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Diffeomorphisms with infinitely many irrational invariant curves

LEONARDO MORA and BLADISMIR RUIZ

Departamento de Matemática, Facultad de Ciencias, La Hechicera, Universidad de los Andes, Mérida, 5101, Venezuela
(e-mail: lmora@ula.ve, bladismir@ula.ve)

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Abstract. For a surface diffeomorphism $f \in \mathrm{Diff}^l(M)$, with $l \geq 8$, we prove that if f exhibits a non-transversal heteroclinic cycle composed of two fixed saddle points Q_1 and Q_2, one dissipative and the other expansive, then there exists an open set $\mathcal{V} \subset \mathrm{Diff}^l(M)$ such that $f \in \overline{\mathcal{V}}$ and there exists a dense set $\mathcal{D} \subset \mathcal{V}$ such that for all $g \in \mathcal{D}$, g exhibits infinitely many invariant periodic curves with irrational rotation numbers. Moreover, these curves are C^1 conjugated to an irrational rotation on $\mathbb{S}^1$.

1. *Introduction*

In the late 1970s, regions in the space of diffeomorphisms exhibiting persistence of homoclinic tangencies were discovered by Newhouse, and subsequently called Newhouse regions. The first example of such regions was associated with a diffeomorphism exhibiting a quadratic homoclinic tangency between the invariant manifolds of a dissipative hyperbolic periodic orbit; see [**N79**]. Inside such regions, Newhouse found residual sets in the C^2 topology, with each diffeomorphism inside one of that residual sets, exhibiting infinitely many periodic sinks, a very unstable phenomenon. Then Colli [**Col98**] showed the existence of a dense set, now in the C^∞ topology, such that each diffeomorphism in that dense set exhibited infinitely many Hénon-like strange attractors. Indeed, Newhouse regions are places in the space of diffeomorphisms where very degenerate and unstable dynamical behavior appears; see [**GST07**] for another example.

In order to prove the above results Newhouse and Colli use a non-transversal heteroclinic cycle (see definition below) and the creation of thick Cantor sets which are used in two different ways. Newhouse shows essentially that two thick linked Cantor sets always meet, a result now known as Newhouse's gap lemma. On the other hand, Colli shows that two Cantor sets which move with positive relative velocity can be perturbed to another pair of Cantor sets having subsets stably linked. See §2 for a complete statement of these results. Both results allow two methods of proof to be implemented in which, using Newhouse's results, residual sets of diffeomorphisms are obtained exhibiting infinitely many dynamical behaviors which are open in the Whitney topology, for example existence

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of hyperbolic periodic sinks. On the other side, using Colli's results, dense (not necessarily residual) sets of diffeomorphisms can be obtained by exhibiting infinitely many dynamical behaviors which are persistent in a metric sense, like Hénon-like strange attractors.

We observe that Newhouse and Colli consider a non-transversal heteroclinic cycle with dissipative periodic points Q_1 and Q_2. In [**GSS05**], Gonchenko *et al* consider the same heteroclinic cycle but now Q_1 is a dissipative periodic point and Q_2 is an expansive one. This dynamical configuration allows them to construct Newhouse regions, in the C^l topology ($l \geq 3$), with a dense set of diffeomorphisms having infinitely many smooth invariant periodic curves normally attracting. Here, for an invariant curve for a diffeomorphism we understand an embedded C^s submanifold S which is diffeomorphic with $\mathbb{S}^1$, the unit circle. The integer s is the degree of smoothness of the invariant curve. It will be a periodic one, if it is an invariant curve for some diffeomorphism's iterate. In order to find the last sets they use the Newhouse approach. Unhappily, the dynamics on each of these curves can be very simple, i.e., a Morse–Smale diffeomorphism of the circle, given as a total result a semiglobal dynamic not too complicated but very unstable. It is well known that a dynamics on the circle is classified by its rotation number and its differentiability. So, if the dynamics has an irrational rotation number, it can be considered complicated, although it is well understood. We observe that with enough differentiability the last fact is metrically persistent, so we ask if we could get Newhouse regions with a dense set of diffeomorphisms exhibiting infinitely many smooth invariant periodic curves normally attracting and with irrational rotation number. This paper is devoted to solving this problem, using the Colli approach.

The next dynamical configuration will appear throughout the paper. Let us consider a diffeomorphism $f \in \mathrm{Diff}^l(M)$, with M a compact surface, which has a non-transversal heteroclinic cycle, composed of two fixed saddle points Q_1 and Q_2. The invariant manifolds associated to Q_i are supposed to meet in the following way: $W^u(Q_1)$ and $W^s(Q_2)$ intersect transversally at a heteroclinic orbit Γ_{12} and $W^s(Q_1)$ and $W^u(Q_2)$ have a quadratic tangency along the heteroclinic orbit Γ_{21}. Let λ_i and γ_i be the contractive and the expanding eigenvalues of $Df(Q_i)$. And let $\sigma_i = |\lambda_i \gamma_i|$, where we suppose that $\sigma_1 < 1 < \sigma_2$. From now on, for a diffeomorphism f as above we will say that it exhibits a *dissipative-contractive non-transversal heteroclinic 2-cycle*. The main result of this paper is as follows.

MAIN THEOREM. *Let $l \geq 8$ and $f \in \mathrm{Diff}^l(M)$ exhibiting a dissipative-contractive non-transversal heteroclinic 2-cycle. Then there exists an open set $\mathcal{V} \subset \mathrm{Diff}^l(M)$ such that:*

(i) $f \in \overline{\mathcal{V}}$;

(ii) *there exists a dense set $\mathcal{D} \subset \mathcal{V}$ such that for all $g \in \mathcal{D}$, g exhibits infinitely many invariant periodic curves with irrational rotation numbers and smoothness of class C^l. Moreover, these curves are C^1 conjugated to an irrational rotation on $\mathbb{S}^1$.*

Since, as we have mentioned above, we follow the Colli construction in [**Col98**], we will assume that the reader is familiar with [**Col98**].

Finally, the paper is organized as follows. In §2 we recall some facts about thick Cantor sets, in particular the gap lemma and linking lemma. In §3 we present a rescaling result for non-transversal heteroclinic 2-cycles by Gonshenko *et al*, which is an extension of a Colli

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rescaling result. In §4 we present the Hopf bifurcation and some refinements needed in the proof of the main theorem. Section 5 is devoted to computations of some quantities of the rescaling map of a non-transversal heteroclinic 2-cycle, needed to get the conclusions given by Hopf's theorem. In §6 we gather together all the information in the previous sections in order to prove main theorem.

2. *Preliminaries*

In this section, we follow ideas and rewrite some results in [**PT93**].

A *Cantor set* in $\mathbb{R}$ is a compact, perfect and totally disconnected set. Let K be a Cantor set and I its *convex hull*, i.e. the minimal (closed) interval of $\mathbb{R}$ containing K. A *gap* of K is a connected component of $\mathbb{R}\backslash K$. A *presentation* of K is an ordering $\mathcal{U} = \{U_n\}_{n\geq 1}$ of the bounded gaps. An *ordered presentation* of K is a presentation $\mathcal{U}$ such that $\ell(U_n) \leq \ell(U_m)$ for all $n > m$, where $\ell(U_n)$ denotes the length of U_n. The *bridge* at $u \in \partial U_n$, $U_n \in \mathcal{U}$, is the component of $I\backslash(U_1 \cup \cdots \cup U_n)$ that contains u. The *thickness* of K is the number

$$\tau(K) = \inf\{\tau(K, \mathcal{U}, u) : u \in K\},$$

where $\mathcal{U}$ is any ordered presentation of K,

$$\tau(K, \mathcal{U}, u) = \frac{\ell(C)}{\ell(U_n)},$$

and C is the bridge at $u \in \partial U_n$. This definition of thickness does not depend on the ordered presentation $\mathcal{U}$ (see [**PT93**]). Let $k \in K$. The *local thickness* of K at k is the number

$$\tau(K, k) = \lim_{\varepsilon\to 0}(\sup\{\tau(\tilde{K}) : \tilde{K} \subset K \cap B_\varepsilon(k) \text{ a Cantor set}\}).$$

Let K_1, K_2 be Cantor sets and I_1, I_2 their convex hulls. We say that the pair $\langle K_1, K_2\rangle$ is *linked* if $I_1 \cap I_2 \neq \emptyset$ and I_1 is not inside a gap of K_2 and I_2 is not inside a gap of K_1. The link is called *stable* if the same condition is satisfied by the interiors $\text{int}(I_1)$ and $\text{int}(I_2)$ of I_1 and I_2, respectively. Further, we say that a pair $\langle K_1, K_2\rangle$ has a *sublink* if there are proper intervals $C_1 \subset I_1$ and $C_2 \subset I_2$, bridges of K_1 and K_2 respectively, such that the pair $\langle C_1 \cap K_1, C_2 \cap K_2\rangle$ is linked. Finally, we say that $\langle K_1, K_2\rangle$ *has two sublinks* if there are two pairs of distinct proper subintervals forming independent sublinks.

