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AN EXPLICIT DIAGONAL RESOLUTION FOR A NON-ABELIAN METACYCLIC GROUP

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Abstract. We consider the notion of a free resolution. In general, a free resolution can be of any length depending on the group ring under investigation. The metacyclic groups $G(pq)$ however admit periodic resolutions. In the particular case of $G(21)$ it is possible to achieve a fully *diagonalized resolution*. In order to achieve a diagonal resolution, we obtain a complete list of *indecomposable modules* over Λ. Such a list aids the decomposition of the augmentation ideal (the first *syzygy*) into a direct sum of indecomposable modules. Therefore, we are able to achieve a diagonalized map here. From this point it is possible to decompose all of the remaining syzygies in terms of indecomposable modules, leaving a diagonal resolution.

§1. *Introduction.* Throughout this paper, our concern is to investigate free resolutions of group rings $\mathbb{Z}[G]$ of certain finite groups G. We look at the metacyclic groups $G(pq)$:

$$G(pq) = \langle x, y \mid x^p = y^q = 1,\ yx = x^r y \rangle,$$

where $r^q \equiv 1 \pmod{p}$ and $p,\ q$ are prime with $q/(p-1)$. Specifically, we show that for $p = 7$ and $q = 3$ the group $G(21)$ has a completely *diagonalized* minimal free resolution. Such diagonal resolutions, hitherto unknown, significantly simplify certain problems.

Diagonal resolutions aid the attack on a problem in low-dimensional topology, namely the $\mathcal{R}(2)$–$\mathcal{D}(2)$ problem that originates from Wall's $\mathcal{D}(n)$ problem [13]. The $\mathcal{R}(2)$–$\mathcal{D}(2)$ problem is examined elsewhere, and the reader is referred to Johnson's book [7] which discusses the problem for dihedral and quaternionic groups.

Any module M has a free resolution

$$\cdots \longrightarrow F_{n+1} \xrightarrow{\partial_{n+1}} F_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\epsilon} M \longrightarrow 0,$$

$$K(n+1) \qquad\qquad K(1)$$

where F_m is a finitely generated free module over a group ring Λ as established by Hilbert [6]. Here $K(m)$ represents the mth *syzygy*. Given two resolutions

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$\mathfrak{F}, \mathfrak{F}'$, we have exact sequences at each kernel, namely $K(m)$, $K(m)'$. Then by Schanuel's lemma there exists an isomorphism $K(m) \oplus F' \cong K(m)' \oplus F$ for finitely generated free modules F, F'. We therefore say that $K(m)$ and $K(m')$ are *stably equivalent*. We denote the class of all modules stably equivalent to $K(m)$ by $\Omega_m(M)$ (the mth syzygy). Details on tree structures and stable modules are found in [7].

Constructing explicit free resolutions is not a straightforward task. Most groups do not have cohomological periodicity. Artin–Tate–Zassenhaus gave conditions for when a free resolution may be periodic [1, Ch. XII, p. 262]. The group $G(21)$ has cohomological period 6. More details on groups of periodic cohomology can be found in Swan's paper [12] and Wolf [14, Ch. 5].

Using Fox's method [4], we can obtain ∂_1 and ∂_2. However, the task of computing the higher differentials is more difficult. Instead of using Fox's method, we look directly at the syzygies at the minimal level. We decompose each syzygy into a direct sum of indecomposable modules. Thus, by describing ∂_1 as a direct sum of polynomials, we manage to show completely diagonalized minimal free resolution for the group $G(21)$.

Denote $\Lambda = \mathbb{Z}[G(21)]$. We define a Λ-lattice to be a Λ-module that is finitely generated and free over $\mathbb{Z}$. When we discuss modules here we automatically imply lattices unless clearly stated otherwise. Denote $\mathcal{F}(\Lambda)$ as the category of all Λ-lattices. It may be regarded as a full subcategory of Λ-modules. Lastly we denote by $\mathcal{D}\mathrm{er} = \mathcal{D}\mathrm{er}(\Lambda)$ the derived module category [8, Ch. 5] of $\mathcal{F}(\Lambda)$, which is obtained by setting any Λ-homomorphism that factors through a projective module equal to zero.

