be expressed in the form

$$g_{ij}(t, r, z) = a_{ij}(t, r, z) + h_{ij}(t, r, z); \tag{9}$$

this equation is to be read as defining h_{ij}. The symmetry conditions ensure that $g_{\alpha 3} = h_{\alpha 3} = 0$, while the initial conditions imply that $a_{ij}(0, r, z)$ is the Weyl metric, and $h_{ij}(0, r, z) = 0$.

Now one cannot hope to find exact solutions of the vacuum field equations, and the approximation procedure adopted depends upon the assumption of slow motions. Accordingly it will be assumed that the metric tensor is an analytic function of a small parameter ϵ, that is

$$g_{ij} = \overset{0}{g}_{ij} + \epsilon \overset{1}{g}_{ij} + \epsilon^2 \overset{2}{g}_{ij} + \ldots,$$

where ϵ is to be identified with a characteristic velocity of the source. Moreover, $\epsilon = 0$ corresponds to a static situation so that g_{ij} must be the initial Weyl metric.

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It will also be assumed that differentiation of a function f with respect to t increases the order of f by unity, thus

$$a_{ij} \sim O_0,\ h_{ij} \sim O_1,\ a_{ij,0} \sim O_1,\ h_{ij,0} \sim O_2, \tag{10}$$

where $f \sim O_n$ denotes that the quantity f is of order n in ϵ.

The discussion which follows will be confined to the gravitational fields outside the sources, and so it will be sufficient to specify the boundary conditions at infinity. Accordingly it is assumed that space is asymptotically flat, and that as $R \to \infty$

$$u \sim O(R^{-1}),\quad h_{\alpha\beta} \sim O(R^{-1}),\quad h_{33} \sim O(R),\quad k \sim O(R^{-2}) \tag{11}$$

so that the coordinate system goes over to cylindrical polar coordinates in a Lorentz frame.

In order to calculate the components of the Ricci tensor, it is convenient to define a_{ij}, h_{ij} and H_{ijk} by

$$a^{ij}a_{jk} = \delta^i_k;\ h^{ij} = g^{ij} - a^{ij};\ H_{ijk} = \tfrac{1}{2}(h_{ik,j} + h_{jk,i} - h_{ij,k});$$

and denoting by a subscript 0 a quantity corresponding to the pseudo-Weyl metric, one obtains.

$$\Gamma^i_{jk} = \underset{0}{\Gamma^i_{jk}} + a^{ip}H_{jkp} - a^{ir}h_{rs}\underset{0}{\Gamma^s_{jk}} + O_2$$

and

$$\begin{aligned} R_{rm} = {} & \underset{0}{R_{rm}} + h_{kj}\,a^{pj}[\underset{0}{R^k}_{rpm} - \underset{0}{\Gamma^q}_{mp}\,\Gamma^k_{rq} + \underset{0}{\Gamma^q}_{qp}\,\underset{0}{\Gamma^k}_{rm}] \\ & + h_{kj}[a^{pj}{}_{,p}\,\underset{0}{\Gamma^k}_{rm} - a^{pj}{}_{,m}\,\underset{0}{\Gamma^k}_{rp}] - H_{msq}\,a^{ps}\,\underset{0}{\Gamma^q}_{rp} + H_{psq}\,a^{ps}\,\Gamma^q_{rm} \\ & + H_{rps}[a^{ps}{}_{,m} + \underset{0}{\Gamma^p}_{mq}\,a^{qs}] - H_{rms}[a^{ps}{}_{,p} + a^{ps}\,\underset{0}{\Gamma^q}_{qp}] + a^{ps}(H_{rps,m} - H_{rms,p}) + O_2. \end{aligned} \tag{12}$$

It should be noted that although the expressions (12) have been linearized with respect to h_{ij}, they will contain non-linear terms of order ϵ^2 through time derivatives.

3. *The vacuum field equations.* The evolution of the system is, to the first order, governed by the time dependence of u and k. This means that the time development is completely specified if the functions A_n and B_n are given. However, it is not to be expected that all these functions can be arbitrary, but rather that they will be restricted in some way by conservation laws. In the first instance one will therefore be concerned with determining what sort of slow motions are possible, and the exclusion of those which are not must yield information concerning the mass and momentum of the system.

Now because of (6), the vacuum field equations are satisfied to order zero, while the first-order equations divide naturally into two groups:

Group I. $R_{00} = R_{11} = R_{22} = R_{33} = R_{12} = 0.$

These five equations involve only the five functions h_{00}, h_{11}, h_{22}, h_{33}, h_{12} and the space derivatives of u and k.

