Ergod. Th. & Dynam. Sys. (2017), **37**, 995–1008 © *Cambridge University Press, 2015*
doi:10.1017/etds.2015.63

Whirly 3-interval exchange transformations

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(*Received* 21 *May* 2014 *and accepted in revised form* 17 *June* 2015)

Abstract. Irreducible interval exchange transformations are studied with regard to the whirly property, a condition for a non-trivial spatial factor. A uniformly whirly transformation is defined and is further studied. An equivalent condition is introduced for the whirly transformation. We will prove that almost all 3-interval exchange transformations are whirly, using a combinatorics approach with application of the Rauzy–Veech induction. It is still an open question whether the whirly property is a generic property for m-interval exchange transformations for $m \geq 4$.

1. *Introduction*

Interval exchange transformations, as a set of important dynamical systems, have been actively studied for decades. We recall some of the key theorems, either the results or the methods of which are related to the current study or possible extensions of this paper in the future. The proof of the unique ergodic property of measure theoretical generic interval exchange transformations was achieved by Masur [**14**] and Veech [**18**] independently using geometric methods, and was proved later using mainly combinatorial methods by Boshernitzan [**2**]. Avila and Forni [**1**] showed that weak mixing is a measure theoretical generic property for irreducible m-interval exchange transformations ($m \geq 3$). Chaika [**4**] developed a general result showing that any ergodic transformation is disjoint with almost all interval exchange transformations. Chaika and Fickenscher [**5**] showed that topological mixing is a topologically residual property for interval exchange transformations.

The concept of whirly transformation was introduced and studied in Glasner and Weiss [**9**] and Glasner *et al* [**7**]. In Glasner and Weiss [**8**], Proposition 1.9 states that the near action (weak closure of all the powers) of a transformation admits no non-trivial spatial factors if and only if it is whirly. In §2, we recall the relevant notions and facts about interval exchange transformations and whirly transformation. A new notion introduced in this section is that of uniformly whirly transformation. In §3, we study the space of 3-interval exchange transformations and deduce facts about the visitation times of Rauzy–Veech induction. In §4, we complete the proof of the major theorem, which states that

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almost all 3-interval exchange transformations are whirly and thus admit no non-trivial spatial factor. To prove this main theorem, first we establish an equivalent definition of whirly transformation based on the assumption of ergodicity. Then the major key facts are deduced as Claims 1–3, which show the whirly property for the base of the Rohlin tower associated with the Veech induction map. Finally, we apply a density point argument to extend the property to arbitrary general non-null measurable sets.

2. *Background*

An interval exchange transformation perturbs the half-closed half-open subintervals of a half-closed half-open interval. The subintervals have lengths corresponding to the vector $\lambda = (\lambda_1, \ldots, \lambda_m)$, $\lambda_i > 0$, $1 \le i \le m$. All such vectors form a positive cone $\Lambda_m \subset R^m$. The subintervals are thus $[\beta_{i-1}, \beta_i)$, $1 \le i \le m$, with $\bigcup[\beta_{i-1}, \beta_i) = [0, |\lambda|)$, where

$$|\lambda| = \sum_{i=1}^{m} \lambda_i \quad \text{and} \quad \beta_i(\lambda) = \begin{cases} 0, & i = 0, \\ \sum_{j=1}^{i} \lambda_j, & 1 \le i \le m. \end{cases} \tag{2.1}$$

Let $\mathcal{G}_m$ be the group of m-permutations, and $\mathcal{G}_m^0$ be the subset of $\mathcal{G}_m$ which contains all the irreducible permutations on $\{1, 2, \ldots, m\}$. A permutation π is irreducible if and only if for any $1 \le k < m$, $\{1, 2, \ldots, k\} \ne \{\pi(1), \ldots, \pi(k)\}$ or, equivalently, $\sum_{j=1}^{k}(\pi(j) - j) > 0$ $(1 \le k < m)$. Given $\lambda \in \Lambda_m$, $\pi \in \mathcal{G}_m^0$, the corresponding interval exchange transformation is defined by

$$\begin{gathered} T_{\lambda,\pi}(x) = x - \beta_{i-1}(x) + \beta_{\pi i-1}(\lambda^\pi) \quad (x \in [\beta_{i-1}(\lambda),\ \beta_i(\lambda))), \\ \text{where } \lambda^\pi = (\lambda_{\pi^{-1}1}, \lambda_{\pi^{-1}2}, \ldots, \lambda_{\pi^{-1}m}). \end{gathered} \tag{2.2}$$

Obviously, $\beta_{\pi i-1}(\lambda^\pi) = \sum_{j=1}^{\pi i-1} \lambda_{\pi^{-1}j}$, and the transformation $T_{\lambda,\pi}$, which is also denoted by (λ, π), sends the ith interval to the $\pi(i)$th position.

In Keane [**12**], the infinite distinct orbits condition (i.d.o.c.) is raised for the sufficient condition of minimality: λ, π is said to satisfy the i.d.o.c. if:

(i) for any $0 \le i < m$, $\{T^k\beta_i, k \in \mathbb{Z}\}$ is an infinite set;

(ii) $\{T^k\beta_i, k \in \mathbb{Z}\} \cap \{T^k\beta_j, k \in \mathbb{Z}\} = \emptyset$ whenever $i \ne j$.

Suppose that $m > 1$, $(\lambda, \pi) \in \Lambda_m \times \mathcal{G}_m^*$, where $\mathcal{G}_m^*$ is the set of irreducible permutations with the property that $\pi(j+1) \ne \pi(j) + 1$ for all $1 \le j \le m-1$. Let I be an interval of the form $I = [\xi, \eta)$, $0 \le \xi < \eta \le |\lambda|$. Since T is defined on $[0, |\lambda|)$, and T is Lebesgue measure preserving, we know that Lebesgue almost all points of I return to I infinitely often under iteration of T. We use $T|_I$ to denote the induced transformation of T on I. By Veech [**16**], $T|_I$ is an interval exchange transformation with $m-2$, $m-1$ or m discontinuities.