§2. *Wedderburn decomposition.* We first describe the Wedderburn decomposition of $\mathbb{C}[G(21)]$. We need to obtain the conjugacy classes of $\mathbb{C}[G(21)]$, which are: $\{1\}, \{x, x^2, x^4\}, \{x^3, x^5, x^6\}, \{y, xy, \ldots, x^6y\}, \{y^2, xy^2, \ldots, x^6y^2\}$. In total, there are five conjugacy classes, leaving only one option for the Wedderburn decomposition:

$$\mathbb{C}[G(21)] \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times M_3(\mathbb{C}) \times M_3(\mathbb{C}).$$

We move on to $\mathbb{Q}[G(21)]$. Taking the cyclic group $C_7 = \langle x | x^7 = 1 \rangle$, the following is a square for the rational group ring $\mathbb{Q}[C_7]$:

$$\begin{array}{ccc} \mathbb{Q}[C_7] & \longrightarrow & \mathcal{I}^* \\ \downarrow & & \downarrow \\ \mathbb{Q} & \longrightarrow & 0 \end{array}$$

Therefore, $\mathbb{Q}[C_7] \cong \mathbb{Q} \times \mathcal{I}^*$. Over $\mathbb{Z}[C_7]$, we clearly see that $\mathcal{I}^* = \mathbb{Q}(\zeta)$, where $\zeta = e^{2\pi i/7}$, the primitive seventh root of unity. We denote $\mathcal{I}^* = K$.

In order to describe $\mathbb{Q}[G(21)]$, we apply the cyclic algebra construction to the above diagram. We take $\mathcal{R}$ to be a commutative ring and $\theta : \mathcal{R} \to \mathcal{R}$; then the cyclic algebra construction $\mathscr{C}(\mathcal{R}, \theta, a)$ of an involuted ring $(\mathcal{R}, \theta, a)$ is

$$\mathscr{C}_n(\mathcal{R}, \theta, a) = \mathcal{R} \dot{+} \mathcal{R}y \dot{+} \cdots \dot{+} \mathcal{R}y^{n-1},$$

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where $a \in \mathcal{R}$, $y\lambda = \theta(\lambda)y$ and $y^n = a$. Using this construction, take $\mathscr{C}_3(\mathbb{Q}[C_7], \theta_{[3]}, 1)$, where $y\zeta = \zeta^2 y$, to obtain the following square for $\mathbb{Q}[G(21)]$:

$$\begin{array}{ccc} \mathbb{Q}[G(21)] & \longrightarrow & \mathscr{C}_3(K, \theta_{[3]}, 1) \\ \downarrow & & \downarrow \\ \mathbb{Q}[C_3] & \longrightarrow & 0 \end{array}$$

This leaves us with the decomposition $\mathbb{Q}[G(21)] \cong \mathbb{Q}[C_3] \times \mathscr{C}_3(K, \theta_{[3]}, 1)$. We shall now obtain a description for $\mathscr{C}_3(K, \theta_{[3]}, 1)$.

PROPOSITION 2.1. *The following surjective ring homomorphism exists:*

$$\rho_\alpha : \mathbb{Q}[G(21)] \to M_3(K_0),$$

where $K_0 = \mathbb{Q}(\alpha)$ *and* $\alpha = i\sqrt{7}$.

Proof. Return to $\mathbb{Q}[C_7]$. Here we have $\mathbb{Q}[C_7] \cong \mathbb{Q}[x]/(x-1) \times \mathbb{Q}[x]/c_7(x)$, where $c_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$. From the fact that $\mathbb{Q}[x]/c_7(x) \cong \mathbb{Q}(\zeta)$, we have

$$\begin{aligned} c_7(x) &= x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\ &= (x-\zeta)(x-\zeta^2)(x-\zeta^3)(x-\zeta^4)(x-\zeta^5)(x-\zeta^6) \\ &= (x^3 - \alpha x^2 + \overline{\alpha}x - 1)(x^3 - \overline{\alpha}x^2 + \alpha x - 1), \end{aligned}$$

where $\alpha = \zeta + \zeta^2 + \zeta^4$ and $\overline{\alpha} = \zeta^3 + \zeta^5 + \zeta^6$. By equating coefficients, we obtain α and $\overline{\alpha}$ explicitly.

$$\begin{aligned} \alpha + \overline{\alpha} &= -1, \\ \alpha + \alpha\overline{\alpha} + \overline{\alpha} &= 1. \end{aligned}$$

This can be rearranged to give $\alpha^2 + \alpha + 2 = 0$. So, $\alpha = (-1/2 + \sqrt{-7}/2)$ and $\overline{\alpha} = (-1/2 - \sqrt{-7}/2)$. Thus, $K_0 = \mathbb{Q}(i\sqrt{7})$. This leaves us with the following representation:

$$\rho_\alpha : \mathbb{Q}[G(21)] \to M_3(\mathbb{Q}(\alpha)),$$

$$\rho_\alpha(x) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & \alpha+1 \\ 0 & 1 & \alpha \end{pmatrix}, \quad \rho_\alpha(y) \mapsto \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

This representation has the properties $x^7 = y^3 = 1$ and $yx = x^2$. □

We see from the above that $\dim_{K_0}(K) = 3$ and $\dim_{\mathbb{Q}}(K_0) = 2$. Hence, $\mathscr{C}_3(K, \theta_{[3]}, 1) \cong M_3(\mathbb{Q}(i\sqrt{7}))$. This results in the full Wedderburn decomposition of $\mathbb{Q}[G(21)]$:

$$\mathbb{Q}[G(21)] \cong \mathbb{Q} \times \mathbb{Q}(\omega) \times M_3(\mathbb{Q}(i\sqrt{7})),$$

where $\omega = e^{2\pi i/3}$, the third root of unity.

