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A NOTE ON CHAINS AND BOUNDING PAIRS OF DEHN TWISTS

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Abstract. Let N_g^k be a nonorientable surface of genus g with k punctures. In the first part of this note, after introducing preliminary materials, we will give criteria for a chain of Dehn twists to bound a disc. Then, we will show that automorphisms of the mapping class groups map disc bounding chains of Dehn twists to such chains. In the second part of the note, we will introduce bounding pairs of Dehn twists and give an algebraic characterization for such pairs.

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1. Introduction. Algebraic characterizations of elements of the mapping class groups and their subgroups play important role in low-dimension al topology. One can find such studies in [**1**, **2**, **6**, **7**, **10**], and [**5**]. In this note, we will concentrate on Dehn twists about two-sided simple closed curves with nonorientable complements, on nonorientable surfaces. In the next section, we will present preliminary materials. Then, we will introduce chains of Dehn twists and give criteria for a chain of Dehn twists to bound a disc. This will be the content of Lemma 3.1. Then, we will show that these criteria are algebraic in the sense of Theorem 3.3 stated below:

THEOREM 1.1. *Let $N = N_g^k$ be a connected nonorientable surface of genus $g \geq 5$ with k punctures. Then the image of a disc separating chain under an automorphism of* $\mathrm{M}(N)$ *is again a chain which separates a disc.*

In Section 4, we will introduce bounding pairs of Dehn twists and give an algebraic characterization for such pairs in Theorem 4.3:

THEOREM 1.2. *Let $N = N_g^k$ be a nonorientable surface of genus $g \geq 5$ with $k \geq 0$ punctures and a_1 and a_2 be disjoint, nonisotopic, nonseparating two-sided simple closed curves with nonorientable complements. Then (t_{a_1}, t_{a_2}) is a bounding pair if and only if the following conditions are satisfied:*

(1) *For any Dehn twist t_b satisfying the braid relation $t_{a_1} t_b t_{a_1} = t_b t_{a_1} t_b$, we have $t_b \notin C_{\mathrm{M}(N)}(t_{a_2})$;*
(2) *In the even genus case, the twists t_{a_1} and t_{a_2} are contained in a free generating set, whose elements are all two-sided nonseparating simple closed curves with nonorientable complement, of a maximal abelian subgroup K in* $\mathrm{M}(N)$*, of rank $r = \frac{3g-6-2s}{2} + k$, where $s = 1$ or $s = 2$.*

Moreover, if the conditions of theorem are satisfied, then $s = 2$ if both components of the surface N cut along the curves a_1 and a_2 are nonorientable of even genus, and $s = 1$ otherwise.

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Indeed, in [2], it has been shown that any automorphism of the mapping class group is induced by a homeomorphism of the surface and therefore any chain of Dehn twists bounding a disc or a bounding pairs of Dehn twists are mapped to elements of this type by any automorphism of the mapping class group. Our results provide independent proofs of these facts.

Bounding pairs of Dehn twists form natural elements of the Torelli group of the mapping class group of nonorientable surfaces, which is recently introduced by Hirose and Kobayashi ([4], see also [8]). It is natural to ask what the automorphism group of the Torelli group is, and we think that the above algebraic criteria might be useful for this problem as well as any other problem about the Torelli group or some other subgroups of the mapping class group.

2. Preliminaries. Let N denote a connected, nonorientable surface of genus g with k punctures. The mapping class group of N, the group of isotopy classes of all diffeomorphisms of N, where diffeomorphisms and isotopies fix each point on the boundary, is denoted by $\mathrm{Mod}(N)$. If we restrict ourselves to the diffeomorphisms and isotopies to those which do not permute the punctures, then we obtain the pure mapping class group $\mathrm{PMod}(N)$. The subgroup $\mathrm{PMod}^+(N)$ of the pure mapping class group consists of the pure mapping classes that preserve the local orientation around each puncture. Let $\mathrm{M}(N)$ be any of three groups $\mathrm{PMod}^+(N)$, $\mathrm{PMod}(N)$, and $\mathrm{Mod}(N)$.

Let H be a group. If $G \leq H$ is a subgroup and $h \in H$ is an element of H, then the center of H, the centralizer of G in H, and the centralizer of h in G will be denoted by $C(H)$, $C_H(G)$, and $C_G(h)$, respectively.