Definition 2.1. (Admissible interval: [**15**, **16**]) Suppose that (λ, π) satisfies the i.d.o.c., and $I = (\xi, \eta)$, where $\xi = T^k\beta_s$ $(1 \le s < m)$, $\eta = T^l\beta_t$ $(1 \le t < m)$ and $\tau \in \{k, l\}$ have the following property: if $\tau \ge 0$, there is no j, $0 < j < \tau$, such that $T^j\beta_s \in I$; if $\tau < 0$, there is no j, $0 \ge j > \tau$, such that $T^j\beta_s \in I$. Then we say that I is an admissible subinterval of (λ, π).

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2.1. *Rauzy–Veech induction.* For $T_{\lambda,\pi}$, the Rauzy map sends it to its induced map on $[0, |\lambda| - \min\{\lambda_m, \lambda_{\pi^{-1}m}\})$, which is the largest admissible interval of the form $J = [0, L)$, $0 < L < |\lambda|$.

Given any permutation, two actions a and b are

$$a(\pi)(i) = \begin{cases} \pi(i), & i \le \pi^{-1}m, \\ \pi(i-1), & \pi^{-1}m + 1 < i \le m, \\ \pi(m), & i = \pi^{-1}m + 1 \end{cases} \tag{2.3}$$

and

$$b(\pi)(i) = \begin{cases} \pi(i), & \pi(i) \le \pi(m), \\ \pi(i) + 1, & \pi(m) + 1 < \pi(i) < m, \\ \pi(m) + 1, & \pi(i) = m. \end{cases} \tag{2.4}$$

The Rauzy–Veech map $\mathcal{Z}(\lambda, \pi) : \Lambda_m \times \mathcal{G}_m^0 \to \Lambda_m \times \mathcal{G}_m^0$ is determined by

$$\mathcal{Z}(\lambda, \pi) = (A(\pi, c)^{-1}\lambda, c\pi), \tag{2.5}$$

where $c = c(\lambda, \pi)$ is defined by

$$c(\lambda, \pi) = \begin{cases} a, & \lambda_m < \lambda_{\pi^{-1}m}, \\ b, & \lambda_m > \lambda_{\pi^{-1}m}. \end{cases} \tag{2.6}$$

$\mathcal{Z}(\lambda, \pi)$ is almost everywhere defined on $\Lambda_m \times \{\pi\}$ for each $\pi \in \mathcal{G}_m^0$.

The matrices $A = A(\pi, c)$ in (2.5) are defined as follows:

$$A(\pi, a) = \left(\begin{array}{c|ccccc} & 0 & 0 & \cdots & 0 & 0 \\ & 0 & 0 & \cdots & 0 & 0 \\ I_{\pi^{-1}m} & . & . & \cdots & . & . \\ & 0 & 0 & \cdots & 0 & 0 \\ & 1 & 0 & \cdots & 0 & 0 \\ \hline & & & & & \\ & 0 & 1 & \cdots & 0 & 1 \\ & 0 & 0 & \cdots & 0 & 0 \\ 0 & . & . & \cdots & . & . \\ & 0 & 0 & \cdots & 0 & 1 \\ & 1 & 0 & \cdots & 1 & 0 \end{array}\right), \tag{2.7}$$

$$A(\pi, b) = \left(\begin{array}{ccccccc|c} & & & I_{m-1} & & & & 0 \\ \hline \underbrace{0 & \cdots & 0 & 1 & 0 & \cdots & 0}_{1 \text{ at the } j\text{th position}} & & 1 \end{array}\right), \tag{2.8}$$

where I_k is the k-identity matrix and $j = \pi^{-1}m$.

And, the normalized Rauzy map $\mathcal{R} : \Delta_{m-1} \times \mathcal{G}_m^0 \to \Delta_{m-1} \times \mathcal{G}_m^0$ is defined by

$$\mathcal{R}(\lambda, \pi) = \left(\frac{A(\pi, c)^{-1}\lambda}{|A(\pi, c)^{-1}\lambda|}, c\pi\right) = \left(\frac{\pi_1^* \mathcal{Z}(\lambda, \pi)}{|\pi_1^* \mathcal{Z}(\lambda, \pi)|}, \pi_2^* \mathcal{Z}(\lambda, \pi)\right), \tag{2.9}$$

where π_1^* and π_2^* are the projections to the first coordinate and the second coordinate, respectively.

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Iteratively,

$$\mathcal{Z}^n(\lambda, \pi) = ((A^{(n)})^{-1}\lambda, c^{(n)}\pi) = (\lambda^{(n)}, \pi^{(n)}), \tag{2.10}$$

where

$$c^{(n)} = c_n c_{n-1} \cdots c_1, (c_1, \ldots, c_n \in \{a, b\}, c_i = c(\mathcal{Z}^{i-1}(\lambda, \pi))) \tag{2.11}$$

and

$$A^{(n)} = A(\pi, c_1)A(c^{(1)}\pi, c_2)A(c^{(2)}\pi, c_3) \cdots A(c^{(n-1)}\pi, c_n). \tag{2.12}$$

The Rauzy class $\mathcal{C} \subseteq \mathcal{G}_m$ of π is a set of orbits for the group of maps generated by a and b. On the $\mathcal{R}$ invariant component $\Delta_{m-1} \times \mathcal{C}$, we have the following result.

THEOREM 2.2. [**14**, **18**] *Let $\pi \in \mathcal{G}_m^0$, the set of irreducible permutations. For Lebesgue almost all $\lambda \in \Lambda_m$, normalized Lebesgue measure on I^λ is the unique invariant Borel probability measure for $T_{(\lambda,\pi)}$. In particular, $T_{(\lambda,\pi)}$ is ergodic for almost all λ.*

2.2. *Whirly action and whirly automorphism.* In this paper, we assume weak topology as defined in the following definition (Definition 2.3). For fundamental concepts and methods related to analysis and ergodic theory, please refer to [**6**, **10**, **11**].