Group II. $R_{01} = R_{02} = 0.$

These two equations involve only the two functions h_{01}, h_{02}, and the space and time derivatives of u and k.

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From this it may be concluded that the equations in group I do not contain any information concerning the time development of the system. Moreover, the equations can be satisfied by the choice

$$h_{00} = h_{11} = h_{22} = h_{33} = h_{12} = 0, \tag{13}$$

which may be regarded as a set of coordinate conditions. The argument which follows will, however, be independent of these conditions.

The equations in group II may be put in the form

$$0 = R_{01} = S_1 + \Lambda_{,1} \tag{14a}$$

$$0 = R_{02} = S_2 - (1/r)\,(r\Lambda)_{,1} \tag{14b}$$

where
$$\Lambda = e^{2u-2k}(\tfrac{1}{2}h_{02,1} - \tfrac{1}{2}h_{01,2} + h_{01}u_{,2} - h_{02}u_{,1}), \tag{14c}$$

and
$$S_1 = k_{,01} - 2u_{,01} + 2u_{,1}u_{,0} - (1/r)\,k_{,0}$$

$$S_2 = k_{,02} - 2u_{,02} + 2u_{,2}u_{,0}.$$

In order to discuss the integrability conditions of these equations, it is convenient to rewrite the equations (14) as a single-vector equation

$$\mathbf{R} = \mathbf{S} + \operatorname{curl} \boldsymbol{\Lambda} = \mathbf{0}, \tag{15}$$

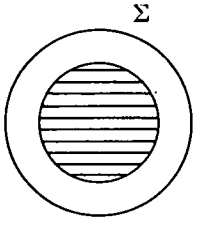
where
$$\mathbf{R} = R_{01}\hat{\mathbf{r}} + R_{02}\hat{\mathbf{z}}, \quad \boldsymbol{\Lambda} = \Lambda\hat{\boldsymbol{\phi}},$$

$\mathbf{S} = S_1\hat{\mathbf{r}} + S_2\hat{\mathbf{z}}$, and $\hat{\mathbf{r}}$, $\hat{\mathbf{z}}$, $\hat{\boldsymbol{\phi}}$ are unit vectors along the coordinate curves with respect to the metric $dr^2 + dz^2 + r^2 d\phi^2$, and curl is the usual vector operator with respect to this metric.

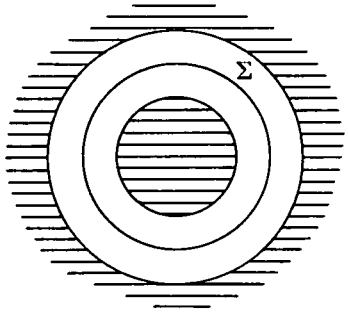
The local integrability condition for the existence of $\boldsymbol{\Lambda}$, is $\operatorname{div} \mathbf{S} = 0$, which is automatically satisfied as a consequence of the zero-order field equations. In addition, to ensure the existence of $\boldsymbol{\Lambda}$ in a region R, it is necessary that

$$I = \int_{\Sigma} \mathbf{S}.\mathbf{d\Sigma} = 0,$$

where Σ is the boundary of R.



Case (i)



Case (ii)

This integral will be evaluated over a closed surface Σ located entirely in empty space, and for the condition $I = 0$ to be non-trivial, Σ must enclose at least one of the spheroids (see diagram). In fact, since $\operatorname{div} \mathbf{S} = 0$ *in vacuo*, the integral is surface independent and Σ may be chosen as the surface of a 'sphere' of radius R surrounding the origin.

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In 'spherical polar coordinates', again defined by equation (8), the surface integral condition on S becomes

$$4\pi R^2 \int_0^\pi \left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial R} - \frac{\partial^2 u}{\partial R\,\partial t} + \frac{1}{2}\frac{\partial^2 k}{\partial R\,\partial t} - \tfrac{1}{2}R^{-1}\frac{\partial k}{\partial t}\right]\sin\theta\,d\theta. \tag{16}$$

Substituting for k in terms of u, taking advantage of the fact that k vanishes on the axis of symmetry (5), and that u satisfies Laplace's equation, one obtains after some reduction

$$-\dot{A}_0 + \frac{1}{2}\sum_{n=0}^{\infty}(A_n\dot{B}_n - \dot{A}_n B_n) + \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)\frac{d}{dt}(AnB_n) = 0. \tag{17}$$

The condition $I = 0$ is thus a non-trivial restriction on the evolution of the system, and may be interpreted as the law of conservation of energy. Moreover, (17) is rather similar in form to (3), and this leads to the following tentative identifications:

(*a*) $-\frac{1}{2}\sum_{n=0}^{\infty}(A_n\dot{B}_n - \dot{A}_n B_n)$ is the rate of flow of energy through Σ into R, and leads to the definition

$$8\pi\mathbf{p} = u\,\mathrm{grad}\,\dot{u} - \dot{u}\,\mathrm{grad}\,u$$

as the Poynting vector of the gravitational field.