2.1. Simple closed curves on N. Let a be a simple closed curve on N. If a regular neighborhood of a is an annulus or a Möbius strip, then we call a a two-sided or a one-sided simple closed curve, respectively. The curve a will be called trivial, if it bounds a disc with at most one puncture or a Möbius band on N (or if it is isotopic to a boundary component). Otherwise, it is called nontrivial. A collection of curves $\mathcal{C} = \{a_1, \ldots, a_n\}$ is called a multicurve if the curves a_i are nontrivial, pairwise disjoint, pairwise nonisotopic, and none is isotopic to a boundary component of N. Let $N^{\mathcal{C}}$ denote the surface obtained by cutting N along all curves of $\mathcal{C}$. In particular, we denote by N^a the result of cutting of N along the simple closed curve a. The simple closed curve a is called nonseparating if N^a is connected. Otherwise, it is called separating.

2.2. Chains of Dehn twists. A sequence of Dehn twists $(t_{a_1}, \ldots, t_{a_n})$ is called a chain if the geometric intersection $i(a_i, a_{i+1}) = 1$, for $1 \leq i \leq n-1$ and $i(a_i, a_j) = 0$ for $|i-j| > 1$. The integer $n \geq 1$ is called the length of the chain. For a chain of Dehn twists, we always fix an orientation for a tubular neighborhood of the union of simple closed curves (which is always orientable) and consider Dehn twists using this orientation. It is known that two Dehn twists t_a, t_b satisfy the braid relation $t_a t_b t_a = t_b t_a t_b$ if and only if $i(a, b) = 1$ (see [3, 9]). In this note, unless we state otherwise a chain in a nonorientable surface N will mean a chain of Dehn twists about nonseparating two-sided simple closed curves with nonorientable complement.

3. A criteria for disc bounding chains of Dehn twists. In this section, we prove that automorphisms of the mapping class groups map disc bounding chains of Dehn twists to such chains which generalizes a result in [1] to punctured surfaces. We will include the

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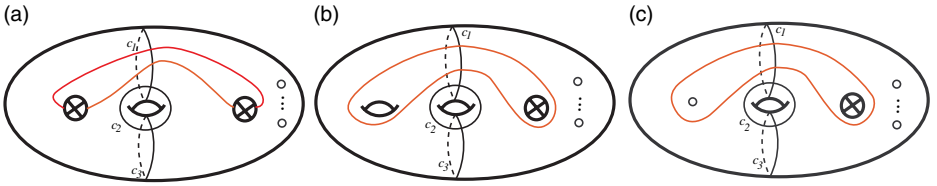


Figure 1. Three special cases of the second part of Lemma 3.1.

proof since the one in [1] has a gap, indicated by B. Szepietowski. For this purpose, we need the result stated in Lemma 3.5 of [1], and for the sake of completeness, we include the lemma and its proof.

LEMMA 3.1. *Let N be a connected nonorientable surface of genus g with k punctures.*

- *If a chain of two-sided nontrivial simple closed curves $\{c_1, \ldots, c_l\}$ on the surface N is maximal then l is an odd integer. Moreover, if g is an odd integer, then the chain is maximal if and only if the chain separates the surface into two components. In case of surfaces of even genus g, the chain $\{c_1, \ldots, c_l\}$ is maximal if and only if it is either separating, or nonseparating with orientable complement.*
- *Let $\{c_1, \ldots, c_{2k+1}\}$ be a separating maximal chain of two-sided nontrivial curves on N. Then it separates a disc if and only if there is no Dehn twist, which commutes with t_{c_i}, for all $i \neq 1$, and which does not commute with t_{c_1}.*

Proof. The key observation for the first part is that the tubular neighborhood of a chain of even length $l = 2r$ is the orientable surface of genus r with one boundary component $\Sigma_{r,1}$, which is embedded into N. It is readily obvious to see that such a chain is not maximal.

For the second part, note that Figure 1 proves the statement in three special cases, which indeed provides the proof of the only if part of the statement. The other direction can be seen as follows. Suppose that there is a curve d, which intersects the chain transversally at finitely many points of only c_1 and does not meet any other member of the chain. Let S denote the disc component of the two subsurfaces obtained by cutting the surface N along the chain. Since c_1 is the first element of the chain, the c_1 part of the boundary of S is connected. Note that $S \cap d$ is a disjoint union of arcs. Now, since S is a disc, we can isotope these arcs outside the surface S so that d does not intersect the chain anymore. Hence, t_d and t_{c_1} commute. □

REMARK 3.2. We know an algebraic characterization for Dehn twists about nonseparating simple closed curves with nonorientable complements (see [1, 2]). Therefore, by the paragraph in Subsection 2.2, any automorphism $\Psi : \mathrm{M}(N) \to \mathrm{M}(N)$ maps a chain of Dehn twists of length at least two to another chain of Dehn twists of the same length.