Definition 2.3. (Weak topology on automorphism group) Let $(\mathbb{X}, \mathcal{B}, \mu)$ be a standard probability Borel space, and $G = \mathrm{Aut}(\mathbb{X}, \mathcal{B}, \mu)$ be the group of all non-singular measurable automorphisms of $(\mathbb{X}, \mathcal{B}, \mu)$. Suppose that (E_n) is a countable family of measurable subsets generating $\mathcal{B}$. The weak topology of G is generated by the metric $d(S, T)$ for any $S, T \in G$, where $d(S, T) = \sum_{n=1}^{\infty} 2^{-n}\mu(SE_n \Delta TE_n)$.

Utilizing the weak topology defined as above, one can project the concept of whirly action (Definition 2.7) to whirly automorphism (Definition 2.10). This is included in the following review of the definitions and fundamental propositions about whirly action and whirly automorphisms.

Whirly action was introduced by Glasner *et al* [**7**, Definition 3.1]. The purpose is to study the condition for a Polish group action to admit a spatial model. In the same paper, they translated the concept of whirly from the group action to automorphisms, since the weak closure of a rigid automorphism is a near action. They showed that in the group G of automorphisms on a finite Lebesgue space, whirly (in the sense of Z action) is a topologically generic property, i.e. the set of whirly automorphisms is residual in G. The concept of 'whirly transformation' is inherited from the theory about general group actions, and implies weak mixing. It is interesting to ask whether whirly is a generic property in the space of interval exchange transformations. Theorem 2.13 gives a positive answer for 3-interval exchange transformations.

Without considering the measure, we have the Borel action, satisfying a similar condition as in Definition 2.6, defined below.

Definition 2.4. (Borel action) Suppose that G is a Polish group and $(\mathbb{X}, \mathcal{B}, \mu)$ is a standard probability Borel space. We say that a Borel map $G \times \mathbb{X} \to \mathbb{X}$ $((g, x) \to gx)$ is a Borel action of G on $(\mathbb{X}, \mathcal{B}, \mu)$ if it satisfies the following properties:

(i) $ex = x$ for all $x \in \mathbb{X}$, where e is the identity element of G;

(ii) $g(hx) = (ghx)$ for all $x \in \mathbb{X}$, where $g, h \in G$.

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Definition 2.5. (Spatial G action: [**7**]) A spatial G-action on a standard Lebesgue space $(\mathbb{X}, \mathcal{B}, \mu)$ is a Borel action of $\mathbb{P}$ on the space such that each $g \in \mathbb{P}$ preserves μ.

The concept of near action is introduced measure theoretically.

Definition 2.6. (Near action: [**7**]) Suppose that $\mathbb{P}$ is a Polish group and $(\mathbb{X}, \mathcal{B}, \mu)$ is a standard probability Borel space. We say that a Borel map $\mathbb{P} \times \mathbb{X} \to \mathbb{X}$ $((g, x) \to gx)$ is a near action of $\mathbb{P}$ on $(\mathbb{X}, \mathcal{B}, \mu)$ if it satisfies the following properties:

(i) $ex = x$ for almost every $x \in \mathbb{X}$, where e is the identity element of $\mathbb{P}$;

(ii) $g(hx) = (ghx)$ for almost every $x \in \mathbb{X}$, where $g, h \in \mathbb{P}$;

(iii) each $g \in G$ preserves the measure μ.

Note. The set of measure one in Definition 2.6(ii) may depend on the pair g, h. It is easy to see that a near action is a continuous homomorphism from $\mathbb{P}$ to G (G is the automorphism group of $\mathbb{X}$). Now we define the key concept of this paper.

Definition 2.7. (Whirly action: [**7**]) Given $\varepsilon > 0$, if, for all sets $E, F \in \mathcal{B}$ with $\mu(E), \mu(F) > 0$, there exists $\gamma \in N_\varepsilon(\mathrm{Id})$ (the ε neighborhood of the identity $\mathrm{Id} = e$ in $\mathbb{P}$) such that $\mu(E \cap \gamma F) > 0$, then we say that the near action of $\mathbb{P}$ on $(\mathbb{X}, \mathcal{B}, \mu)$ is whirly.

THEOREM 2.8. [**7**, Proposition 3.3] *A whirly action does not admit a non-trivial spatial factor and thus has no spatial model.*

Remark 2.9. If an automorphism $(\mathbb{X}, \mathcal{B}, T, \mu)$ is rigid, then its weak closure '$\mathrm{Wcl}(T)$' is a closed subgroup of $G = \mathrm{Aut}(\mathbb{X}, \mathcal{B}, \mu)$. With the induced topology, $\mathrm{Wcl}(T)$ is also a Polish space. Based on this fact, the whirly transformation is a concept induced from whirly action.

Let $(\mathbb{X}, \mathcal{B}, \mu)$ be the standard Lebesgue probability space, $\mathbb{X} = [0, 1]$, and denote by $G = \mathrm{Aut}(\mathbb{X})$ the Polish group of its automorphism.

Definition 2.10. (Whirly automorphism) We say that a rigid system $(X, \mathcal{B}, \mu, T)$ is whirly if, given $\epsilon > 0$ for any μ-positive measure sets E and F $(\mu(E), \mu(F) > 0)$ in $\mathcal{B}$, there exists n such that $T^n \in U_\epsilon$ (the ϵ-neighborhood of the identity map in the weak topology of G) and $\mu(T^n E \cap F) > 0$.

Whirly implies rigid. Glasner and Weiss [**9**, Corollary 4.2] showed that if $(\mathbb{X}, \mathcal{B}, \mu, T)$ is whirly, then it is weak mixing. In the same paper (as Theorem 5.2), the following result is proved.

THEOREM 2.11. [**9**, Theorem 5.2] *The set of all the the whirly transformations is residual (dense G_δ subset) in G.*

Next we introduce a new notion of uniformly whirly, which is stronger than or equivalent to whirly.

Definition 2.12. (Uniformly whirly) A rigid system $(\mathbb{X}, \mathcal{B}, \mu, T)$ is uniformly whirly if, given $\varepsilon > 0$ for any $0 < \alpha, \beta < 1$, we have

$$\inf_{\mu(E)=\alpha, \mu(F)=\beta} \sup_{T^n \in U_\varepsilon} \{\mu(T^n E \cap F)\} > 0.$$

Uniformly whirly implies whirly.

