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Note on analytic spread and asymptotic sequences

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1. *Introduction and notation.* Since the foundational paper (**10**) by Northcott and Rees in 1954 there have been quite a few papers concerning reductions of ideals and the analytic spread of an ideal. One particular line of investigation concerning the analytic spread $l(I)$ of an ideal I in a local ring (R, M) was begun in 1972 by Burch in (**5**), where it was shown that $l(I) \leqslant$ altitude $R - \min\{\text{grade } R/I^n;\ n \geqslant 1\}$. This result was sharpened in 1980–81 by Brodmann in three papers, (**2, 3, 4**). Therein he showed that the sets $\{\text{grade } R/I^n;\ n \geqslant 1\}$ and $\{\text{grade } I^{n-1}/I^n;\ n \geqslant 1\}$ stabilize for all large n, and calling the stable values t and t^*, respectively, it holds that $t \leqslant t^*$ and $l(I) \leqslant$ altitude $R - t^*$ when I is not nilpotent. He then gave a case (involving R being quasi-unmixed) when equality holds. In 1981 in (**20**) Rees used two new approaches to Burch's inequality, and he proved two nice results which may both be stated as: $l(I) \leqslant$ altitude $R - s(I)$ with equality holding when R is quasi-unmixed; here, $s(I) = \min\{\text{height } P;\ P \text{ is a minimal prime divisor of } (M, u)\, R[tI, u]\} - 1$ (in the first theorem), and $s(I)$ is the length of a maximal asymptotic sequence over I (in the second theorem).

My original goal in starting this paper was to see if Rees' two inequalities in the general case could be strengthened to equalities by changing 'altitude R' to 'altitude R^*/z for some minimal prime ideal z in the completion R^* of R'. This is possible for his first theorem (if $\min\{\text{height } P\}$ is changed to $\max\{\text{depth } P\}$), as is shown in (2·1). However, it soon became apparent that this is not always possible for the second theorem. (In fact, the class of all local rings for which equality always holds is determined in (4·2). This result follows quite directly from a new opposite inequality for $l(I)$; $l(I) \geqslant d^* - s(I)$, where $d^* = \min\{\text{depth } z;\ z \text{ is a minimal prime ideal in } R^*\}$ and $s(I)$ is as in Rees' second theorem.) However, it is shown in (5·2) that Rees' second inequality can also be changed to an equality by making this substitution together with replacing '$s(I)$' by '$s((IR^* + Z)/Z)$', where Z is the intersection of certain minimal prime ideals in R^*. Both of these new results reduce to Rees' results in the quasi-local case.

To state efficiently and prove the results mentioned above a fair amount of notation is needed, so this section will be closed by summarizing this notation and giving a brief remark concerning it.

(1·1) *Notation.* Let I be an ideal in a local ring (R, M).

(1·1·1) $m\,\text{Ass}\, R = \{z \in \text{Ass}\, R;\ z \text{ is minimal}\}$.

(1·1·2) The *integral closure* I_a *of* I *in* R is given by $I_a = \{x \in R;\ x \text{ is a root of a polynomial of the form } X^k + a_1 X^{k-1} + \ldots + a_k, \text{ where } a_i \in I^i\}$.

(1·1·3) $\hat{A}^*(I) = \{p \in \text{Spec}\, R;\ p \text{ is a prime divisor of } (I^n)_a \text{ for all large } n\}$.

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(1·1·4) The *Rees ring* $\mathscr{R} = \mathscr{R}(R, I)$ *of* R *with respect to* I is the graded subring $R[tI, u]$ of $R[t, u]$, where t is an indeterminate and $u = 1/t$.

(1·1·5) $l(I) = \operatorname{depth}(M, u)\mathscr{R}(R, I)$ is the *analytic spread* of I.

(1·1·6) An element b in M is *asymptotically prime* to I in case $(I^n)_a : bR = (I^n)_a$ for all $n \geqslant 1$. Elements $b_1, \ldots, b_g$ in M are an *asymptotic sequence over* I in case b_i is asymptotically prime to $(I, b_1, \ldots, b_{i-1})R$ for $i = 1, \ldots, g$. $s(I) = \max\{g$; there exists g elements in M which are an asymptotic sequence over $I\}$.

(1·2) *Remark.* If I is an ideal in a local ring (R, M), then the following statements hold:

(1·2·1) I_a is an ideal in R such that $I \subseteq I_a \subseteq \operatorname{Rad} I$, by ((11), section 6).