Let Λ be a non-trivial basic set of a C^2 diffeomorphism on a two-dimensional manifold $\varphi : M \mapsto M$. Let $z \in W^s(\Lambda)$ and $\phi : [-a, a] \mapsto M$ be a C^1 embedding transverse to $W^s(\Lambda)$ at $z = \phi(0)$. The *local stable thickness of* Λ *at* z is $\tau^s(\Lambda, z) = \tau(\phi^{-1}(W^s(\Lambda)), 0)$. This is independent of the choice of ϕ, as a consequence of the fact that the holonomy maps (i.e., the projections along the leaves of the stable foliation of Λ) can be extended to C^1 maps. Actually, this smoothness of the holonomy of $W^s(\Lambda)$, together with the transitivity of $\varphi|_\Lambda$, also implies that $\tau^s(\Lambda, z)$ has the same value at every $z \in W^s(\Lambda)$. We denote this constant value by $\tau^s(\Lambda)$ and call it the *local stable thickness* of Λ. This is a strictly positive (finite) number and depends continuously on the diffeomorphism, in the sense that if Λ_ψ denotes the smooth continuation of Λ for a diffeomorphism ψ which is C^2-close to φ, then $\tau^s(\Lambda_\psi)$ is close to $\tau^s(\Lambda)$. The *local unstable thickness* $\tau^u(\Lambda, z)$ and $\tau^u(\Lambda)$ are defined in a similar way.

In the proof of main theorem we will use the following two important results involving thick Cantor sets, whose proofs can be found in [**Col98, N79**] respectively.

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PROPOSITION 1. (Newhouse's gap lemma) *Let K_1, K_2 be Cantor sets in $\mathbb{R}$ such that $\tau(K_1)\cdot\tau(K_2)>1$ and $\langle K_1, K_2\rangle$ is linked. Then $K_1\cap K_2\neq\emptyset$.*

PROPOSITION 2. (Linking lemma) *Let K_1, K_2 be Cantor sets in $\mathbb{R}$, with $\tau(K_1)\cdot\tau(K_2)>1$, and I_1, I_2 the convex hull of K_1, K_2, respectively. Let $\vartheta_\beta : I_1\mapsto\mathbb{R}$ and $\widetilde{\vartheta}_\beta : I_2\mapsto\mathbb{R}$ be such that:*

(a) *ϑ_β and $\widetilde{\vartheta}_\beta$ are topological embeddings, for all $\beta\in\mathbb{R}$;*
(b) *$\vartheta_\beta(x)$ and $\widetilde{\vartheta}_\beta(y)$ are differentiable with respect to β, for all $x\in K_1$ and $y\in K_2$;*
(c) *$\partial_\beta[\upsilon_\beta(x)-\widetilde{\upsilon}_\beta(y)]\geq c>0$, for all $x\in K_1$ and $y\in K_2$;*
(d) *if $\widetilde{K}_1\subset K_1$ and $\widetilde{K}_2\subset K_2$ are Cantor subsets, and letting $\beta_0\in\mathbb{R}$ be such that the pair $\langle\vartheta_{\beta_0}(\widetilde{K}_1), \widetilde{\vartheta}_{\beta_0}(\widetilde{K}_2)\rangle$ is linked, then, for any $\varepsilon>0$, there is β such that:*
  (i) *$|\beta-\beta_0|<\varepsilon$;*
  (ii) *the pair $\langle\vartheta_\beta(\widetilde{K}_1), \widetilde{\vartheta}_\beta(\widetilde{K}_2)\rangle$ has two stable sublinks.*

3. *Renormalization along non-transversal heteroclinic 2-cycles*

Let us consider in this section a diffeomorphism $f\in\mathrm{Diff}^l(M)$, with M a compact surface, which exhibits a dissipative-contractive non-transversal heteroclinic 2-cycle; see Figure 1.

Now consider a two-parameter family $f_{(\mu_1,\mu_2)}$ such that $f_{(0,0)}=f$. We choose as the parameter μ_1 that which controls the splitting of the manifolds $W^s(Q_1)$ and $W^u(Q_2)$ along the orbit Γ_{21} and require that it does so in a generic way. As for the second parameter, we choose that which controls the evolution of the ratio $-\ln(\sigma_2)/\ln(\sigma_1)$ so that

$$\mu_2=-\frac{\ln(\sigma_2(\mu))}{\ln(\sigma_1(\mu))}+\frac{\ln(\sigma_2(0))}{\ln(\sigma_1(0))},$$

in order that it is enough to assume that

$$\frac{\partial}{\partial\mu_2}\left(\frac{\ln(\sigma_2(\mu))}{-\ln(\sigma_1(\mu))}\right)\neq 0.$$

Let U_1 and U_2 be two disks around Q_1 and Q_2, respectively. Let r be the last point of the orbit Γ_{12} in U_1 and q be the first point of Γ_{21} in U_1. Also let k_{12} be the first positive integer such that $r_{12}=f^{k_{12}}(r)\in U_2$ and k_{21} be the first positive integer such that $q_{21}=f^{-k_{21}}(q)\in U_2$; see Figure 1.

PROPOSITION 3. **[GSS05]** *Let $l\geq 5$. For U_i small enough, there exist new coordinates (x, y) near q depending on μ and a new parameter $\nu=(a, b)$ such that the map*

$$\phi_\mu^{i,j}=f_\mu^{k_{21}}\circ f_\mu^j\circ f_\mu^{k_{12}}\circ f_\mu^i,$$

for i and j big enough, in these new coordinates and parameters is given by

$$H_{i,j}:\begin{array}{l}\overline{x}=y,\\ \overline{y}=a-bx-y^2+e\lambda_1^i\lambda_2^j xy+f\gamma_1^{-i}\gamma_2^{-j}y^3+R_{i,j}=l_{i,j}(x, y),\end{array}\qquad(1)$$

where

$$a=m(\mu_1+O(\lambda_2^j)+O(\gamma_1^{-i}))\gamma_1^{2i}\gamma_2^{2j},$$
$$b=h\sigma_1^i\sigma_2^j$$

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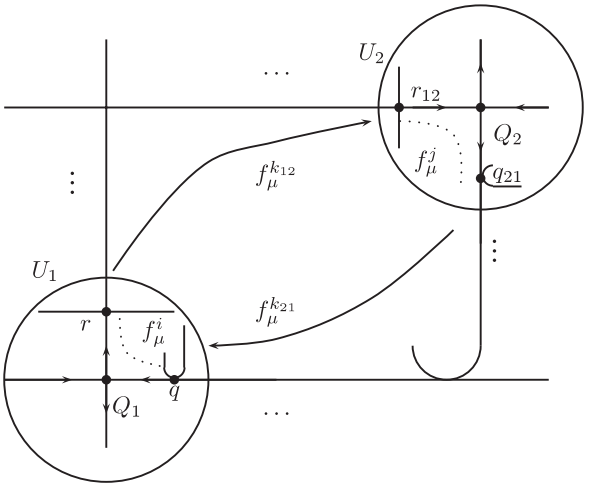


FIGURE 1. A dissipative–contractive non-transversal heteroclinic 2-cycle.

with $e,\ f \in \mathbb{R}$ *and* h, m *are non-vanishing functions.* $R_{i,j}$ *is a function such that*

$$\|R_{i,j}(a, b, x, y)\|_{C^{r-2}} \leq o(|\gamma_1^{-i}\gamma_2^{-j}| + |\lambda_1^i\lambda_2^j|).$$

Also the domain of the map $H_{i,j}(x, y)$ *includes disks of radius* $\rho_{i,j}$ *such that* $\rho_{i,j} \to \infty$.

Remark 1. This proposition extends the rescaling result in [**Col98**, §4]. Moreover, it simplifies the Colli proof slightly, since it drops the assumption of linearization in [**Col98**].

4. *Hopf's bifurcation theorem revisited*

Let $H_b : U \subset \mathbb{R}^2 \hookleftarrow$ be a C^l one-parameter family of maps with $l \geq 5$. Assume that $H(b, x, y) = H_b(x, y)$ is C^2 and:

(1) $H_0(0) = 0$;

(2) $DH_0(0)$ has eigenvalues $\lambda, \overline{\lambda}$, with $|\lambda| = 1$ and $(d/db)|\lambda|_0 \neq 0$.

In the coordinates $z = x + iy$ and $\overline{z} = x - iy$, we can write $H_b(z, \overline{z})$ as

$$H_b(z, \overline{z}) = \lambda(b)z + \sum_{i=2}^{4} \sum_{p+q=i} \xi_{pq}(b) z^p \overline{z}^q + O(|z|^5).$$

The following theorem is the standard normal form for the family of maps H. The proof can be found in [**I79**, Ch. III].