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Questions. Is uniformly whirly equivalent to whirly? If not, is the collection of uniformly whirly automorphisms a dense G_δ subset of G?

It is interesting to ask whether the whirly property is a generic property in the space of m-interval exchange transformations ($m \geq 3$). The major theorem (Theorem 2.13) of this paper provides a positive answer for $m = 3$. Below is the main result and an outline of the proof.

THEOREM 2.13. *Let $\pi = (3, 2, 1)$ for Lebesgue almost all $\lambda \in \Lambda_3$. The three-dimensional cone of positive real numbers, the interval exchange transformation $T_{(\lambda,\pi)}$, is whirly.*

Outline of the proof of Theorem 2.13. First, we raise an equivalent definition for whirly transformation (Definition 4.1). This definition enables us to use the cyclic approximation of rank-one [**3**, **13**] stacking structure [**19**, §3] associated with Rauzy–Veech induction more effectively.

Second, for a symmetric 3-permutation π, we study the Veech induction map $\mathcal{T}_2$: $\lambda \to \alpha/|\alpha|$, $|\alpha| = \max\{\lambda_1, \lambda_3\}$, and we observe that the visitation times (a_1, a_2, a_3) of each subinterval of $\alpha/|\alpha|$ admit the equation $a_2 = a_1 + a_3 - 1$. Considering the cyclic approximation of rank-one stacking structure, we construct a series of cyclic approximations with the base interval to be the second subinterval of α. Together with the relation $a_2 = a_1 + a_3 - 1$, we demonstrate the fundamental structure for the whirly property, summarized as Lemma 4.4.

The last part of the proof is to use a density point argument to extend the fundamental structure based on the Veech induction to general measurable subsets.

CONJECTURE 2.14. *Let $\pi \in \mathcal{G}_m^0$, $m \geq 3$, for Lebesgue almost all $\lambda \in \Lambda_m$. Then the m-dimensional cone of positive real numbers, the interval exchange transformation $T_{(\lambda,\pi)}$, is whirly.*

3. *The space of 3-interval exchange transformations*

In Veech [**19**], key results in the theory about interval exchange transformation space are established. We will utilize the results in Veech [**18**, **19**]. Let $m > 1$ and, specifically here, let π be the symmetric permutation (i.e. $\pi = (m, m-1, \ldots, 1)$). In Veech [**17**], it is proved that for almost every λ the induced transformation of $T_{\lambda,\pi}$ on $[0, \max\{\lambda_1, \lambda_m\})$ is an (α, π) interval exchange transformation with $|\alpha| = \max\{\lambda_1, \lambda_m\}$ and π still the same symmetric permutation. That is a transformation $\mathcal{T}_2 : (\lambda, \pi) \to (\alpha/|\alpha|, \pi)$ or, simply, $\mathcal{T}_2(\lambda) \sim \mathcal{T}_2(\lambda, \pi)$. So, without confusion, let $\mathcal{T}_2(\lambda) = \mathcal{T}_2(\lambda, \pi)$. When $m = 3$, $f_2(\lambda) = (1/(1-\lambda_1) + 1/(1-\lambda_3)) \prod_{j=1}^{2} 1/(\lambda_j + \lambda_{j+1})$ is the density of a conservative ergodic invariant measure for $\mathcal{T}_2$ by Veech [**18**, Equation (1.2)].

We claim that if (λ, π) satisfies the i.d.o.c. and $\pi(j) = m - j + 1$, there exists some k such that $\mathcal{Z}^k(\lambda, \pi) = (\alpha, \pi)$ with $|\alpha| = \max\{\lambda_1, \lambda_m\}$. To verify this, we need the following lemma.

LEMMA 3.1. *If $\lambda \in \Lambda_m$, $m \geq 3$, $T_{(\lambda,\pi)}$ satisfies the i.d.o.c., and $\mathcal{Z}^k(\lambda, \pi) = (\lambda', \pi')$, where k is the largest integer such that $|\lambda'| > \max\{\lambda_1, \lambda_m\}$, then $J = [0, \max\{\lambda_1, \lambda_m\})$ is an admissible interval of (λ', π').*

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Proof. If $\lambda_1 > \lambda_m$, then λ_1 is a discontinuous point of $T_{(\lambda',\pi')}$ and $[0, \lambda_1)$ is an admissible interval of (λ', π'). If $\lambda_m > \lambda_1$, let $\beta'_t = \beta'_t(\lambda') = \sum_{i=1}^{t} \lambda'_i$. Since $\lambda_m = T(\beta_{m-2})$ and $T_{(\lambda',\pi')}$ is the induced transformation of $T_{(\lambda,\pi)}$ on $[0, |\lambda'|)$, we have that there exist $1 \le t \le m-1$ and $k_t > 0$ such that $\lambda_m = T_{(\lambda',\pi')}(\beta'_t) = T^{k_t}_{(\lambda,\pi)}(\beta'_t)$. By the definition of admissible interval, $[0, \lambda_m)$ is an admissible interval associated with $T_{\lambda',\pi'}$. □

PROPOSITION 3.2. *Suppose that* $\lambda \in \Lambda_{m-1}$, $\pi(j) = m - j + 1$, $1 \le j \le m$, *and* (λ, π) *satisfies the i.d.o.c. Then there exists* $k_0 \in \mathbb{N}$ *such that* $\mathcal{Z}^{k_0}(\lambda, \pi) = (\alpha, \pi)$, *where* $|\alpha| = \max\{\lambda_1, \lambda_m\}$. *Therefore,* $\mathcal{T}_2(\lambda, \pi) = \mathcal{R}^{k_0}(\lambda, \pi)$.