(*b*) $-A_0$ is the monopole coefficient in the Weyl metric, and therefore represents the gravitational mass inside Σ, the term $-\dot{A}_0$ being the relativistic correction to the Newtonian conservation law for which the gravitational mass is constant.

(*c*) $\frac{1}{2}\sum_{n=0}^{\infty}(2n+1)\frac{d}{dt}(A_nB_n)$ is the sum (rate of increase of potential energy inside Σ) + (rate at which the gravitational forces are doing work within Σ); there being no material energy flow across Σ, the surface being located entirely in empty space.

If the surface Σ surrounds *all* the matter present, then $B_n = 0$ and (17) reduces to $\dot{A}_0 = 0$. That is, a quasi-static system can, to the first order, only lose energy in the presence of a receiver! This result for near field energy transfer may be contrasted with the result in radiation theory that energy can be lost to infinity. This accords very nicely with the electromagnetic analogy.

4. *Multipole solution of the field equations.* In this final section the multipole solution of the field equations will be examined for the case of an isolated source; and in particular the possibility of further restrictions on the evolution of the system is discussed. The boundary conditions are given by (11), and it follows that the solutions for u, k and Λ (defined by equation (14*c*)) must be of the form:

$$\left.\begin{aligned} u &= \frac{A_0}{R} + \frac{A_1}{R^2}\cos\theta + \frac{A_2 P_2}{R^3} + \ldots, \\ k &= -\frac{A_0^2\sin^2\theta}{R^2} - 2\frac{A_0 A_1\cos\theta\sin\theta}{R^3} - \ldots, \\ \Lambda &= \sum_{n=2}^{\infty} R^{-n}\overset{(n)}{\Lambda}(\theta, t), \end{aligned}\right\} \tag{18}$$

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where $\overset{(n)}{\Lambda}$ are functions of θ and t only. After substituting the expansions (18) into the field equations (15) one obtains

$$\left.\begin{aligned} &O(R^{-2}): \dot{A}_0 = 0 \\ &O(R^{-3}): \overset{(2)}{\Lambda} = 2\dot{A}_1 \sin\theta, \\ &O(R^{-4}): \overset{(3)}{\Lambda} = -A_0\dot{A}_1 \sin\theta\cos 2\theta + \tfrac{3}{2}\dot{A}_2 \sin 2\theta, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{19}$$

thus the time dependence of A_n, apart from A_0, is arbitrary. This result may, however, be due to the lack of a precise definition of the function h_{ij}, and the effects of possible coordinate transformations will now be examined. It can be shown (3) that the 'allowed coordinate transformations' (that is coordinate transformations which preserve the form (9)) must be of the form

$$\bar{x}^\alpha = x^\alpha + \xi^\alpha(t, r, z),$$

where a transformed quantity is denoted by a bar, and where $\xi^\alpha \sim 0_1, \xi^\alpha(0, r, z) = 0$. Applying this infinitesimal coordinate transformation, one obtains

$$\bar{h}_{ij}(x) = h_{ij}(x) - a_{ij,\alpha}\xi^\alpha - a_{i\alpha}\xi^\alpha{}_{,j} - a_{j\alpha}\xi^\alpha{}_{,i} + O_2, \tag{20}$$

where all the functions are evaluated at the unbarred point x^j. From this transformation one immediately deduces $\Lambda(x) = \bar{\Lambda}(x)$, where

$$\bar{\Lambda} = e^{2u-2k}(\tfrac{1}{2}\bar{h}_{02,1} - \tfrac{1}{2}\bar{h}_{01,2} + \bar{h}_{01}u_{,2} - \bar{h}_{02}u_{,1}).$$

That is, Λ is an invariant function under infinitesimal coordinate transformations which preserve the functional form of a_{ij}. In particular it is not possible to set $\overset{(n)}{\Lambda} = 0$ for any integer n, so that the time dependence of A_n for $n \geqslant 1$ *must*, to the first order, be arbitrary. This result does not mean that there are no further conservation laws, but that the conservation of energy is the only one which operates to this order. If one adopts the Newtonian interpretation of A_n, then the conservation of momentum is given by $\ddot{A}_1 = 0$, which is a second-order equation, and it is quite reasonable that this should also be so in the relativistic case. This result has recently been confirmed by Morgan (4), though his results are dependent on the first-order coordinate conditions (13), and which are not easy to understand.

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