THEOREM 4. *Let* $\lambda(b) = |\lambda(b)| \exp(2\pi\phi(b)i)$ *and suppose that* $\lambda^i \neq 1$ *for* $i = 1, 2, 3, 4, 5$. *Then:*

(1) *there exist* C^{l-4} *coordinates, depending on b, such that H in these new coordinates is written as*

$$H_b(w, \overline{w}) = \lambda(b)w + \alpha(b)w^2\overline{w} + R_5(w, \overline{w}, b),$$

with $R_5(w, \overline{w}, b) = O(|w|^5)$ *and of class* C^{l-5}. *Moreover,*

$$\alpha(0) = \xi_{21} + \frac{2|\xi_{02}|^2}{\lambda^2 - \overline{\lambda}} + \frac{|\xi_{11}|^2}{1 - \overline{\lambda}} + \frac{\xi_{11}\xi_{02}(1 - 2\lambda)}{\lambda^2 - \lambda},$$

where $\xi_{pq} = \xi_{pq}(0)$;

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(2) *there exists a C^∞ change of coordinates $(\theta, r) \mapsto (x, y)$ which is C^{l-5} on b, and in these new coordinates H_b is rendered in the form $H_b(\theta, r) = (\Theta, R)$ with*

$$\Theta = \theta + \phi(b) + d(b)r^2 + O(r^4),$$
$$R = |\lambda(b)|r(1 + c(b)r^2) + O(r^5).$$

The functions c and d at $b = 0$ are given by

$$c = c(0) = \mathrm{Re}\left(\overline{\lambda}\xi_{21} + \frac{2\overline{\lambda}|\xi_{02}|^2}{\lambda^2 - \overline{\lambda}} + \frac{\overline{\lambda}|\xi_{11}|^2}{1 - \overline{\lambda}} + \frac{\xi_{11}\xi_{20}\overline{\lambda}(1 - 2\lambda)}{\lambda^2 - \lambda}\right),$$
$$d = d(0) = \frac{1}{2\pi}\mathrm{Im}\left(\overline{\lambda}\xi_{21} + \frac{2\overline{\lambda}|\xi_{02}|^2}{\lambda^2 - \overline{\lambda}} + \frac{\overline{\lambda}|\xi_{11}|^2}{1 - \overline{\lambda}} + \frac{\xi_{11}\xi_{20}\overline{\lambda}(1 - 2\lambda)}{\lambda^2 - \lambda}\right);$$

(3) *if $c < 0$ (>0), with an affine change of the coordinate R, which holds for $b > 0$ ($b < 0$) and depends differentially on $\sqrt{b}$ ($\sqrt{-b}$), we get that $H_b(\theta, x) = (\Theta, X)$ has the form*

$$\Theta = \theta + \theta_0 + b\theta_2 + |b|^{3/2}\Theta_1(\theta, x, b),$$
$$X = \kappa(b)x + |b|^{3/2}X_1(\theta, x, b)$$

with X_1 and Θ_1 being C^{l-5} on $(\theta, x, |b|^{1/2})$ and with $\kappa(b)$ and θ_2 given by

$$\kappa(b) = 1 - 2b\,\mathrm{Re}(\lambda_1),$$
$$\theta_2 = \frac{1}{2\pi}\mathrm{Im}(\lambda_1) - \frac{1}{2\pi}(\mathrm{Im}(\alpha(0)\overline{\lambda})) \cdot \frac{\mathrm{Re}(\lambda_1)}{\mathrm{Re}(\alpha(0)\overline{\lambda})},$$

where $\lambda_1 = \overline{\lambda}\partial_b\lambda(0)$.

THEOREM 5. *For a family of maps as above:*

(1) *there exists a family of maps $u(\theta, b)$, C^l on θ, such that $\{(\theta, r) : r = u(\theta, b)\}$ is an invariant set for the map H_b;*
(2) *$b \mapsto u(\cdot, b) \in \mathrm{Diff}^l(\mathbb{S}^1)$ is a continuous function;*
(3) *if $l - 5 \geq 2$, then u is $C^{1,1}$ on $\mathbb{S}^1 \times I$, on an interval I as close to 0 as we please;*
(4) *the dynamics on the graph of u is given by*

$$f_b : \theta \mapsto \theta + \theta_0 + b\theta_2 + b^{3/2}\Theta_1(\theta, u(\theta, b), b);$$

(5) *the rotation number of f_b is given by*

$$\rho(b) = \rho(f(\cdot, b)) = \theta_0 + b\theta_2 + O(b^{3/2}).$$

The above theorems are usually known as Hopf's bifurcation theorem (HBT); see, for example, [**I79, R89**]. We know that when $\rho(b)$ is rational, then this condition continues to hold generically in a neighborhood of the parameter b. For the irrational case, this is no longer true. We would like to know how big is the set of parameters b such that $\rho(b)$ is an irrational number. For this purpose we need to recall a theorem due to Herman. Let $\mathrm{Diff}^k(\mathbb{S}^1)$ be the set of C^k diffeomorphisms on the unit circle endowed with the C^k-Whitney topology; if $k = 0$ we have the uniform convergence topology.

THEOREM 6. (Herman [**H76**]) *Assume that we have a continuous curve $\alpha : t \mapsto \alpha(t) \in \mathrm{Diff}^k(\mathbb{S}^1)$ which is a C^1 function valued in $\mathrm{Diff}^0(\mathbb{S}^1)$. If $\rho(\alpha(t))$ is not constant and $k \geq 3$,*

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then

$$M_\alpha = m(\{t : \alpha(t) \text{ is } C^{k-2}\text{-conjugated to an irrational rotation}\})$$

is positive, where m denotes the Lebesgue measure.

THEOREM 7. (Addendum to HBT) *If $l - 5 \geq 3$ and $\theta_2 \neq 0$, then we can affirm that E, the set of parameters b for which $\rho(b)$ is an irrational number, has Lebesgue positive measure.*

Proof. As $\theta_2 \neq 0$ we have that $b \mapsto \rho(b)$ is not a constant, so the same applies to the curve $b \mapsto f_b$. This curve is continuous since item (2) of Theorem 5 and its C^1 differentiability, when we consider the C^0 topology, follow from item (3) of Theorem 5 and the following lemma, whose proof we leave to the reader.

LEMMA 8. *For $b \in (b_1, b_2)$ and $\theta \in [\theta_1, \theta_2]$, let $f(\theta, b)$ be a bivariate function with values in $\mathbb{R}$ such that $\partial_b f$ exists and is continuous. Then the map $b \mapsto f(\cdot, b) \in C^0([\theta_1, \theta_2])$ is a C^1 curve.*

As for the estimate of the size of E, we observe first that item (5) in Theorem 5 and the assumption $l - 5 \geq 3$ imply that $\| f_b - R_{\rho(b)} \|_{C^3}$ converges to zero when b converges to zero. Then by [**H79**, Theorem 2.2, Complement 2.5, p. 204] the conjugation h between f_b and $R_{\rho(b)}$, when ρ_b is a Diophantine number, has C^1 norm converging to one when b converges to zero. Then the estimative follows from the argument in [**H76**, §7.1]. □

5. *First return map*

The map $H_{i,j}$ has a fixed point $P = P^+_{i,j}$, from (1), which has the form $(p,\ p) = (p_{i,j},\ p_{i,j})$ where p satisfies $l_{i,j}(p,\ p) = p$. It is shown in [**GSS05**] that P is a weak focus, i.e., $DH_{i,j}(P)$ has two complex eigenvalues $e^{\pm \iota \psi}$ for $\psi \in (0, \pi)$ and where $\iota = \sqrt{-1}$, along a curve in the parameter space

$$L_{i,j} : \begin{array}{l} a = \cos^2(\psi) - 2\cos(\psi) + \cdots, \\ b = 1 - e\lambda_1^i \lambda_2^j \cos(\psi) + \cdots, \end{array} \tag{2}$$

where $\cdots$ denotes terms that converge to zero as $i,\ j \to \infty$. We want to show that, for $\psi \in (0, \pi)$ except for a finite set of points, P and $H_{i,j}$ meet the conditions of the HBT. As a first step, we render the map $H_{i,j}$ in the normal form given in Theorem 4, item (1).

LEMMA 9. *There exist coordinates z and $\overline{z}$ such that*

$$H_{i,j}(z, \overline{z}) = \lambda z + \zeta_{20} z^2 + \zeta_{11} |z|^2 + \zeta_{02} \overline{z}^2 + \zeta_{21} z |z|^2 + \cdots,$$

where $\cdots$ stands for terms of degree three different from $z|z|^2$ or higher, and

$$\zeta_{21} = -3\lambda \frac{fh\lambda_1^i \lambda_2^j}{\sin(\psi)} \iota,$$

$$\zeta_{20} = \frac{\iota}{4\sin(\psi)} (\lambda^2 - \lambda e \lambda_1^i \lambda_2^j),$$

$$\zeta_{11} = \frac{\iota}{4\sin(\psi)} (2 - (\overline{\lambda} + \lambda) e \lambda_1^i \lambda_2^j),$$

$$\zeta_{02} = \frac{\iota}{4\sin(\psi)} (\overline{\lambda}^2 - \overline{\lambda} e \lambda_1^i \lambda_2^j).$$

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Proof. It follows from (1) and (2) that, along $L_{i,j}$, the map $H_{i,j}$ can be written as

$$H_{i,j}: \begin{array}{l} \overline{x} = y, \\ \overline{y} = a - bx - y^2 + e\lambda_1^i\lambda_2^j xy + fh\lambda_1^i\lambda_2^j(1 + \cdots)y^3 + R_{i,j}. \end{array} \tag{3}$$