Proof. Assume that for all $k \in \mathbb{N}$, $\mathcal{Z}^k(\lambda, \pi) = (\alpha^{(k)}, \pi)$ is such that $|\alpha^{(k)}| \neq \max\{\lambda_1, \lambda_m\}$. Since (λ, π) satisfies the i.d.o.c., $|\pi_1^*(\mathcal{Z}^k(\lambda, \pi))| \to 0$ as $k \to \infty$ (see [**20**, Corollary 5.2] for a detailed proof), there exists $k_0 \ge 0$ such that $|\pi_1^*(\mathcal{Z}^{k_0}(\lambda, \pi))| > \max\{\lambda_1, \lambda_m\}$ and $|\pi_1^*(\mathcal{Z}^{k_0+1}(\lambda, \pi))| < \max\{\lambda_1, \lambda_m\}$. By Lemma 3.1, for any $r > |\pi_1^*(\mathcal{Z}^{k_0+1}(\lambda, \pi))|$, $[0, r)$ is not an admissible interval of $(\lambda', \pi') = \mathcal{Z}^{k_0}(\lambda, \pi)$, which is a contradiction to the fact that $[0, \max\{\lambda_1, \lambda_m\})$ is an admissible interval of (λ', π'). □

The above argument assures us that essential general results about the iteration of Rauzy–Veech induction may be applied to $\mathcal{T}_2$. For convenience, let us denote the induced map of $T_{\lambda,\pi}$ on $[0, \max\{\lambda_1, \lambda_m\})$ by (α, π), and define $\mathcal{Z}_* : \Lambda_m \times \{\pi\} \to \Lambda_m \times \{\pi\}$ by $\mathcal{Z}_*(\lambda, \pi) = (\alpha, \pi)$ with $|\alpha| = \max\{\lambda_1, \lambda_m\}$.

Next we limit the discussion to the case $m = 3$. Recall §2 for the visitation matrix associated with $\mathcal{Z}^n(\lambda, \pi)$, $\mathcal{Z}^n(\lambda, \pi) = (\alpha^{(n)}, \pi)$. We have $\lambda = A^{(n)}\alpha^n$, and the summation of the ith column of $A^{(n)}$, $a_i^{(n)}$, is the first return time of the ith subinterval of $[0, |\alpha^{(n)}|)$ under $T_{(\lambda,\pi)}$. It will be shown that for all $n \in \mathbb{N}$, $a_2^{(n)} = a_1^{(n)} + a_3^{(n)} - 1$. In fact, we will verify the same equality for a more general case. It is done by looking at the Rauzy graph for the closed paths based at $\pi = (3, 2, 1)$. The Rauzy class of $\pi = (3, 2, 1)$ is $\{\pi, \pi_1, \pi_2 | \pi_1 = a\pi = (3, 1, 2), \pi_2 = b\pi = (2, 3, 1)\}$. We have

$$A(\pi, a) = A(\pi_1, a) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$A(\pi, b) = A(\pi_1, b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

$$A(\pi_2, a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A(\pi_2, b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

LEMMA 3.3. *If* $\mathcal{Z}_*(\lambda, \pi) = (\alpha, \pi)$, $\lambda \in \Lambda_3$, $\pi = (3, 2, 1)$ *and the visitation matrix is* A *(i.e.* $\lambda = A\alpha$*), then* $a_2 + 1 = a_1 + a_3$.

Proof. To prove this lemma, we look into the following two cases.

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Case 1. [ab^la or ba^lb]

(1) Starting from π, go along the path ab^la and come back to π. Then the associated visitation matrix is $A^{(l+2)}$. We want to show that

$$a_2^{(l+2)} = a_1^{(l+2)} + a_3^{(l+2)} - 1.$$

Since

$$A^{(1)} = A(\pi, a) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$a_1^{(1)} = a_3^{(1)} = 1, \quad a_2^{(1)} = 2,$$

$$A^{(2)} = A^{(1)} \cdot A(\pi, b) = (A_1^{(1)}, A_2^{(1)}, A_3^{(1)}) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= (A_1^{(1)} + A_3^{(1)}, A_2^{(1)}, A_3^{(1)}),$$

where $A_i^{(n)}$ is the ith column vector of $A^{(n)}$,

$$\begin{aligned} &\vdots \\ A^{(l+1)} &= (A_1^{(1)} + lA_3^{(1)}, A_2^{(1)}, A_3^{(1)}), \\ A^{(l+2)} &= A^{(l+1)} \cdot A(\pi_1, a) \\ &= (A_1^{(1)} + lA_3^{(1)}, \ A_2^{(1)}, \ A_3^{(1)}) \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (3.1) \\ &= (A_1^{(1)} + lA_3^{(1)}, A_1^{(1)} + (l+1)A_3^{(1)}, A_2^{(1)}). \end{aligned}$$

Therefore,

$$\begin{aligned} a_1^{(l+2)} &= a_1^{(1)} + la_3^{(1)} = l + 1, \\ a_2^{(l+2)} &= a_1^{(1)} + (l+1)a_3^{(1)} = l + 2, \qquad (3.2) \\ a_3^{(l+2)} &= a_2^{(1)} = 2. \end{aligned}$$

Thus, $a_2 = a_1 + a_3 - 1$ is proved for the path ab^la.

(2) Similar to the argument in (1), if we replace the path of ab^la by the path of ba^lb, the associated matrix $A^{(l+2)}$ satisfies

$$a_2^{(l+2)} = a_1^{(l+2)} + a_3^{(l+2)} - 1.$$

Case 2. [p_0ab^la or p_0ba^lb]

(1) Suppose that the closed path is $p = p_0ab^la$, where p_0 is a closed path based at $\pi = (3, 2, 1)$, p_0 admits length n_0 and associated with p_0 is the matrix $A^{(n_0)}$ with column summations $a_1^{(n_0)}, a_2^{(n_0)}, a_3^{(n_0)}$ satisfying $a_2^{(n_0)} + 1 = a_1^{(n_0)} + a_3^{(n_0)}$. Then, by a similar computation as Case 1, we have the conclusion that, after going along p, the return times satisfy

$$a_2^{(n_0+l+2)} = a_1^{(n_0+l+2)} + a_3^{(n_0+l+2)} - 1. \qquad (3.3)$$

(2) Similar to (1) above, the same relation on the three return times is true for the path $p = p_0ba^lb$.

By Cases 1 and 2, we have proved Lemma 3.3. □

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4. *Whirly 3-interval exchange transformations*

Before discussing the 3-interval exchange transformations, let us introduce another way to define the concept of whirly automorphism and verify the equivalence between the two definitions.