(1·2·2) It was shown in ((17), (2·4) and (2·7)) that the sets $\operatorname{Ass} R/(I^n)_a$ are increasing and stabilize for large n, so $\hat{A}^*(I)$ is well defined.

(1·2·3) If R/M is infinite, then $l(I)$ is the number of elements in a minimal basis of a minimal reduction B of I, and if $b_1, \ldots, b_h$ are such a basis, then $b_1, \ldots, b_h$ are analytically independent in R and $l(B) = l(I) = h$, by ((10), theorem 1, p. 150).

(1·2·4) If $b \in M$, then it follows from (1·2·2) that b is asymptotically prime to I if and only if $b \notin \cup \{p; p \in \hat{A}^*(I)\}$. Therefore, $b_1, \ldots, b_g$ in M are a maximal asymptotic sequence over I if and only if $b_1, \ldots, b_g$ are an asymptotic sequence and

$$M \in \hat{A}^*((I, b_1, \ldots, b_g)R).$$

Finally, it is shown in (7) that any two maximal asymptotic sequences over I have the same number of elements.

2. *The first formula on the analytic spread of an ideal.* The single theorem in this section gives a formula for the analytic spread of an ideal in a local ring. This formula is closely related to (and was motivated by) Rees' theorem 3·3 in (20).

(2·1) THEOREM. *Let I be an ideal in a local ring (R, M), let $\mathscr{R} = \mathscr{R}(R, I)$, let $P_1, \ldots, P_h$ be the minimal prime divisors of $(M, u)\mathscr{R}$ that have depth $l(I)$, and let $h_i = \operatorname{height} P_i - 1$. Then for each $i = 1, \ldots, h$ there exists $z \in m \operatorname{Ass} R^*$ such that $l(I) = \operatorname{depth} z - h_i$.*

Proof. Let $\mathscr{S} = \mathscr{R}(R^*, IR^*)$. Then for $i = 1, \ldots, h$ there exists a minimal prime divisor Q_i of $(MR^*, u)\mathscr{S}$ such that $Q_i \cap \mathscr{R} = P_i$, $\operatorname{height} Q_i = \operatorname{height} P_i$, and $\operatorname{depth} Q_i = \operatorname{depth} P_i$, by ((15), (4·1)), so $\operatorname{height} Q_i - 1 = h_i$ and $\operatorname{depth} Q_i = l(I)$. Since the prime divisors of zero in $\mathscr{S}$ are the ideals $z^* = zR^*[t, u] \cap \mathscr{S}$ with $z \in \operatorname{Ass} R^*$, fix i and let $z \in m \operatorname{Ass} R^*$ such that $z^* \subseteq Q_i$ and $\operatorname{height} Q_i/z^* = \operatorname{height} Q_i$. Then $\operatorname{height} Q_i/z^* - 1 = h_i$ and $\operatorname{depth} Q_i/z^* = \operatorname{depth} Q_i = l(I)$. Also, $(Q_i/z^*) \cap (R^*/z) = MR^*/z$, so $\operatorname{trd}((\mathscr{S}/z^*) \div (Q_i/z^*))/((R^*/z)/(MR^*/z)) = \operatorname{depth} Q_i/z^*$, by ((9), (14·6)). Therefore it follows from the altitude formula for Q_i/z^* over R^*/z that $\operatorname{height} Q_i/z^* + l(I) = \operatorname{height} MR^*/z + \operatorname{trd}(\mathscr{S}/z^*)/(R^*/z)$. Now $\operatorname{height} MR^*/z = \operatorname{depth} z$ and $\operatorname{trd}(\mathscr{S}/z^*)/(R^*/z) = 1$, since $\mathscr{S}/z^* \cong \mathscr{R}(R^*/z, (IR^* + z)/z)$, by ((19), lemma 1·1), so it follows that $l(I) = \operatorname{depth} z - h_i$, q.e.d.