From the fact that the trace of $DH_{i,j}(P)$ equals $2\cos(\psi)$, we find that

$$p = -\cos(\psi)\left(1 + \frac{e}{2}\lambda_1^i\lambda_2^j - \frac{3fh}{2}\lambda_1^i\lambda_2^j\cos(\psi)\right) + o(\lambda_1^i\lambda_2^j).$$

Let $\eta = x - p$ and $\zeta = y - p$ be new coordinates around the fixed point P. If $H_{i,j}(\eta, \zeta)$ denotes the map $H_{i,j}$ in these coordinates, then along $L_{i,j}$ we have

$$\begin{array}{l} \overline{\eta} = \zeta, \\ \overline{\zeta} = -\eta + 2\cos(\psi)\zeta - (1 + 3fh\cos(\psi)\lambda_1^i\lambda_2^j)\zeta^2 + e\lambda_1^i\lambda_2^j\eta\zeta + fh\lambda_1^i\lambda_2^j\zeta^3 + o(\lambda_1^i\lambda_2^j). \end{array}$$

This last map is smoothly conjugated to the map

$$\begin{array}{l} \overline{\eta} = \zeta, \\ \overline{\zeta} = -\eta + 2\cos(\psi)\zeta - \zeta^2 + e\lambda_1^i\lambda_2^j\eta\zeta + fh\lambda_1^i\lambda_2^j\zeta^3 + o(\lambda_1^i\lambda_2^j). \end{array}$$

Now the linear change of coordinates

$$S_{\mathbb{R}} = \begin{pmatrix} 1 & 0 \\ \cos(\psi) & -\sin(\psi) \end{pmatrix}$$

brings $DH_{i,j}(0, 0)$ into the real Jordan canonical form. So in the new coordinates given by $S_{\mathbb{R}}(\overline{x}, \overline{y})^t = (\eta, \zeta)^t$ we get that (3) is transformed into

$$\begin{aligned} H_{i,j}(\overline{x}, \overline{y}) = {} & \begin{pmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{pmatrix}\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} - (\cos(\psi)\overline{x} - \sin(\psi)\overline{y})^2 S_{\mathbb{R}}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ & + e\lambda_1^i\lambda_2^j\overline{x}(\cos(\psi)\overline{x} - \sin(\psi)\overline{y})S_{\mathbb{R}}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ & + fh\lambda_1^i\lambda_2^j(\cos(\psi)\overline{x} - \sin(\psi)\overline{y})^3 S_{\mathbb{R}}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(\lambda_1^i\lambda_2^j). \end{aligned}$$

Let $z = \overline{x} + \iota\overline{y}$, $\overline{z} = \overline{x} - \iota\overline{y}$ and $\lambda = e^{\iota\psi}$. Since

$$S_{\mathbb{R}}^{-1} = \begin{pmatrix} 1 & 0 \\ \cos(\psi)/\sin(\psi) & -1/\sin(\psi) \end{pmatrix},$$

we finally get the normal form required in Theorem 4 for $H_{i,j}(z, \overline{z})$:

$$\begin{aligned} H_{i,j}(z, \overline{z}) = {} & \lambda z + \frac{\iota}{\sin(\psi)}\left(\cos(\psi)\left(\frac{z + \overline{z}}{2}\right) - \sin(\psi)\left(\frac{z - \overline{z}}{2\iota}\right)\right)^2 \\ & - \frac{e\lambda_1^i\lambda_2^j\iota}{\sin(\psi)}\left(\cos(\psi)\left(\frac{z + \overline{z}}{2}\right) - \sin(\psi)\left(\frac{z - \overline{z}}{2\iota}\right)\right)\left(\frac{z + \overline{z}}{2}\right) \\ & - \frac{fh\lambda_1^i\lambda_2^j\iota}{\sin(\psi)}\left(\cos(\psi)\left(\frac{z + \overline{z}}{2}\right) - \sin(\psi)\left(\frac{z - \overline{z}}{2\iota}\right)\right)^3 \\ & + o(\lambda_1^i\lambda_2^j) \\ = {} & \lambda z + \zeta_{20}z^2 + \zeta_{11}|z|^2 + \zeta_{02}\overline{z}^2 + \zeta_{21}z|z|^2 + \cdots, \end{aligned}$$

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where $\cdots$ stands for terms of degree three different from $z|z|^2$ or higher, and

$$\zeta_{21} = -3\lambda \frac{fh\lambda_1^i\lambda_2^j}{\sin(\psi)}\iota,$$
$$\zeta_{20} = \frac{\iota}{4\sin(\psi)}(\lambda^2 - \lambda e\lambda_1^i\lambda_2^j),$$
$$\zeta_{11} = \frac{\iota}{4\sin(\psi)}(2 - (\bar{\lambda} + \lambda)e\lambda_1^i\lambda_2^j),$$
$$\zeta_{02} = \frac{\iota}{4\sin(\psi)}(\bar{\lambda}^2 - \bar{\lambda}e\lambda_1^i\lambda_2^j).$$

LEMMA 10. *The $H_{i,j}$ satisfy HBT.*

Proof. (1) From the second equation of (2) we get $\partial_b|\lambda| \neq 0$.

(2) $\lambda^i \neq 1$, for $i = 1, 2, 3, 4, 5$ except for a finite number of parameters, since the parametrization of $L_{i,j}$ in terms of ψ is one-to-one.

(3) We calculate c as follows:

$$\begin{aligned}
&\zeta_{21} + \frac{2|\zeta_{02}|^2}{\lambda^2 - \bar{\lambda}} + \frac{|\zeta_{11}|^2}{1 - \bar{\lambda}} + \frac{\zeta_{11}\zeta_{20}(1 - 2\lambda)}{\lambda^2 - \lambda} \\
&\quad = -3\lambda \frac{fh\lambda_1^i\lambda_2^j}{\sin(\psi)}\iota + \frac{1}{4\sin^2(\psi)}\left(\frac{1}{2}\frac{1}{\lambda^2 - \bar{\lambda}} + \frac{1}{1 - \bar{\lambda}}\right)(1 - 2\cos(\psi)e\lambda_1^i\lambda_2^j) \\
&\qquad - \frac{\lambda}{8\sin^2(\psi)}\frac{1 - 2\lambda}{\lambda^2 - \lambda}(\lambda - (1 + \lambda\cos(\psi))e\lambda_1^i\lambda_2^j) + o(\lambda_1^i\lambda_2^j) = \alpha,
\end{aligned}$$

so

$$\begin{aligned}
\alpha\bar{\lambda} &= -3\frac{fh\lambda_1^i\lambda_2^j}{\sin(\psi)}\iota + \frac{1}{4\sin^2(\psi)}\left(\frac{1}{2}\frac{1}{\lambda^3 - 1} + \frac{1}{\lambda - 1}\right)(1 - 2\cos(\psi)e\lambda_1^i\lambda_2^j) \\
&\quad - \frac{1}{8\sin^2(\psi)}\frac{1 - 2\lambda}{\lambda^2 - \lambda}(\lambda - (1 + \lambda\cos(\psi))e\lambda_1^i\lambda_2^j) + o(\lambda_1^i\lambda_2^j)
\end{aligned}$$

and

$$\begin{aligned}
c = \mathrm{Re}(\alpha\bar{\lambda}) &= \frac{1}{8\sin^2(\psi)}\mathrm{Re}\left(\frac{1}{\lambda^3 - 1}\right)(1 - 2\cos(\psi)e\lambda_1^i\lambda_2^j) \\
&\quad + \frac{1}{4\sin^2(\psi)}\mathrm{Re}\left(\frac{1}{\lambda - 1}\right)(1 - 2\cos(\psi)e\lambda_1^i\lambda_2^j) - \frac{1}{8\sin^2(\psi)}\mathrm{Re}\left(\frac{1 - 2\lambda}{\lambda - 1}\right) \\
&\quad + \frac{1}{8\sin^2(\psi)}\mathrm{Re}\left(\frac{1 - 2\lambda}{\lambda^2 - \lambda}(1 + \lambda\cos(\psi))e\lambda_1^i\lambda_2^j\right) + o(\lambda_1^i\lambda_2^j) \\
&= \frac{3\cos(\psi)e\lambda_1^i\lambda_2^j}{8\sin^2(\psi)} + \frac{\cos(\psi)e\lambda_1^i\lambda_2^j}{8\sin^2(\psi)}\left(1 + \frac{\cos(2\psi) - \cos(\psi)}{2(1 - \cos(\psi))}\right) \\
&\quad - \frac{3\cos(\psi)e\lambda_1^i\lambda_2^j}{16\sin^2(\psi)} + o(\lambda_1^i\lambda_2^j) \\
&= \frac{\cos(\psi)e\lambda_1^i\lambda_2^j}{8\sin^2(\psi)}\left(\frac{3}{2}\cos(\psi) + 1 + \frac{\cos(2\psi) - \cos(\psi)}{2(1 - \cos(\psi))}\right) \\
&= \frac{e\lambda_1^i\lambda_2^j}{16(1 - \cos(\psi))} + o(\lambda_1^i\lambda_2^j).
\end{aligned}$$

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This formula is presented in [**GSS05**] without proof. Its proof is included for the sake of completeness.