Definition 4.1. (Whirly automorphism) A rigid ergodic automorphism $T \in G$ is said to be whirly if, given $\varepsilon > 0$, for any $l \in \mathbb{N}$ (or for any $-l \in \mathbb{N}$) and a μ-positive measure set $E \in \mathcal{B}$, there exists $n \in \mathbb{N}$ such that $T^n \in U_\varepsilon$ and $\mu(T^n E \cap T^l E) > 0$.

THEOREM 4.2. *Conditions in Definitions 2.10 and 4.1 for an automorphism to be whirly are equivalent to each other.*

Proof. Suppose that $T \in G$ satisfies the condition in Definition 4.1 (without loss of generality, we take the case that $-l \in \mathbb{N}$); then we claim that for any $E, F \in \mathcal{B}$ with $\mu(E), \mu(F) > 0$ there exists $n \in \mathbb{N}$ such that $T^n \in U_\varepsilon$ and $\mu(T^n E \cap F) > 0$. Since T is ergodic, there exists $-q \in \mathbb{N}$ such that

$$\mu(T^q E \cap F) > 0.$$

Then

$$\mu(E \cap T^{-q} F) > 0,$$

so therefore there exists $n \in \mathbb{N}$ such that $T^n \in U_\varepsilon$ and

$$\mu(T^n(E \cap T^{-q} F) \cap T^q(E \cap T^{-q} F)) > 0.$$

Thus, $\mu(T^n E \cap F) > 0$.

The opposite direction is obvious. □

Let π be the symmetric m-permutation. According to Veech [**19**, §3], there exist $n > 1$ and $c_1, c_2, \ldots, c_n \in \{a, b\}$ such that: if $A^{(i)} = A(\pi^{(i-1)}, c_i)$ $(1 \le i \le n)$, $c_n \circ c_{n-1} \circ \cdots \circ c_1(\pi) = \pi$, then $\pi^{(0)} = \pi$, $\pi^{(1)} = c_1\pi^{(0)}$, $\pi^{(2)} = c_2\pi^{(1)}, \ldots, \pi^{(n)} = c_n\pi^{(n-1)} = \pi$ and $B = A^{(1)}A^{(2)} \cdots A^{(n)}$ is a positive $m \times m$ matrix.

Remark 4.3. If $\lambda \in \Lambda_m$, then $\mathcal{Z}^n(B\lambda, \pi) = (\lambda, \pi)$, and the orbit of $(B\lambda, \pi)$ under $\mathcal{Z}$ passes the same sequence of permutations $\{\pi^j, 0 \le j \le n\}$.

Let $\nu(A) = \max_{1 \le i,j,k \le m}\{a_{ij}/a_{ik}\}$, where A is a positive matrix. Then

$$a_i \le \nu(A) a_j, \quad 1 \le i, j \le m \ (a_i \text{ is the } i\text{th column sum of } A), \tag{4.1}$$

$$\nu(MA) \le \nu(A) \quad \text{for any non-negative matrix } M \text{ with at least one non-zero element.} \tag{4.2}$$

We see that $\nu(B)$ and $\nu(B^t)$ are both positive numbers greater than one. Let $r = \nu(B)$ and $r' = \nu(B^t)$.

Next we fix $m = 3$, while still keeping notation as above. Let l be a given positive integer; we will set up an open set in $\Lambda_3 \times \{\pi\}$ and carry out some computation on the approximation by the Kakutani tower associated with the Rauzy induction.

Let $\varepsilon_1, \varepsilon_2$ be two small positive numbers to be specified for our purpose later. Let $Y^*(\varepsilon_1, \varepsilon_2) = \{\alpha | \alpha \in \Lambda_m, (1 - (\varepsilon_1/2))|\alpha| > \alpha_2 > (1 - \varepsilon_1)|\alpha|$ and $(1 + \varepsilon_2)\alpha_3 > \alpha_1 > \alpha_3\}$, an open subset of Λ_m. Let $W(\varepsilon_1, \varepsilon_2) = B^2 Y^*(\varepsilon_1, \varepsilon_2) \times \{\pi\}$, an open subset of $\Lambda_m \times \{\pi\}$.

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Suppose that $(\lambda, \pi) \in \Delta_2 \times \{\pi\}$ and there exists $k \in \mathbb{N}$ such that $\mathcal{Z}^k(\lambda, \pi) \in W(\varepsilon_1, \varepsilon_2)$. We know that $\xi = B^2\alpha$ for some $\alpha \in Y^*(\varepsilon_1, \varepsilon_2)$. Then $\lambda = A^{(k)}\xi$, where $A^{(k)}$ is the visitation matrix associated with $\mathcal{Z}^k(\lambda, \pi)$, $\mathcal{Z}^{k+2n}(\lambda, \pi) = \mathcal{Z}^{2n}(\xi, \pi) = (\alpha, \pi)$ and $\lambda = A^{(k)}B^2\alpha$. Let $A = A^{(k)}B^2$. Since $A^{(k)}$ is a non-negative matrix (no zero-row, no zero-column), by (4.2) we have $\nu(A) \le \nu(B) = r$. Therefore, the following arguments may give us a clear view of the stack structure associated with the Veech induction map $\mathcal{T}_2$.