In (2·1), every minimal prime divisor of $(M, u)\mathscr{R}$ has depth at most $l(I)$ and at least one has this depth, by the definition of $l(I)$, so the P_i in (2·1) are the prime divisors of $(M, u)\mathscr{R}$ of maximum depth. In ((20), theorem 3·3), Rees considered the minimal prime divisors of $(M, u)\mathscr{R}$ of minimum height. Note that if R is quasi-unmixed, then $\mathscr{R}_{(M,u,tI)\mathscr{R}}$ satisfies the first chain condition for prime ideals, by ((12), proposition 2·1),

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and every homogeneous ideal in $\mathscr{R}$ is contained in $(M, u, tI)\mathscr{R}$. Also, their depth may be computed in this ring, by ((15), (3·7)), so the prime divisors of $(M, u)\mathscr{R}$ of maximum depth are the same as the prime divisors of $(M, u)\mathscr{R}$ of minimum height. That is, (2·1) agrees with ((20), theorem 3·3) in the quasi-unmixed case, and in the general local ring case it is a different generalization (from that in (20)) of the quasi-unmixed case. This version of the result has the advantage that it replaces the inequality in the general case in (20) with an equality. The version in (20) has the advantage that some useful information concerning certain associated graded rings is obtained.

3. *Some basic lemmas on $l(I)$ and $s(I)$.* To prove the second formula on analytic spread, (5·2), quite a few facts concerning the behaviour of $l(I)$ and $s(I)$ in flat extension rings and in passing to factor rings modulo a minimal prime ideal are needed. These results are proved in this section.

Throughout this section, R is a local ring with maximal ideal M, I is an ideal in R, and R^* is the M-adic completion of R.

(3·1) LEMMA. *Let X be an indeterminate and let $R(X) = R[X]_{MR[X]}$. Then the following statements hold*:

(3·1·1) $l(IR(X)) = l(I)$.

(3·1·2) $\hat{A}^*(IR(X)) = \{pR(X);\quad p \in \hat{A}^*(I)\}$, *so* $\{\text{height } P;\quad P \in \hat{A}^*(IR(X))\} = \{\text{height } p;\ p \in \hat{A}^*(I)\}$ *and* $\{\text{depth } P;\ P \in \hat{A}^*(IR(X))\} = \{\text{depth } p;\ p \in \hat{A}^*(I)\}$.

(3·1·3) $s(IR(X)) = s(I)$.

Proof. (3·1·1) follows readily from the definition, (1·1·5), and the fact that $\mathscr{R}(R(X), IR(X)) = \mathscr{R}(R, I)[X]_{(R[X]-MR[X])}$.

For (3·1·2), it is readily seen that $(I^k)_a R[X] = (I^k R[X])_a$, for all $k \geqslant 1$, so it follows that $(I^k)_a R(X) = (I^k R(X))_a$, since $J_a A_S = (JA_S)_a$ holds for all ideals J and all multiplicatively closed sets S $(0 \notin S)$ in a ring A. Therefore a normal primary decomposition of $(I^k)_a$ extends to a normal primary decomposition of $(I^k R(X))_a$, and the conclusions readily follow from this.

(3·1·3) follows readily from (3·1·2), the definition of $s(I)$, and (1·2·4), q.e.d.

(3·2) specifies the relation between $l(I)$ and $l((I+z)/z)$ with $z \in m \operatorname{Ass} R$.

(3·2) LEMMA. *For all $z \in m \operatorname{Ass} R$ it holds that $l(I) \geqslant l((I+z)/z)$ and equality holds for some such z.*

Proof. Let $L = R[X]_{MR[X]}$, so $L/zL = (R/z)[X]_{(M/z)(R/z)[X]}$, for all $z \in m \operatorname{Ass} R$. Also $l(I) = l(IL)$ and $l((I+z)/z) = l(((I+z)/z)(L/zL))$, by (3·1·1), so it may be assumed that R/M is infinite. Therefore let B be a minimal reduction of I. Then $(B+z)/z$ is a reduction of $(I+z)/z$, so $l(B) = l(I)$ and $l((B+z)/z) = l((I+z)/z)$, by (1·2·3). Therefore it may be assumed that $I = B$. Then $I = B$ is generated by $l(I)$ analytically independent elements, by (1·2·3), say by $b_1, \ldots, b_h$. Then $u, b_1, \ldots, b_h$ are analytically independent in $D = R[u]_{(M, u)R[u]}$ where u is an indeterminate. Therefore $N^* = (M, u)D[tb_1, \ldots, tb_h]$ is a depth h prime ideal and the N^*-residue classes of the tb_i are algebraically independent over $R[u]/(M, u)R[u]$, by ((13), lemma 4·3). Thus $N^* \cap \mathscr{R}(R, I) = (M, u)\mathscr{R}(R, I) =$ (say) N is a depth h prime ideal, by ((13), lemma 4·2). Also, with $z \in m \operatorname{Ass} R$, the ideals $z^* = zR[t, u] \cap \mathscr{R}(R, I)$ are the minimal prime ideals in $\mathscr{R}(R, I)$, and the images of the generators of I are analytically independent in R/z if and only if $z^* \subseteq N$, by ((13), remark 4·4 (i)). Therefore, since $\mathscr{R}(R, I)/z^* \cong \mathscr{R}(R/z, (I+z)/z)$, by