(4) θ_2 is given by

$$\theta_2 = \frac{1}{2\pi}\left(\mathrm{Im}(\lambda_1) - \mathrm{Im}(\alpha\bar{\lambda})\frac{\mathrm{Re}(\lambda_1)}{\mathrm{Re}(\alpha\bar{\lambda})}\right), \tag{4}$$

where $\lambda_1(a, b)$, for (a, b) on the curve $L_{i,j}$, is defined by $\bar{\lambda}\partial_b\lambda(0)$.

From (1) we get

$$DH_{i,j} = \begin{pmatrix} 0 & 1 \\ \partial_x l_{i,j} & \partial_y l_{i,j} \end{pmatrix},$$

so its characteristic polynomial is $X^2 - \partial_y l_{i,j} X - \partial_x l_{i,j}$. We can conclude that

$$\partial_b\lambda = \frac{\partial^2_{bx} l_{i,j} + \partial^2_{by} l_{i,j}\lambda}{2\lambda - \partial_y l_{i,j}}.$$

Thus, for parameters (a, b) on $L_{i,j}$, we get

$$\begin{aligned}\partial_b\lambda &= \frac{-1 + o(\lambda_1^i\lambda_2^j)}{2\lambda + 2p - e\lambda_1^i\lambda_2^j p - 3f\gamma_1^{-i}\gamma_2^{-j}p^2 - \partial_y R_{i,j}} \\ &= \frac{-1 + o(\lambda_1^i\lambda_2^j)}{2i\,\sin(\psi) + o(\lambda_1^i\lambda_2^j)} \\ &= \frac{i}{2\,\sin(\psi)} + o(\lambda_1^i\lambda_2^j).\end{aligned}$$

Thus, we have

$$\lambda_1 = \frac{i\bar{\lambda}}{2\,\sin(\psi)} + o(\lambda_1^i\lambda_2^j).$$

Also

$$\mathrm{Im}(\alpha\bar{\lambda}) = \frac{1}{2\,\sin^3(\psi/2)\cos(\psi/2)(1 - \cos(\psi))} + o(\lambda_1^i\lambda_2^j).$$

Putting together these terms in (4) gives

$$\theta_2 \simeq -\frac{4}{e\,\sin^3(\psi/2)\cos(\psi/2)}\frac{1}{\lambda_1^i\lambda_2^j}. \tag{5}$$

Finally, since we already know that $H_{i,j}$ are in the conditions of HBT and, from (5) we get the following corollary.

COROLLARY 11. *Given a small square Δ above the line $b = 1$ in the parameter space, for i, j big enough, there exists a prolongation $\Delta_{i,j}$ of this square such that the following statement holds: the set $E_{i,j}$ of parameters $(a, b) \in \Delta_{i,j}$, for which there exists an irrational invariant curve C^1 conjugated to an irrational rotation, satisfy*

$$m(E_{i,j}) \geq \frac{95}{100}m(\Delta_{i,j}).$$

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Remark 2. For $\epsilon > 0$, if $\widetilde{H}$ is near enough in the C^l topology to some $H_{i,j}$ as above, we can conclude that there exist a set $\Delta(H)$ with the property that

$$m(E(H)) \geq \left(\frac{95}{100} - \epsilon\right) m(\Delta_{i,j})$$

where $E(H) \subset \Delta(H)$ is the analogous set to $E_{i,j}$ as above.

Proposition 3 and Corollary 11 allow us to conclude the following result.

THEOREM 12. *Let f be a C^l diffeomorphism, $l \geq 7$, with a dissipative-contractive non-transversal heteroclinic 2-cycle. Let $f_{(\mu_1,\mu_2)}$ be a two-parameter family with $f_{0,0} = f$ satisfying the condition above. Then there is a compact set E in the parameter space (μ_1, μ_2) with $m(E) > 0$, where m denote the Lebesgue measure, such that for all $\mu = (\mu_1, \mu_2) \in E$ the corresponding diffeomorphism f_μ exhibits a periodic irrational invariant curve C^1 conjugated to an irrational rotation on $\mathbb{S}^1$. The curve is of class C^l.*

6. *Proof of the main result*

The proof of the main result follows closely the strategy implemented by Colli in [**Col98**]. So we only recall the parts of that strategy when these are different from that in [**Col98**].

6.1. *Special perturbation.*

We first consider the following result.

PROPOSITION 13. *Let p be a dissipative periodic saddle point for f, a C^l diffeomorphism, with eigenvalues λ and γ and such that $W^s(p)$ and $W^u(p)$ have a point of transversal intersection. Then, for any $\epsilon > 0$, the point p belongs to a hyperbolic set $\Lambda = \Lambda(\epsilon)$ which satisfies the following condition: if $x \in \Lambda$ is a periodic point of period k, then the eigenvalues of $Df^k(x)$ are between $(\lambda - \epsilon)^k$ and $(\lambda + \epsilon)^k$ and between $(\gamma - \epsilon)^k$ and $(\gamma + \epsilon)^k$.*

Proof. This follows from [**Col98**, Proposition 3.4, Corollary 3.5]. □

Let us consider in this section a diffeomorphism $f \in \mathrm{Diff}^l(M)$, with M a compact surface, which has a dissipative-contractive non-transversal heteroclinic 2-cycle. Following ideas from [**PT93**], we can find a diffeomorphism g arbitrarily near to f such that:

(1) there exist Q_{1g} and Q_{2g}, the continuations of Q_1 and Q_2 respectively, satisfying the condition that the manifolds $W^s(Q_{1g})$ and $W^u(Q_{2g})$ and the manifolds $W^u(Q_{2g})$ and $W^s(Q_{1g})$ have a transverse intersection;
(2) the manifolds $W^u(Q_{2g})$ and $W^s(Q_{1g})$ have a quadratic tangency;
(3) $\sigma_{1g} < 1 < \sigma_{2g}$ where $\sigma_{ig} = |\lambda_{ig} \cdot \gamma_{ig}|$ and λ_{ig} and γ_{ig} are the contractive and expanding eigenvalues of $\mathrm{D}f_{Q_{ig}}$, $i = 1, 2$.

From item (1) above, there is a point of transversal intersection between $W^s(Q_{2g})$ and $W^u(Q_{2g})$. Then, by Proposition 13, Q_{2g} belongs to a hyperbolic set $\Lambda_2(g)$ such that if $x \in \Lambda_2(g)$ is a periodic point of period k, the eigenvalues of $Df^k(x)$, λ^k and γ^k satisfy $\sigma = |\lambda \cdot \gamma| > 1$.

Now, from items (1) and (2) above we can consider a new diffeomorphism g_1 arbitrarily close to g that satisfies the same items above except for item (2). Instead of (2), the following should be satisfied:

(2)′ the manifolds $W^u(Q_{1g_1})$ and $W^s(Q_{1g_1})$ have a quadratic tangency.

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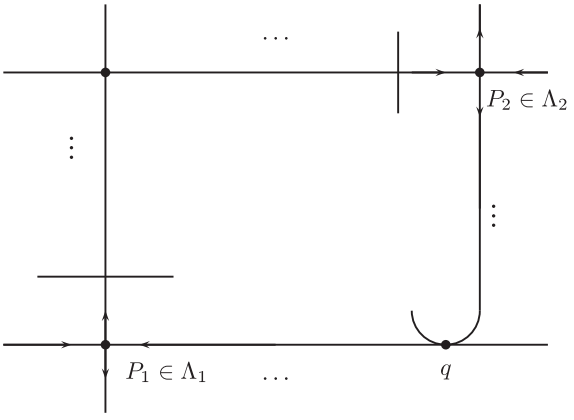


FIGURE 2.

Using ideas from [**PT93**, Ch. 3 and 6], we have that there exists a diffeomorphism f_0, arbitrarily near to g_1, satisfying the following properties:

- f_0 has a hyperbolic set $\Lambda_1(f_0)$ and a periodic point $Q_{0f_0} \in \Lambda_1(f_0)$ such that the stable thickness of $\Lambda_1(f_0)$ is big, in particular $\tau^s(\Lambda_1) \cdot \tau^u(\Lambda_2) > 1$;
- the manifolds $W^s(Q_{1f_0})$ and $W^u(Q_{2g})$ have a transverse intersection, while $W^u(Q_{1f_0})$ and $W^s(Q_{2g})$ have a quadratic tangency;
- if $x \in \Lambda_{1f_0}$ is a periodic point of period k, the eigenvalues of $Df_0^k(x)$, λ^k and γ^k satisfy $\sigma = |\lambda \cdot \gamma| < 1$ (see [**Col98**, Ch. 3]).

We conclude this section with the following result which is a summary of this section.