THEOREM 3.3. *Let $N = N_g^k$ be a connected nonorientable surface of genus $g \geq 5$ with k punctures. Then the image of a disc separating chain under an automorphism of* $\mathrm{M}(N)$ *is again a chain which separates a disc.*

Proof. If the genus of the surface is odd then a chain is separating if and only if it is maximal in $\mathrm{M}(N)$. However, being maximal is clearly preserved under an automorphism. Now Lemma 3.1 finishes the proof.

Now assume that the genus $g \geq 6$ is an even integer. Further assume that $t_{c_1}, \ldots, t_{c_{2l+1}}$ is a disc separating chain in $\mathrm{M}(N)$. Hence, when we delete a tubular neighborhood of the

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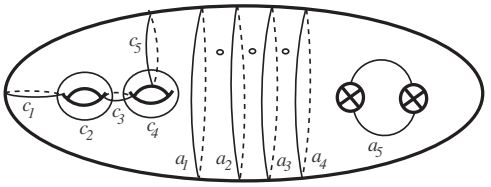


Figure 2. An example of the abelian subgroup K of rank 5.

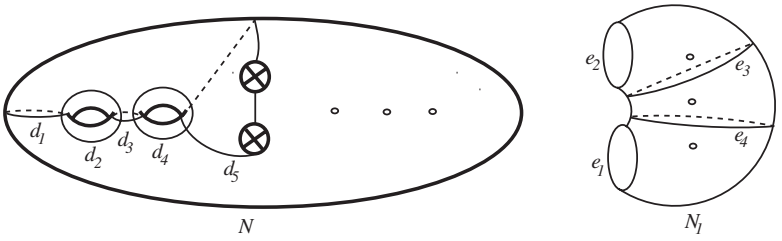


Figure 3. The characteristic curves e_1 and e_2.

chain $c_1, \ldots, c_{2l+1}$ from the surface, we obtain a disjoint union of a disc and a nonorientable surface, say N_0, of genus $g-2l$ with k punctures and one boundary component. By an Euler characteristic calculation and inspection, we see that the group $\mathrm{M}(N_0)$ has an abelian subgroup K of rank $\dfrac{3(g-2l)-6}{2}+k+2$, contained in each $C_{\mathrm{M}(N)}(t_{c_i})$, which is freely generated by Dehn twists about pairwise nonisotopic simple closed curves (see Figure 2 for $g=6$, $k=3$, $l=2$, rank(K)=5, $K=\langle t_{a_i}\rangle$).

Now suppose on the contrary that the image $t_{d_1}, \ldots, t_{d_{2l+1}}$ of the chain $t_{c_1}, \ldots, t_{c_{2l+1}}$ under an automorphism is not separating. So by Lemma 3.1, the complement of a tubular neighborhood of the chain $d_1, \ldots, d_{2l+1}$ in N is an orientable surface of genus $\dfrac{g-2l}{2}-1$ with k punctures and two boundary components, say e_1 and e_2 (see Figure 3). Let us call this surface N_1. (Since the surface N is nonorientable, the curves e_1 and e_2 are both characteristic curves, that is, nonseparating two-sided simple closed curves with orientable complement in the surface N.) The image of the abelian subgroup K under the same automorphism is again a maximal subgroup in $\mathrm{M}(N)$, and it lies in each $C_{\mathrm{M}(N)}(t_{d_i})$, $i=1, \ldots, 2l+1$.

However, the orientable subsurface N_1 can support an abelian subgroup K_0 in $\mathrm{M}(N)$, which lies in each $C_{\mathrm{M}(N)}(t_{d_i})$, of rank at most rank$(K)-1$ (in the example in Figure 3, we may take K_0 as the group generated by e_i, $i=1, 2, 3, 4$). This finishes the proof. □

4. Bounding pairs of Dehn twists. In this section, we start with the following definition and a technical result. Then, we give an algebraic characterization for bounding pairs of Dehn twists.

DEFINITION 4.1. A bounding pair (a_1, a_2) is defined to be a pair of disjoint, nonseparating simple closed curves on N (so that together they separate the surface). We call (t_{a_1}, t_{a_2}) a bounding pair of Dehn twists about (a_1, a_2).

We note that our definition is slightly different than the usual one in the literature since we do not require that the two twists in a bounding pair are in different directions.