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((19), lemma 1·1), it follows that depth $(M/z, u)\mathscr{R}(R/z, (I+z)/z) \leqslant h$ and equality holds if and only if $z^* \subseteq N$. Hence $l((I+z)/z) \leqslant l(I)$ and equality holds for all z such that $z^* \subseteq N$, q.e.d.

(3·4) is the analogue of (3·2) for $s(I)$. To prove this result (and several others below) we need the following known result concerning $\hat{A}^*(I)$ and $\hat{A}^*((I+z)/z)$. The result is numbered for ease of reference.

(3·3) *Remark.* ((17), (8·3)). If $P \in \hat{A}^*(I)$, then there exists $z \in m\,\mathrm{Ass}\,R$ such that $z \subseteq P$ and $P/z \in \hat{A}^*((I+z)/z)$; conversely, if $z \in m\,\mathrm{Ass}\,R$ and $Q \in \hat{A}^*((I+z)/z)$, then there exists $P \in \hat{A}^*(I)$ such that $z \subseteq P$ and $P/z = Q$.

(3·4) LEMMA. *For all $z \in m\,\mathrm{Ass}\,R$ it holds that $s(I) \leqslant s((I+z)/z)$ and equality holds for some such z.*

Proof. Let $b_1, \ldots, b_s$ be a maximal asymptotic sequence over I. Then it follows from (3·3) and (1·2·4) that for all $z \in m\,\mathrm{Ass}\,R$ the images of the b_i in R/z are an asymptotic sequence over $(I+z)/z$, so $s(I) \leqslant s((I+z)/z)$. Also, since $M \in \hat{A}^*((I, b_1, \ldots, b_s)R)$, by (1·2·4), it follows from (3·3) and (1·2·4) that the images of the b_i in R/z are a maximal asymptotic sequence over $(I+z)/z$ for some $z \in m\,\mathrm{Ass}\,R$, so $s(I) = s((I+z)/z)$ for all such z, q.e.d.

(3·5) gives some information concerning the behaviour of l and s when I is extended to a faithfully flat extension ring.

(3·5) LEMMA. *If S is a local ring which is a faithfully flat R-module, then the following statements hold:*

(3·4·1) $l(IS) = l(I)$.

(3·5·2) *If $P \in \hat{A}^*(I)$ and P^* is a minimal prime divisor of PS, then $P^* \in \hat{A}^*(IS)$; and, if $Q^* \in \hat{A}^*(IS)$, then $Q^* \cap R \in \hat{A}^*(I)$.*

(3·5·3) $s(IS) \geqslant s(I)$ *and equality holds if* altitude S = altitude R.

Proof. For (3·5·1), $S(X)$ is a faithfully flat $R(X)$-module, so it may be assumed that R/M is infinite, by (3·1·1). Let B be a minimal reduction of I and let $b_1, \ldots, b_h$ be a minimal basis of B, so the b_i are analytically independent in S, since if $F(b_1, \ldots, h_h) = 0$ is an S-relation of $b_1, \ldots, b_h$ then it is a linear combination of R-relations of $b_1, \ldots, b_h$, since S is R-flat. Also, $BI^k = I^{k+1}$ for some $k \geqslant 1$, so $BSI^kS = I^{k+1}S$, and so BS is a reduction of IS. Therefore, since the b_i are analytically independent in S, it follows from (1·2·3) that $l(I) = l(B) = l(BS) = l(IS)$.

(3.5.2) is given in ((17), (8·5) and (8.8)).