THEOREM 14. *Let $f \in \mathrm{Diff}^l(M)$ be a diffeomorphism C^l, with M a compact surface and $l \geq 2$, which has a dissipative-contractive non-transversal heteroclinic 2-cycle. Then, there exists a C^r diffeomorphism f_0 arbitrarily near f such that:*

(a) *f_0 has hyperbolic basic set Λ_1 and Λ_2 with $\tau^s(\Lambda_1) \cdot \tau^u(\Lambda_2) > 1$;*

(b) *there are periodic points $P_1 \in \Lambda_1$ with period k_1 and $P_2 \in \Lambda_2$ with period k_2 such that $W^u(P_1)$ has a transversal intersection with $W^s(P_2)$ and $W^u(P_2)$ has quadratic tangency with $W^s(P_1)$ at a point q;*

(c) *$\sigma_1(P_1) < 1 < \sigma_2(P_2)$, where $\sigma_i = |\lambda_i \gamma_i|$ and $\lambda_i = \lambda_i(P_i)$, $\gamma_i = \gamma_i(P_i)$ are the contractive and expansive eigenvalue of $Df_0^{k_i}(P_i)$, for all $P_i \in \Lambda_i$ periodic points of period k_i, $i = 1, 2$.*

Note that f_0 given in the theorem above has a non-transversal heteroclinic cycle associated with two periodic points, $P_1 \in \Lambda_1$ and $P_2 \in \Lambda_2$. See Figure 2.

6.2. *Making a 2-cycle fulfilling a generic condition.* In this section we show how to perturb the heteroclinic 2-cycles as above in order to fulfill the conditions of the renormalization, so that we can obtain invariant curves. Let the heteroclinic 2-cycle under consideration be formed by two periodic points Q_1 and Q_2 of period k_1 and k_2 respectively, with λ_i and γ_i the respective contractive and expanding eigenvalues of $Df_{(\mu_1,\mu_2)}^{k_i}(Q_i)$

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for $i = 1, 2$. Let $\sigma_i = \lambda_i \gamma_i$, so the condition which we have in mind is

$$\partial_{\mu_2}\left(\frac{\ln(\sigma_2(\mu))}{-\ln(\sigma_1(\mu))}\right) \neq 0. \tag{6}$$

In general this condition can be obtained generically, but we require that the perturbation that achieves this keeps the dynamical context fixed. We will show that this is possible. Let W_i be neighborhoods of Q_i such that $f^j_{(\mu_1,\mu_2)}(W_i) \cap W_i \neq \emptyset$ for $j = 1, \ldots, k_i - 1$. Let ζ be a C^∞ function on $\mathbb{R}$ satisfying

$$\begin{cases} \zeta(s) = 0, & s \geq 2 \\ \zeta(s) = 1, & s \leq 1 \\ 0 \leq \zeta \leq 1, & \forall s. \end{cases}$$

Let $\phi_i : W_i \to \mathbb{R}$ be a local chart defined on W_i with $\phi_i(Q_i) = 0$, and $B_\alpha \subset \phi_i(W_i)$ be a ball of radius α centered at the origin and $\widetilde{\zeta}_i(x) = \zeta(4\|x\|/\alpha)$. The perturbation $f_{(t,\mu_1,\mu_2)}$ of the family $f_{(\mu_1,\mu_2)}$ is defined as $\iota_t \circ f_{(t,\mu_1,\mu_2)}$, where

$$\begin{cases} \iota_t = x, & \text{if } x \in M \backslash (W_1 \cup W_2) \\ \iota_t(x) = \phi_1^{-1}((1 + t\widetilde{\zeta}(|\phi_1|))\phi_1(x)), & \text{if } x \in W_1 \\ \iota_t(x) = \phi_2^{-1}((1 + t\widetilde{\zeta}(|\phi_2|))\phi_2(x)), & \text{if } x \in W_2. \end{cases}$$

Now, the eigenvalues of $Df^{k_i}_{(t,\mu_1,\mu_2)}(Q_i)$ are $(1 + t)\lambda_i$ and $(1 + t)\gamma_i$. If we let $\sigma_i = \sigma_{t,\mu_1,\mu_2}$, then

$$\begin{aligned} \partial_{\mu_2}\left(-\frac{\ln(\sigma_1)}{\ln(\sigma_2)}\right) &= \partial_{\mu_2}\left(-\frac{2\ln(1+t) + \ln(\sigma_1)}{2\ln(1+t) + \ln(\sigma_2)}\right) \\ &= -\frac{(2\ln(1+t) + \ln(\sigma_2))\partial_{\mu_2}\ln(\sigma_1) - (2\ln(1+t) + \ln(\sigma_1))\partial_{\mu_2}\ln(\sigma_2)}{(2\ln(1+t) + \ln(\sigma_2))^2}. \end{aligned} \tag{7}$$

If this expression vanishes then

$$\frac{(2\ln(1+t) + \ln(\sigma_2))}{(2\ln(1+t) + \ln(\sigma_1))} = \frac{\partial_{\mu_2}\ln(\sigma_1)}{\partial_{\mu_2}\ln(\sigma_2)},$$

but this equality can be broken for t small enough. So we can perturb the family $f_{(\mu_1,\mu_2)}$ in such a way that (6) is fulfilled and without destroying the dynamics outside $f^j_{(\mu_1,\mu_2)}(W_i)$ with $i = 1, 2$ and $j = 1, \ldots k_i - 1$.

6.3. *Fixing notation and control of the orbits.* In this section we recall the construction in [**Col98**]. Let $f \in \mathrm{Diff}^l(M)$ be a diffeomorphism, with M a compact surface and $l \geq 7$, which has a dissipative-contractive non-transversal heteroclinic 2-cycle. Then, by Theorem 14, there exists $f_0 \in \mathrm{Diff}^l(M)$ arbitrarily close to f, exhibiting hyperbolic basic sets Λ_1, Λ_2 and periodic points $P_1 \in \Lambda_1$ and $P_2 \in \Lambda_2$ satisfying items (a) to (c). Let $\mathcal{U}$ be a small neighborhood of f_0 yielding hyperbolic continuations of Λ_1 and Λ_2.

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We denote by r_0 the point of transversal intersection between $W^s(P_1, f_0)$ and $W^u(P_2, f_0)$. Let q be the point of quadratic tangency between $W^u(P_1)$ and $W^s(P_2)$ and U be a small neighborhood of q. It is possible to take $\delta > 0$, $\mathcal{U}$ and U sufficiently small such that there is a tangency line l_ψ inside U (see [**Col98**, §6.3]), for all $\psi \in \mathcal{U}$. Moreover, all possible tangencies in l_ψ between $W^u(x)$ and $W^s(y)$ are quadratic, for $x \in B(P_1, \delta) \cap \Lambda_1$ and $y \in B(P_2, \delta) \cap \Lambda_2$; see [**Col98**, §6.1]. And δ, U and $\mathcal{U}$ can be taken such that the orbit of every periodic invariant circle meets U only once, so any perturbation done inside U but outside a neighborhood of the periodic invariant circle does not affect the periodic invariant curve.

Suppose that there are periodic points $q_1 \in B(P_1, \delta) \cap \Lambda_1$ and $q_2 \in B(P_2, \delta) \cap \Lambda_2$ such that for $\psi \in \mathcal{U}$, $W^u(q_1, \psi)$ and $W^s(q_2, \psi)$ have a (quadratic) tangency q inside U. Now, consider a two parameter family $f_{(\mu_1,\mu_2)}$ such that $f_{(0,0)} = \psi$. We choose as the parameter μ_1 that which controls the splitting of the manifolds $W^s(q_2)$ and $W^u(q_1)$ along the orbit of the point q. As for the second parameter, we choose that which controls the evolution of the ratio $-\ln(\sigma_2)/\ln(\sigma_1)$ so that

$$\mu_2 = \frac{\ln(\sigma_2(\mu))}{-\ln(\sigma_1(\mu))} - \frac{\ln(\sigma_2(0))}{-\ln(\sigma_1(0))}.$$

Then, by Theorem 12, there exist a sequence of connected compact sets $\Delta_n \to (0, 0)$ in the parameter space (μ_1, μ_2), a sequence of subsets $E_n \subset \Delta_n$ with $m(E_n) > 0$ and a sequence of integers $k_n \to \infty$ as $n \to \infty$ such that for $(\mu_1, \mu_2) \in E_n$, the diffeomorphisms $f^{k_n}_{\mu_1,\mu_2}$ have an irrational invariant curve $S_n = S_n(\mu_1, \mu_2)$ inside U, and S_n is topologically conjugate to irrational rotation on S^1. It is possible, as was observed above, to take U, δ and $\mathcal{U}$ sufficiently small such that

$$\{f_\mu(S_n), f^2_\mu(S_n), \ldots, f^{k_n-1}_\mu(S_n)\} \cap U = \emptyset.$$

This implies that any perturbation done inside U but outside a neighborhood of S_n does not affect the periodic invariant curve $\bigcup_0^{k_n-1} f^i_\mu(S_n)$. The proof of this fact is the same as in [**Col98**, §6.2].

6.4. *Final argument.* In this section we follow ideas and some notation from [**Col98**]. We consider a diffeomorphism f_0 with the condition given by Theorem 14, with periodic points $P_1 \in \Lambda_1(f_0)$ and $P_2 \in \Lambda_2(f_0)$ such that $W^u(P_2)$ and $W^s(P_1)$ are tangent (quadratically) inside U. We take U, δ and $\mathcal{U}$ sufficiently small as in the previous section. Then, for each $f \in \mathcal{U}$ there is a C^1 line of tangency l_f inside U corresponding to the unstable leaves of $\Lambda_1(f)$ and stable leaves of $\Lambda_2(f)$; see [**PT93**].