CLAIM 1. *T^{a_2} translates the subinterval I_2^α (i.e. the second subinterval of I^α) to the left by $\alpha_1 - \alpha_3$. That is,*

$$I_2^\alpha \cap T^{a_2}(I_2^\alpha) = [\alpha_1, \alpha_1 + \alpha_2 - l(\alpha_1 - \alpha_3)).$$

Since l is a fixed positive integer, and ε_1, ε_2 are small enough, we have

$$\begin{aligned}\mu(I_2^\alpha \cap T^{la_2}(I_2^\alpha)) &= \alpha_2 - l(\alpha_1 - \alpha_3) > \alpha_2 - l\varepsilon_2\alpha_3 \\ &> \alpha_2 - l\varepsilon_1\varepsilon_2|\alpha| > \alpha_2 - l\frac{\varepsilon_1\varepsilon_2}{1-\varepsilon_1}\alpha_2 \\ &= \left(1 - l\frac{\varepsilon_1\varepsilon_2}{1-\varepsilon_1}\right)\alpha_2. \end{aligned} \tag{4.3}$$

CLAIM 2. *The remainder of the column with base I_2^α and height a_2 has measure*

$$\begin{aligned}|\lambda| - \mu &\left(\bigcup_{i=0}^{a_2} T^i(I_2^\alpha)\right) \\ &= a_1\alpha_1 + a_3\alpha_3 < r\alpha_2(\alpha_1 + \alpha_3) \\ &< ra_2\varepsilon_1|\alpha| < ra_2\frac{\varepsilon_1}{1-\varepsilon_1}\alpha_2 < \frac{\varepsilon_1}{1-\varepsilon_1}|\lambda|. \end{aligned} \tag{4.4}$$

From (4.3) and (4.4), for any $\varepsilon > 0$, we can select ε_1, ε_2 small enough such that $T^{la_2} \in U_\varepsilon(\mathrm{Id})$.

CLAIM 3. *(The 'whirly part') T^{a_3} sends $[\alpha_1 + \alpha_2 + (\alpha_1 - \alpha_3), |\alpha|)$ to $[\alpha_1 - \alpha_3, \alpha_3)$, which is continuous under T^{a_1}. That is to say, $[\alpha_1 + \alpha_2 + (\alpha_1 - \alpha_3), |\alpha|)$ is continuous under $T^{a_1+a_3} = T^{a_2-1}$.*

Similarly, by induction, let

$$I_\omega^\alpha = [\alpha_1 + \alpha_2 + l(\alpha_1 - \alpha_3), |\alpha|). \tag{4.5}$$

Then T^i are all continuous (linear) on I_ω^α for $i = 1, 2, \ldots, l(a_1 + a_3)$. And, $T^{l(a_1+a_3)}(I_\omega^\alpha) \subset I_3^\alpha \subset I^\alpha$.

Therefore,

$$\begin{aligned}T^{la_2}(I_\omega^\alpha) &= T^{l(a_1+a_3-1)}(I_\omega^\alpha) \\ &= T^{-l}(T^{l(a_1+a_3)}(I_\omega^\alpha)) \subset T^{-l}(I^\alpha),\end{aligned}$$

which implies that

$$T^{la_2}(I_\omega^\alpha) \subset (T^{la_2}(I^\alpha)) \cap T^{-l}(I^\alpha). \tag{4.6}$$

Hence,

$$\begin{aligned}&\mu(T^{la_2}(I^\alpha) \cap T^{-l}(I^\alpha)) \\ &\quad \ge \mu(T^{la_2}(I_\omega^\alpha)) = \alpha_3 - l(\alpha_1 - \alpha_3) > \alpha_3 - l\varepsilon_2\alpha_3 = (1 - l\varepsilon_2)\alpha_3. \end{aligned} \tag{4.7}$$

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Note that Claims 1 and 2 show that T^{la_2} is close to the identity map; (4.7) shows that we are on the right way to the whirly property (Definition 4.1).

By Claims 1–3, choosing a positive constant $\mathfrak{C}_{\varepsilon,l}$ associated with ε, l and small enough, we have the following lemma.

LEMMA 4.4. *Let $\pi = (3, 2, 1)$ and let n and B be specified in the paragraph immediately before Remark 4.3. For almost all $\lambda \in \Lambda_3$, given $0 < \varepsilon < 1/10$ and any $l \in \mathbb{N}$, there exist $\mathfrak{C}_{\varepsilon,l} > 0$ and k_0 so that: if $k > k_0$ and $\mathcal{Z}^k(\lambda, \pi) = (\eta, \pi) \in W(\mathfrak{C}_{\varepsilon,l}, \mathfrak{C}_{\varepsilon,l})$, then the following properties hold for $(\alpha, \pi) = \mathcal{Z}^{k+2n}(\lambda, \pi)$:*

$$\mu(I^\alpha \cap T^{la_2}(I^\alpha)) > (1 - \varepsilon)|\alpha|, \tag{P1}$$

$$|\lambda| - \mu\left(\bigcup_{i=0}^{a_2-1} T^i(I_2^\alpha)\right) < \varepsilon|\lambda|, \tag{P2}$$

$$\mu(T^{la_2}(I^\alpha) \cap T^{-l}(I^\alpha)) > \frac{\varepsilon}{3}|\alpha|. \tag{P3}$$

Moreover, for almost every λ, given any $0 < \varepsilon < \frac{1}{10}$ and $l \in \mathbb{N}$, there exist infinitely many such k.

Now let $N_{\varepsilon,l}^{(\lambda)} \subset \mathbb{N}$ be defined by

$$N_{\varepsilon,l}^{(\lambda)} = \{n_t | n_1 < n_2 < \cdots < n_i < \cdots, \mathcal{Z}^{n_t-2n}(\lambda, \pi) \in W(\mathfrak{C}_{\varepsilon,l}, \mathfrak{C}_{\varepsilon,l})\}. \tag{4.8}$$

By Veech's ergodic theorem [**18**, Theorem 1.1] on $\mathcal{T}_2$, we know that for Lebesgue almost every $\lambda \in \Lambda_3$, $T_{(\lambda,\pi)}$ is uniquely ergodic (and thus ergodic with respect to Lebesgue measure), and $N_{\varepsilon,l}^{\lambda}$ is a set with infinitely many elements for any $0 < \varepsilon < 1/10, l \in \mathbb{N}$. Let us continue to study such $T_{(\lambda,\pi)}$. As usual, we use T to denote $T_{\lambda,\pi}$.

We know that $A = A^{(k)}B^2$, $1 \le \nu(A) \le \nu(B) = r$. We need B^2 here instead of B, in order to get a T-stack with the base I^α, which is a relatively large portion of I^λ. The following lemma will be used in the last step, a density point argument, of the proof of Theorem 2.13.