For (3·5·3), it follows from (3·5·2) and the definition of an asymptotic sequence over an ideal, (1·1·6), that if $c_1, \ldots, c_g$ are an asymptotic sequence over I, then $c_1, \ldots, c_g$ are also an asymptotic sequence over IS. Therefore $s(I) \leqslant s(IS)$. Finally, if altitude S = altitude R, then MS is N-primary, so it follows from (3·5·2) and that $s(IS) = s(I)$, q.e.d.

(3·6) COROLLARY. *The following statements hold when $R \subseteq S$ are as in* (3·5):

(3·6·1) $l(I) \geqslant l((IS+z)/z)$ *for all $z \in m\,\mathrm{Ass}\,S$ and equality holds for some such z.*

(3·6·2) $s(I) \leqslant s((IS+z)/z)$ *for all $z \in m\,\mathrm{Ass}\,S$ and equality holds for some such z if* altitude S = altitude R.

Proof. (3·6·1) follows immediately from (3·5·1) and (3·2), and (3·6·2) follows from (3·5·3) and (3·4).

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4. *The opposite inequality for $l(I)$ and local rings which have property* (E). With (3·6·1) and (3·6·2) in mind, it might be hoped that when $S = R^*$ the equalities hold for the same z (depending, of course, on I). For then, since R^*/z is quasi-unmixed, ((20), theorem 4·2 (b)) would imply that $l(I) = \operatorname{depth} z - s(I)$. That is, the inequality in ((20), theorem 4·2 (a)) would be sharpened to an equality. Unfortunately this is not the case, as is shown in (4·2). So in this section a new (opposite) inequality on $l(I)$ is proved and is then used to characterize local rings R which have the following property:

(E) *If I is an ideal in R, then there exists $z \in m \operatorname{Ass} R^*$ such that $l(I) = \operatorname{depth} z - s(I)$.*

We begin by proving the opposite inequality (from Rees' inequality ((20), theorem 4·2 (a))) on $l(I)$.

(4·1) Proposition. *If I is an ideal in a local ring R, then $l(I) \geqslant d^* - s(I)$, where $d^* = \min\{\operatorname{depth} z;\ z \in m \operatorname{Ass} R^*\}$. Moreover, $d^* - s(I) \geqslant 0$ and the inequality holds if* $\operatorname{height} I > 0$.

Proof. By (3·6·2), let $z \in m \operatorname{Ass} R^*$ such that $s(I) = s((IR^* + z)/z)$. Then $l(I) \geqslant l((IR^* + z)/z)$, by (3·6·1). Therefore, since R^*/z is quasi-unmixed, ((20), theorem 4·2 (b)) implies that $l(I) \geqslant l((IR^* + z)/z) = \operatorname{depth} z - s((IR^* + z)/z) = \operatorname{depth} z - s(I) \geqslant d^* - s(I)$.

Now, for each $z \in m \operatorname{Ass} R^*$ and for each minimal prime divisor P^* of $z + IR^*$ it holds that $P^* \in \hat{A}^*(IR^*)$, by ((16), (2·4)). And, for each such P^*, $s(IR^*) \leqslant \operatorname{depth} P^*$, by (18). Therefore, if $\operatorname{depth} z = d^*$ and P^* is a minimal prime divisor of $z + IR^*$, then it follows that $d^* - s(I) = d^* - s(IR^*) \geqslant d^* - \operatorname{depth} P^* = \operatorname{depth} z - \operatorname{depth} P^* \geqslant 0$, and the inequality holds if $\operatorname{height} I > 0$ (so $IR^* \nsubseteq z$), q.e.d.

A corollary of (4·1) concerning when $M \in \hat{A}^*(I)$ is given in (4·5).

(4·2) specifies the class of local rings for which (E) holds. For lack of a better name we say, in (4·2), that the set of minimal prime ideals in a local ring R is *consecutive* if $\{\operatorname{depth} z;\ z \in m \operatorname{Ass} R\} = \{n, n+1, \ldots, \operatorname{altitude} R - 1, \operatorname{altitude} R\}$ for some n such that $0 \leqslant n \leqslant \operatorname{altitude} R$. (Note that if $\operatorname{depth} z = 0$, and $z \in m \operatorname{Ass} R$, then $\operatorname{altitude} R = 0$, so R is unmixed.)