Let $\pi^s_f : W^s_{\text{loc}}(P_1, f) \to l_f$ (respectively $\pi^u_f : W^u(P_2, f) \to l_f$) be the C^1 projection along unstable (respectively stable) leaves of $\Lambda_1(f)$ (respectively $\Lambda_2(f)$). Let $\mathcal{X}_f : l_f \to \mathbb{R}$ be a C^1 chart of the line of tangency. Consider also C^r parametrization $\mathcal{L}^s_f : W^s_{\text{loc}}(P_1, f) \to \mathbb{R}$ and $\mathcal{L}^u_f : W^u_{\text{loc}}(P_2, f) \to \mathbb{R}$ with $\mathcal{L}^s(P_1) = \mathcal{L}^u_f(P_2) = 0$. Consider the dynamically defined Cantor sets

$$K^s_f = \mathcal{L}^s_f(W^s_f(P_1, f) \cap \Lambda_1(f))$$

and

$$K^u_f = \mathcal{L}^u_f(W^u_f(P_2, f) \cap \Lambda_2(f)).$$

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It follows, from Theorem 14, that there is some $t > 0$ such that

$$\tau_{\mathrm{loc}}(K^s_{f_0}) \cdot \tau_{\mathrm{loc}}(K^u_{f_0}) \geq 1 + t.$$

The definition of local thickness and the continuity of thickness imply that for $\mathcal{U}$ small enough, there is $\epsilon_0 > 0$ sufficiently small such that for any $\epsilon_0 > \epsilon > 0$ we can find Cantor sets $\widetilde{K}^u_{f_0} \subset K^u_{f_0} \cap B(0, \epsilon)$ and $\widetilde{K}^s_{f_0} \subset K^s_{f_0} \cap B(0, \epsilon)$ whose continuation $\widetilde{K}^u_f$ and $\widetilde{K}^s_f$ satisfy

$$\tau_{\mathrm{loc}}(\widetilde{K}^u_f) \cdot \tau_{\mathrm{loc}}(\widetilde{K}^s_f) \geq 1 + \frac{3t}{4}, \quad \forall f \in \mathcal{U}. \tag{8}$$

Now we define the functions $\vartheta^u_f : K^u_{f_0} \longrightarrow \mathbb{R}$ with

$$\vartheta^u_f(x) = \mathcal{X}_f \circ \pi^u_f \circ \Phi_1(f) \circ (\mathcal{L}^u_f)^{-1}(x)$$

and $\vartheta^s_f : K^s_{f_0} \longrightarrow \mathbb{R}$ with

$$\vartheta^s_f(x) = \mathcal{X}_f \circ \pi^s_f \circ \Phi_2(f) \circ (\mathcal{L}^s_f)^{-1}(x).$$

Here the maps Φ_i satisfy $\Lambda_i(f) = \Phi_i(f)(\Lambda_i(f_0))$; see [**Col98**, §6.1] for a complete definition. Consider the Cantor sets $\widetilde{K}^u_{f_0} \subset K^u_{f_0} \cap B(0, \epsilon)$ and $\widetilde{K}^s_{f_0} \subset K^s_{f_0} \cap B(0, \epsilon)$ introduced above; for any $\epsilon_0 > \epsilon > 0$ and $f \in \mathcal{U}$, with ϵ_0 and $\mathcal{U}$ small, we have, by (8), that

$$\tau(\vartheta^s_f(\widetilde{K}^s_{f_0})) \cdot \tau(\vartheta^u_f(\widetilde{K}^u_{f_0})) \geq 1 + \frac{t}{2}.$$

Then, by Proposition 2 there is $\hat{f} \in \mathcal{U}$ arbitrarily near f_0 such that the pair $\langle \vartheta^s_{\hat{f}}(\widetilde{K}^s_{f_0}), \vartheta^u_{\hat{f}}(\widetilde{K}^u_{f_0}) \rangle$ is stably linked.

Consider a two-parameter family of diffeomorphisms $F = (f_{(\mu_1,\mu_2)})_\mu$ inside $\mathcal{U}$ with $f_{(0,0)} = \hat{f}$. Here $\mu = (\mu_1, \mu_2) \in (-1, 1) \times (-1, 1)$. Let $\mathcal{Z}$ be a small neighborhood of F in the space of two-parameter families of C^l diffeomorphisms, and let $\eta > 0$ be sufficiently small that for each family $G = (g_\mu)_\mu \in \mathcal{Z}$, and defining the Cantor sets $K^s(G; (\mu)) = \vartheta^s_{g_\mu}(\widetilde{K}^s_{f_0})$ and $K^u(G; (\mu_1, \mu_2)) = \vartheta^u_{g_\mu}(\widetilde{K}^u_{f_0})$, we have that $\langle K^s(G; (\mu)), K^u(G; (\mu)) \rangle$ is a stably linked pair, for all $\mu \in B(0, \eta)$.

We define

$$\mathcal{W} = \{g_\mu \in \mathcal{U} : G = (g_\mu)_\mu \in \mathcal{Z} \text{ and } \mu \in B(0, \eta)\},$$

which is an open set by the openness of the linking property. Observe that $\mathcal{W}$ is what is usually called an *open set of persistence of tangencies*. By construction, $\mathcal{W}$ is arbitrarily close to the diffeomorphism f_0.

LEMMA 15. (Main lemma) *Let $B \subset B(0, \eta)$ be a ball. Then, there exist a residual subset $\mathcal{R}$ of $\mathcal{Z}$ such that, for each family $G = (g_\mu)_\mu \in \mathcal{R}$, there is a parameter value $\mu^* = (\mu^*_1, \mu^*_2) \in B$ such that the corresponding map g_{μ^*} exhibits infinitely many periodic irrational invariant curves.*

Proof. We follow closely the proof of the claim in [**Col98**, p. 572]. Consider a sequence of subsets $\mathcal{R}_N$ of $\mathcal{Z}$ with the following properties:

(1) $\mathcal{Z} \supset \mathcal{R}_1 \supset \cdots \supset \mathcal{R}_N \supset \cdots$.

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(2) For all $N \geq 1$ and each family $G = (G_\mu) \in \mathcal{R}_N$, there exists a compact set $E_N = E_N(G) \subset B$, $m(E_N) > 0$, such that for $\mu \in E_N$, g_μ has N distinct irrational invariant curves $S_1 = S_1(G; \mu), \ldots, S_N = S_N(G; \mu)$. Furthermore:
   (a) for all $i = 1, \ldots, N$, the invariant curve S_i is generated by renormalization (see Theorem 12) and the orbit of S_i intersects U only once, inside B_{r_i}, where $B_{r_i} \subset U$ is a ball of radius r_i, and $B_{r_i} \cap B_{r_j} = \emptyset$ for all $i \neq j$;
   (b) $E_{N+1}(G) \subset E_N(G)$, for $N \geq 1$.

(3) For each $G \in \mathcal{R}_N$ and (μ_1, μ_2) in a neighborhood of the convex hull of $E_N(G)$, there are bridges P_i^s of $\widetilde{K}_f^s$ and P_i^u of $\widetilde{K}_f^u$, $i = 1, \ldots, N$, such that:
   (a) their images $P_i^s(G; \mu) \equiv \vartheta_{g_\mu}^s(P_i^s)$ and $P_i^u(G; \mu) \equiv \vartheta_{g_\mu}^u(P_i^u)$ form a stably linked pair;
   (b) their images on the line of tangency

$$\widetilde{P}_i^s(G; \mu) \equiv \chi_{g_\mu}^{-1}(P_i^s(G; \mu))$$

   and

$$\widetilde{P}_i^u(G; \mu) \equiv \chi_{g_\mu}^{-1}(P_i^u(G; \mu))$$

   satisfy

$$\widetilde{P}_i^s(G; \mu) \cap \widetilde{P}_i^u(G; \mu) \subset B_{r_i}.$$

(4) For each $G \in \mathcal{R}_N$ and μ in a neighborhood of the convex hull of $E_N(G)$ there are bridges Q_N^s of $\widetilde{K}_f^s$ and Q_N^u of $\widetilde{K}_f^u$ such that:
   (a) their images $Q_N^s(G; \mu) \equiv \vartheta_{g_\mu}^s(Q_N^s)$ and $Q_N^u(G; \mu) \equiv \vartheta_{g_\mu}^u(Q_N^u)$ form a linked pair;
   (b) their images on the line of tangency

$$\widetilde{Q}_N^s(G; \mu) \equiv \chi_{g_\mu}^{-1}(Q_N^s(G; \mu))$$

   and

$$\widetilde{Q}_N^u(G; \mu) \equiv \chi_{g_\mu}^{-1}(Q_N^u(G; \mu))$$

   satisfy

$$\widetilde{Q}_i^s(G; \mu) \cap \widetilde{Q}_i^u(G; \mu) \subset B_\epsilon,$$

   where $B_\epsilon \subset U$ is a ball of radius ϵ and $B_\epsilon \cap B_{r_i} = \emptyset$, for any $i = 1, \ldots, N$.