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PROPOSITION 4.2. *Suppose that N is a nonorientable surface of even genus g with k punctures. Then for any integer $s = 0, \ldots, \frac{g-2}{2}$, the group $\mathrm{M}(N)$ has an abelian subgroup of rank $\frac{3g-6-2s}{2} + k$, which is freely generated by Dehn twists about pairwise non-isotopic nonseparating simple closed curves, and so that no abelian subgroup containing this subgroup has bigger rank. Moreover, when we cut the surface along these curves, the resulting surface is a disjoint union of $g + k - 2s - 2$ many pair of pants (sphere with three holes) and $2s$ many two-holed real projective planes.*

Proof. Let $r = \frac{3g-6-2s}{2} + k$. By the maximality condition of the proposition, all the components of the surface cut along these r simple closed curves have Euler characteristic -1. In other words, each component is either a pair of pants or a two-holed real projective plane. Assume that there are l many pair of pants and m many two-holed real projective planes. Hence, considering Euler characteristics, we obtain the equation

$$2 - g - k = \chi(N) = \chi\left(\bigcup_{l} N_0 \cup \bigcup_{m} N_1\right) = -l - m.$$

Now let us count the number of holes in all the resulting surfaces, regarding both a puncture or a boundary component as a hole. The total number of holes in l many pair of pants and m many two-holed real projective planes is $3l + 2m$. On the other hand, cutting the surface, which has already k punctures, along r many two-sided curves introduces $2r$ many boundary components. Hence, we obtain the equation

$$2r + k = 3g - 6 - 2s + 3k = 3l + 2m.$$

These two equations yield $m = 2s$ and $l = g + k - 2s - 2$ as desired. Finally, the existence of such subgroups is readily seen by inspection. □

THEOREM 4.3. *Let $N = N_g^k$ be a nonorientable surface of genus $g \geq 5$ with $k \geq 0$ punctures and a_1 and a_2 be disjoint, nonisotopic, nonseparating two-sided simple closed curves with nonorientable complements. Then (t_{a_1}, t_{a_2}) is a bounding pair if and only if the following conditions are satisfied:*

(1) *For any Dehn twist t_b satisfying the braid relation $t_{a_1} t_b t_{a_1} = t_b t_{a_1} t_b$, we have $t_b \notin C_{\mathrm{M}(N)}(t_{a_2})$;*
(2) *In the even genus case, the twists t_{a_1} and t_{a_2} are contained in a free generating set, whose elements are all two-sided nonseparating simple closed curves with nonorientable complement, of a maximal abelian subgroup K in $\mathrm{M}(N)$, of rank $r = \frac{3g-6-2s}{2} + k$, where $s = 1$ or $s = 2$.*

Moreover, if the conditions of theorem are satisfied, then $s = 2$ if both components of the surface N cut along the curves a_1 and a_2 are nonorientable of even genus, and $s = 1$ otherwise.

Proof. Suppose first that a_1 and a_2 separate the surface. Now if a Dehn twist t_b satisfies the braid relation $t_{a_1} t_b t_{a_1} = t_b t_{a_1} t_b$, then the curves b and a_1 intersect geometrically once (cf. [**9**]). However, since a_1 and a_2 separate the surface, b and a_2 must intersect nontrivially and thus t_b cannot be contained in $C_{\mathrm{M}(N)}(t_{a_2})$. Moreover, it is easy to construct the required free abelian subgroup K, in case the genus of N is even.

For the other direction assume that the conditions of the lemma are satisfied, but on the contrary suppose that the surface cut along the curves a_1 and a_2 is connected. Hence, the surface N cut along only a_2, say N_0, is connected.

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First, we will treat the case where the genus g is an odd integer. Hence, the surface N_0 is a connected nonorientable surface of genus at least 3. Moreover, the curve a_1 is still nonseparating in N_0. Hence, there is a two-sided nonseparating simple closed curve b in N_0, whose geometric intersection with a_1 is one. This is a contradiction to the first condition of the assumption.

Now let us consider the even genus case. By Proposition 4.2, the surface cut along the r many nonseparating curves, about which the Dehn twists generate the subgroup K, has nonorientable components. Thus, the surface N_0 is a connected nonorientable surface of genus at least four, and the curve a_1 is neither separating nor characteristic in N_0. Hence, as above there is a two-sided curve b in N_0, whose geometric intersection with a_1 is one. This finishes the proof for the even genus case.

The final part of the lemma is an immediate consequence of Proposition 4.2. □

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