(4·2) Proposition. *A local ring R has property (E) if and only if $m \operatorname{Ass} R^*$ is consecutive.*

Proof. Assume first that $m \operatorname{Ass} R^*$ is not consecutive and let $m \in D = \{\operatorname{depth} z;\ z \in m \operatorname{Ass} R^*\}$ such that $m < \operatorname{altitude} R$ and $m + 1 \notin D$. Then by ((14), proposition 2·27) there exist $b_1, \ldots, b_{m+1}$ in M such that $M \in \hat{A}^*(I)$ and $\operatorname{height} I = m + 1$, where $I = (b_1, \ldots, b_{m+1})R$. Then clearly $l(I) = m + 1$, and $s(I) = 0$, by (1·2·4), so R does not have property (E).

The converse follows immediately from (4·1) and Rees' inequality ((20), theorem 4·2 (a)), q.e.d.

For a specific example of a local domain that does not have property (E), let R be as in ((9), example 2, pp. 203–205) in the case $m = 0$ and $r = 2$. Then R is a local domain such that $\{\operatorname{depth} z;\ z \in m \operatorname{Ass} R^*\} = \{1, 3\}$. More exotic examples follow from (6), where it is shown that given any finite set S of positive integers there exists a local domain R such that $\{\operatorname{depth} z;\ z \in m \operatorname{Ass} R^*\} = S$.

(4·3) has two further comments concerning property (E).

(4·3) *Remark.* The following statements hold for an ideal I in a local ring (R, M):

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(4·3·1) If $l(I) = l((IR^* + z)/z)$ for all $z \in m \operatorname{Ass} R^*$, then $l(I) = \operatorname{depth} z - s(I)$ for some $z \in m \operatorname{Ass} R^*$.

(4·3·2) If I is generated by an asymptotic sequence over (0) then $l(I) = \operatorname{depth} z - s(I)$ for some $z \in m \operatorname{Ass} R^*$. In particular, this holds if $I = bR$ with $b \notin \cup \{w \mid w \in m \operatorname{Ass} R\}$.

Proof. (4·3·1) By (3·6·2) let $z \in m \operatorname{Ass} R^*$ such that $s(I) = s(\bar{I})$, where $\bar{I} = (IR^* + z)/z$. Then the hypothesis and ((20), theorem 4·2 (*b*)) imply that $l(I) = l(\bar{I}) = \operatorname{depth} z - s(\bar{I}) = \operatorname{depth} z - s(I)$.

(4·3·2) If I is generated by an asymptotic sequence over (0), then clearly $l(I) = \operatorname{height} I$. Also, the generators of I are an asymptotic sequence over (0) in R^*/z for all $z \in m \operatorname{Ass} R^*$, by (3·5·2) and (3·3), so the first statement in (4·3·2) follows from (4·3·1). And the statement concerning bR follows from this, since $\hat{A}^*((0)) = m \operatorname{Ass} R$, q.e.d.

The next result points out that ((20), theorem 4·2 (*b*)) characterizes quasi-unmixed local rings.

(4·4) PROPOSITION. *A local ring R is quasi-unmixed if and only if* $l(I)$ equals altitude $R - s(I)$ *for all ideals I in R.*

Proof. If the equality holds for all ideals, then it holds for $I = (0)$, so $s(I) = \text{altitude } R$. However, this says that there exists a maximal asymptotic sequence over (0) of length equal to altitude R, so $\min\{\operatorname{depth} z;\ z \in m \operatorname{Ass} R^*\} = \text{altitude } R$, by ((16), (2·11)); that is, R is quasi-unmixed.

The converse is given by ((20), theorem 4·2 (*b*)), q.e.d.

This section will be closed with a corollary to (4·1).

(4·5) COROLLARY. *Let I be an ideal in a local ring (R, M). Then if $M \in \hat{A}^*(I)$, then the number of elements in a minimal basis of I is* $\geqslant \min\{\operatorname{depth} z;\ z \in m \operatorname{Ass} R^*\}$.

Proof. If $M \in \hat{A}^*(I)$, then $s(I) = 0$, so $l(I) \geqslant \min\{\operatorname{depth} z;\ z \in m \operatorname{Ass} R^*\}$, by (4·1). Now by (3·1) it may be assumed that R/M is infinite, so a minimal basis for I must have at least $l(I)$ elements, by ((10), theorem 2, p. 151), q.e.d.

5. *The second formula on the analytic spread of an ideal.* Since it is not true that property (*E*) holds for every local ring, by (4·2), we introduce the following definition that will allow us to replace the inequality in ((20), theorem 4·2 (*a*)) with an equality.