LEMMA 4.5. *With notation as above, let $T = T(\lambda, \pi)$; then, for almost every $\lambda \in \Lambda_3$, there exists a positive integer a_* such that T^i $(1 \le i \le a_*)$ are continuous (linear) on I^α, $T^i(I^\alpha) \cap T^j(I^\alpha) = \emptyset$ $(i \ne j, 0 \le i, j < a_*)$ and*

$$a_*|\alpha| > \frac{1}{b_M(1 + 2r \cdot r')}|\lambda|,$$

where $b_M = \max\{b_{11}, b_{12}, b_{13}\}$.

Proof. We know that $A = A^{(k)}B^2$.

Suppose that $\mathcal{Z}^{k+n}(\lambda, \pi) = (\eta, \pi) = (B\alpha, \pi)$, where $\eta = (\eta_1, \eta_2, \eta_3)$, $I_1^\eta = [0, \eta_1)$, $\eta_1 = b_{11}\alpha_1 + b_{12}\alpha_2 + b_{13}\alpha_3$ and

$$I^\alpha \subset I_1^\eta. \tag{4.9}$$

Meanwhile, $\eta_1 < b_M(\alpha_1 + \alpha_2 + \alpha_3) < b_M|\alpha|$; that is,

$$|\alpha| > \frac{1}{b_M}\eta_1. \tag{4.10}$$

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At the same time, since

$$\eta_1 = b_{11}\alpha_1 + b_{12}\alpha_2 + b_{13}\alpha_3,$$
$$\eta_2 = b_{21}\alpha_1 + b_{22}\alpha_2 + b_{23}\alpha_3,$$
$$\eta_3 = b_{31}\alpha_1 + b_{32}\alpha_2 + b_{33}\alpha_3,$$

it follows that

$$\eta_2, \eta_3 < r'\eta_1. \tag{4.11}$$

Remembering that $\lambda = A^{(k)}B\eta$, i.e. $\lambda = a_1^{(k+n)}\eta_1 + a_2^{(k+n)}\eta_2 + a_3^{(k+n)}\eta_3$, by (4.1) and (4.2), we have

$$a_2^{(k+n)}, a_3^{(k+n)} < r a_1^{(k+n)}$$

and, by (4.11), we have

$$a_1^{(k+n)}\eta_1 > \frac{1}{1+2rr'}|\lambda|. \tag{4.12}$$

Equations (4.10) and (4.12) imply that $a_1^{(k+n)}|\alpha| > 1/b_M(1+2rr')|\lambda|$. Combining this with (4.9), the lemma is proved. □

We now prove Theorem 2.13 with a density point argument.

Proof of Theorem 2.13. Let $\pi = (3, 2, 1)$ and let λ be in the full measure subset of Λ_3 as required by Lemma 4.4.

Define

$$\mathfrak{G} = \bigcap_{N=2n+1}^{\infty} \bigcup_{\substack{t\le N \\ n_t \in N_{\varepsilon,l}^{(\lambda)}}} \mathfrak{G}_t,$$

where $\mathfrak{G}_t = \bigcup_{i=0}^{a_*^{(n_t)}-1} T^i(I^{\alpha^{n_t}})$, with $N_{\varepsilon,l}^{(\lambda)}$ as defined in (4.8).

According to Lemma 4.5, $\mu(\mathfrak{G}) \ge (1/b_M)(1/(1+2rr'))|\lambda|$.

Suppose that E is an arbitrary measurable set, $E \subset [0, |\lambda|)$, $\mu(E) > 0$. Then, by the ergodicity of T, there exists $q \in \mathbb{N}$ such that $\mu(T^{-q}(E) \cap \mathfrak{G}) > 0$. Therefore, by the Lebesgue density theorem, there exists a point of density one $x \in T^{-q}(E) \cap \mathfrak{G}$. By definition of $\mathfrak{G}$, we have $x \in T_{(\lambda,\pi)}^{-q}(E) \cap J_k$, where the left closed right open interval $J_k = T^{i_k}(I^{\alpha^{(S_k)}})$, $S_k = n_{t_k}$, $0 \le i_k < a_*^{(S_k)}$, and the approximate density satisfies

$$\lim_{k\to\infty} \frac{\mu(T^{-q}(E) \cap J_k)}{\mu(J_k)} = 1. \tag{4.13}$$

By Lemma 4.4, we know that since $S_k = n_{t_k} \in N_{\varepsilon,l}^{(\lambda)}$,

$$\begin{aligned}
&\mu((T^{la_2^{(S_k)}} J_k) \cap T^{-l}(J_k)) \\
&\quad = \mu(T^{i_k}(T^{la_2^{(S_k)}}(I^{\alpha^{(S_k)}}) \cap T^{-l}(I^{\alpha^{(S_k)}}))) \\
&\quad > \frac{\varepsilon}{3}|\alpha^{(S_k)}|.
\end{aligned} \tag{4.14}$$

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Equation (4.13) implies that there exists k_0 such that

$$\frac{\mu(T^{-q}(E) \cap J_{k_0})}{\mu(J_{k_0})} > \left(1 - \frac{\varepsilon}{10}\right).$$

Therefore, by (4.14), we have

$$\mu(T^{la_2^{(S_k)}}(T^{-q}(E)) \cap (T^{-q}(E)) > 0.$$

Thus, $\mu(T^{la_2^{(S_k)}}(E) \cap T^{-l}(E)) > 0$. Since $S_k = n_{t_k} \in N_{\varepsilon,l}^{(\lambda)}$, together with Theorem 4.2 and Lemma 4.4, we have proved Theorem 2.13.

COROLLARY 4.6. *Let* $\pi = (3, 2, 1)$ *for Lebesgue almost all* $\lambda \in \Lambda_3$. *The interval exchange transformation* $(\mathbb{X}, \mathcal{B}, T_{(\lambda,\pi)})$ *admits no non-trivial spatial factor.*

Proof. This is by Glasner and Weiss [**9**, Proposition 1.9]. □

Acknowledgements. The result about 3-interval exchange transformations is a part of the author's 2006 PhD thesis at Rice University [**21**], Department of Mathematics. The author thanks Rice University for posting his PhD thesis online at the Rice University Digital Scholarship Archive. He would also like to thank his advisor W. A. Veech for directing his thesis.

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