We prove that the sets $\mathcal{R}_{N+1}$ are open and dense in $\mathcal{R}_N$. From here, it follows that $\mathcal{R} = \bigcap_{N \geq 1} \mathcal{R}_N$ is a dense set with the property that for each $G = (g_\mu) \in \mathcal{R}$ there exist a nested sequence $B \supset E_1(G) \supset E_2(G) \cdots \supset E_N(G) \supset \cdots$ of compact sets as in item (1) of the induction. So, if $\mu^* = (\mu_1^*, \mu_2^*) \in \bigcap_{N \geq 1} E_N(G) \subset B$, then g_{μ^*} has infinitely many irrational invariant curves.

The sets $\mathcal{R}_{N+1}$ are easily seen to be open sets by Remark 2. In order to show its denseness in $\mathcal{R}_N$, for a family $G \in \mathcal{R}_N$ we do four successive perturbations of G, each one as small as required, so that we finally get a perturbation $H \in \mathcal{R}_{N+1}$ arbitrarily near to G. To begin with, let $E_0(G) = B$, Q_0^s be the convex hull of $\widetilde{K}_f^s$, and Q_0^u the convex hull of $\widetilde{K}_f^u$. Let $\mu^N \in E_N$ be a density point, i.e.,

$$\lim_{\rho \to 0} \frac{m(E_N \cap B_\rho(\mu^N))}{\pi \rho^2} = 1.$$

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First perturbation. As in [**Col98**] we perturb G to $G^{(1)}$ in order to get two sublinks

$$\langle P^s_{N+1}(G^{(1)};\mu^N),\ P^u_{N+1}(G^{(1)};\mu^N)\rangle,$$
$$\langle Q^s_{N+1}(G^{(1)};\mu^N),\ Q^u_{N+1}(G^{(1)};\mu^N)\rangle$$

of $\langle Q^s_N(G^{(1)};\mu^N),\ Q^u_N(G^{(1)};\mu^N)\rangle$, where

$$P^{s,u}_{N+1}(G^{(1)};\mu) \equiv \vartheta^{s,u}_{g_\mu}(P^{s,u}_{N+1}),$$
$$Q^{s,u}_{N+1}(G^{(1)};\mu) \equiv \vartheta^{s,u}_{g_\mu}(Q^{s,u}_{N+1})$$

for some bridges $P^{s,u}_{N+1}$, $Q^{s,u}_{N+1}$ of $\widetilde{K}^{s,u}_{G^{(1)}}$ respectively. Let $B_{r_{N+1}} \subset B_\epsilon$ be a ball of radius r_{N+1} containing

$$\widetilde{P}^s_{N+1}(G^{(1)};\mu^N) \cap \widetilde{P}^u_{N+1}(G^{(1)};\mu^N)$$

and $B_{\epsilon'} \subset B_\epsilon$ be a ball containing

$$\widetilde{Q}^s_{N+1}(G^{(1)};\mu^N) \cap \widetilde{Q}^u_{N+1}(G^{(1)};\mu^N) \quad \text{with } B_{\epsilon'} \cap B_{r_{N+1}} = \emptyset.$$

This perturbation given here (see [**Col98**]) is of the following form. First we take coordinates around the tangency line l_f as is done in [**Col98**, §6.1], so we state the perturbation around the point $(0,0)$ which corresponds to the point q. Let ζ be a bump function as in §6.2. Consider $\gamma_1 > 0$ to be less than half the distance from

$$\bigcup_{\mu\in E_N} \widetilde{Q}^s_N(G;\mu) \cap \widetilde{Q}^u_N(G;\mu)$$

to $\mathbb{R}^2\backslash B_\epsilon$. Let q_1 be the center of B_ϵ and let

$$\zeta_1(u,v) = \zeta\left(\frac{3}{\gamma_1}\cdot(\|(u,v)-q_1\| - (\epsilon-\gamma_1))\right).$$

Define, for small β, the C^l diffeomorphism

$$\begin{aligned} T_\beta : M &\longrightarrow M \\ (u,v) &\mapsto (u, v+\beta\zeta_1(u,v)) \quad \text{in } U \\ x &\mapsto x \quad \text{in } M\backslash U. \end{aligned}$$

Then, $G^{(1)} = T_{\beta_1}\circ G$ where $T_\beta \circ G$ denotes the family $(T_\beta \circ g_\mu)_\mu$ and β_1 is given by Proposition 2 (Linking lemma); see [**Col98**].

Second perturbation. Since the pair

$$\langle P^s_{N+1}(G^{(1)};\mu^N),\ P^u_{N+1}(G^{(1)};\mu^N)\rangle$$

is linked, there exists a tangency between a stable leaf of Λ_1 and an unstable leaf of Λ_2. The denseness of the periodic points in Λ_1 and Λ_2 implies the existence of two periodic points $p_1 \in \Lambda_1$ and $p_2 \in \Lambda_2$ near enough to Q_1 and Q_2 respectively. Now we perturb as in §6.2 in order to get a new family $G^{(2)}$, where condition (6) holds for p_1 and p_2. Observe that the perturbation does not destroy the link $\langle Q^s_{N+1}(G^{(1)};\mu^N),\ Q^u_{N+1}(G^{(1)};\mu^N)\rangle$ and the local unstable (stable) manifold of $p_{1(2)}$ for the families $G^{(1)}$ and $G^{(2)}$ are close enough to each other in U.

Third perturbation. This perturbation, which we call $G^{(3)}$, creates a tangency between $W^u(p_1)$ and $W^s(p_2)$, and it is localized in $B_{r_{N+1}-\gamma}$ where γ is small enough. Observe that

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for this perturbation μ^N continues to be a density point of $E_N(G^{(3)})$ since we do not destroy what we have until now.

Fourth perturbation. Let ζ be a bump function as above and α a small enough quantity. Let $\zeta_4 = \zeta((3/\alpha).(\|(u, v) - q_2\| - (r_{N+1} - \alpha)))$ and $\widetilde{T}_\beta : M \to M$ be the mapping given by

$$\begin{aligned}(u, v) &\mapsto (u, v + \beta\zeta_2(u, v)) \quad \text{in } U\\ x &\mapsto x \quad \text{in } M\backslash U.\end{aligned}$$

Consider the perturbation $G^{(4)} = (T_\beta \circ G^{(3)})$. As for $\mu = \mu^N$ and $\beta = 0$ the family $G^{(4)}$ has a homoclinic tangency associated with the periodic points p_1 and p_2, we have a surface in the parameter space (μ_1, μ_2, β) containing the point $(\mu^N, 0)$ such that if (μ_1, μ_2, β) belongs to this surface, then $G^{(4)}$ has a homoclinic tangency too. We assume, without loss of generality, that the tangent plane of this surface at $(\mu^N, 0)$ is transversal to the vector $(1, 1, 1)$. We assume that the neighborhood of μ^N is small enough that the distortion produced by the change of coordinates of the renormalization is almost null, so the ratio of measures is almost preserved. Then, by a continuity argument, we can find $\widetilde{\beta}$ such that the square $\Delta(\widetilde{\beta}) = \Delta_{i,j}$, with i and j large enough, provided by Corollary 11 applied to the family $\widetilde{T}_\beta \circ G^{(3)}$ is centered at μ^N, and the corresponding set of parameters $E(\widetilde{\beta}) = E_{i,j}$ satisfies $m(E(\widetilde{\beta})) \geq (92/100)m(\Delta(\widetilde{\beta}))$. As μ^N is a density point of $E_N(G^3)$, we know that, for small ρ,

$$m(E_N(\widetilde{T}_{\widetilde{\beta}} \circ G^{(3)}) \cap B_\rho(\mu^N)) = m(E_N(G^3) \cap B_\rho(\mu^N)) \geq \frac{\pi}{2}\rho^2.$$

Using this, and the fact that we can obtain the diameter of the square $\Delta(\widetilde{\beta})$ small enough, we get that

$$m(E_N(\widetilde{T}_{\widetilde{\beta}} \circ G^{(3)}) \cap E(\widetilde{\beta})) \geq \left(\frac{92}{100}\frac{2}{\pi} - \frac{1}{2}\right)m(B_\rho(\mu^N)) > 0.$$

So we put as the final perturbation $H = \widetilde{T}_{\widetilde{\beta}} \circ G^{(3)}$ and $E_{N+1} = E(\widetilde{\beta}) \cap E_N(\widetilde{T}_{\widetilde{\beta}} \circ G^{(3)})$. □

Proof of the main theorem. Let us consider a diffeomorphism $f \in \mathrm{Diff}^l(M)$, with M a compact surface, which has a dissipative-contractive non-transversal heteroclinic 2-cycle. Then, by Theorem 14, there is $f_0 \in \mathrm{Diff}^l(M)$ arbitrarily near to f satisfying conditions (a) to (c). As we saw in this section, there exists an open set of C^l diffeomorphisms $\mathcal{W} = \mathcal{W}(f_0)$ arbitrarily near to f_0 such that, by the main lemma above, every diffeomorphism ϕ in $\mathcal{W}(f_0)$ can be approximated by another diffeomorphism displaying infinitely many irrational invariant curves. Take $\mathcal{V}$ as the union of these open sets $\mathcal{W}(f_0)$ to obtain main theorem. □

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