(5·1) *Definition.* If I is an ideal in a local ring R, then $w(I) = s((IR^* + Z)/Z)$, where $Z = \cap\{z \in m \operatorname{Ass} R^*;\ l(I) = l((IR^* + z)/z)\}$.

It should be noted that $w(I)$ is well defined, by (3·2) and (1·2·4). Also, note that $s(I) \leqslant w(I)$, by (3·6·2), and equality holds if R is quasi-unmixed; this is explained following (5·2).

(5·2) THEOREM. *Let I be an ideal in a local ring R. Then* $l(I) = \operatorname{depth} z - w(I)$ *for some $z \in m \operatorname{Ass} R^*$.*

Proof. Let $Z = \cap\{z \in m \operatorname{Ass} R^*;\ l(I) = l((IR^* + z)/z)\}$. Then $w(I)$ is well defined and $w(I) = s((IR^* + Z)/Z)$, by (5·1), so there exists $z/Z \in m \operatorname{Ass} R^*/Z$ such that $s((IR^* + Z) \div Z) = s(((IR^* + Z)/Z)/(z/Z))$, by (3·4); that is $w(I) = s((IR^* + z)/z)$. Also, by the choice of Z it follows that $l(I) = l((IR^* + z)/z)$. Therefore, since R^*/z is quasi-unmixed, ((20), theorem 4·2 (*b*)) implies that $l((IR^* + z)/z) = \text{altitude } R^*/z - s((IR^* + z)/z)$; that is $l(I) = \operatorname{depth} z - w(I)$, q.e.d.

Concerning (5·2), note that, if R is quasi-unmixed, then $w(I) = s(I)$, since $w(I) + l(I) = \text{altitude } R = s(I) + l(I)$, by (5·2) and ((20), theorem 4·2 (*b*)). Therefore (5·2)

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coincides with ((20), theorem 4·2 (*b*)) when R is quasi-unmixed, and in the general local ring case it is a different generalization (from that in (20)) of the quasi-unmixed case.

A coarser approach than (5·2), which also leads to an equality, would be to fix $z \in m \operatorname{Ass} R^*$ such that $l((IR^*+z)/z) = l(I)$ and let $w_z(I) = s((IR^*+z)/z)$; then $l(I) = \operatorname{depth} z - w_z(I)$, by ((20), theorem 4·2 (*b*)).

In closing, it should be noted that, at least for many ideals, ((20), theorem 4·2) (and also (5·2)) sharpens Brodmann's inequality (mentioned in the introduction) which, as noted in the introduction, sharpens Burch's inequality. Specifically, this holds for all ideals I such that $s(I) = \min\{\operatorname{depth} P^*;\ P^* \in \hat{A}^*(IR^*)\}$ (and I know of no ideal for which this does not hold). To see this, note that by ((8), corollary 17) and ((17), (2·3) and (2·7)), $\hat{A}^*(IR^*) \subseteq B^*(IR^*) = \{P \in \operatorname{Spec} R^*;\ P \text{ is a prime divisor of } I^{n-1}R^*/I^nR^* \text{ for all large } n\}$, and $B^*(IR^*) \subseteq A^*(IR^*) = \{P \in \operatorname{Spec} R^*;\ P \text{ is a prime divisor of } I^nR^* \text{ for all large } n\}$, by ((1), (3) and (7)). Also, if N is a finite R-module, then it follows from ((21), lemma 1, p. 394) that $\operatorname{grade} N \leqslant \operatorname{depth} P$ for all $P \in \operatorname{Ass} N$. Therefore, by (5·2) and ((20), theorem 4·2) (and for all large n) it holds that $l(I) = \operatorname{depth} z - w(I) \leqslant \operatorname{depth} z - s(I) \leqslant \operatorname{altitude} R - s(I) = \operatorname{altitude} R - \operatorname{depth} P^* \leqslant \operatorname{altitude} R - \operatorname{grade} I^{n-1}R^*/I^nR^*$ (since $P^* \in \hat{A}^*(IR^*) \subseteq B^*(IR^*)) = \operatorname{altitude} R - \operatorname{grade} I^{n-1}/I^n \leqslant \operatorname{altitude} R^* - \operatorname{grade} R^*/I^nR^*$ (since $B^*(IR^*) \subseteq A^*(IR^*)) = \operatorname{altitude} R - \operatorname{grade} R/I^n$.

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