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§3. *Integral representation theory.* Taking the integral group ring of C_7, we get

$$\begin{array}{ccc} \mathbb{Z}[C_7] & \longrightarrow & \mathcal{I}^* \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{F}_7 \end{array}$$

The dual of the augmentation ideal is $\mathcal{I}^* = \mathbb{Z}(\zeta) = R$. Thus, R is the ring of algebraic integers of K. More detail about this type of square can be seen in Milnor [9, Ch. 3]. Taking the cyclic algebra of $\mathbb{Z}[C_7]$, we obtain

$$\begin{array}{ccc} \Lambda & \longrightarrow & \mathscr{C}_3(R, \theta_{[3]}, 1) \\ \downarrow & & \downarrow \\ \mathbb{Z}[C_3] & \longrightarrow & \mathbb{F}_7[C_3] \end{array}$$

where $\mathbb{F}_7[C_3] \cong \mathbb{F}_7 \times \mathbb{F}_7 \times \mathbb{F}_7$. The problem here is how to describe $\mathscr{C}_3(R, \theta_{[3]}, 1)$. We use Rosen's theorem for the specific case of $G(21)$.

THEOREM 3.1 Rosen [11]. *Take a group of the form $G(pq)$. Let $S = \mathbb{Z}[\zeta_p]$; q is a divisor of $p-1$. We then have $\theta_{[q]} = \theta^{(p-1/q)}$ with order q. Therefore, the cyclic algebra construction $\mathscr{C}_q(S, \theta_{[q]}, 1)$ decomposes in the following manner:*

$$\mathscr{C}_q(S, \theta_{[q]}, 1) \cong \mathcal{J}_1 \oplus \cdots \oplus \mathcal{J}_q,$$

where each $\mathcal{J}_i$ is a two-sided ideal. Furthermore, $\mathcal{J}_i \ncong \mathcal{J}_k$ if $i \neq k$. In fact, each of the ideals are of the form $P^e = (\zeta_p - 1)^e \mathbb{Z}[\zeta_p]$, where $0 \leqslant e \leqslant q$.

In the case of $G(21)$, the cyclic algebra construction decomposes, $\mathscr{C}_3(R, \theta_{[3]}, 1) = R \oplus P \oplus P^2$, where $P^e = (\zeta - 1)^e \mathbb{Z}[\zeta]$, and $e = 0, 1, 2$. Using Rosen's theorem we need to verify that the only options for the three non-isomorphic indecomposable modules in $\mathscr{T}_3(R_0, \pi)$ are in fact R, P, P^2. In the process, we will also obtain the duality relations between R, P, P^2.

We first restrict attention to $\mathbb{Z}[C_7]$. Over $\mathbb{Z}[C_7]$, the dual augmentation ideal $\mathcal{I}^* = R$. Inside the ring R there consist ideals of the form $P^e = (\zeta - 1)^e R$ as established earlier; in fact, all modules of the form $P^e \cong R$ over $\mathbb{Z}[C_7]$. Using the cyclic algebra construction we are able look at modules over Λ via the twisting relation $y\zeta = \zeta^2 y$.

Now consider R over Λ. Define $R = \mathbb{Z}[\zeta]/c_7(\zeta)$, where ζ is the image of x under $\mathbb{Z}[x] \to R$. This means that $\zeta^6 = -(1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5)$. So, we can note that the generator x of C_7 in each P^e as a right module acts the same way as ζ:

$$\lambda \cdot x = \lambda \cdot \zeta.$$

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In fact, x can be represented in the following manner: $\rho : G \to \mathrm{GL}_6(\mathbb{Z})$:

$$\rho(x) = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Extend from C_7 to $G(21)$ by allowing y to act as $y \cdot \zeta = \alpha(\zeta)$, where

$$\begin{aligned} \alpha : C_7 &\to C_7, \\ \alpha(x) &= x^2. \end{aligned}$$

So, R, P, $P^2, \ldots$ all become modules over Λ and each P^e can be represented by taking $\rho_e : G \to \mathrm{GL}_6(\mathbb{Z})$. It can be seen that $\rho_e(x) = \rho(x)$ for all $e \geqslant 1$. However, the $\rho_e(y)$ vary. The question of how many distinct non-isomorphic P^e there are in $G(21)$ can now be answered. Firstly, we have the following result.

PROPOSITION 3.2. $R \cong_\Lambda P^6$.

Proof. We have

$$\begin{aligned} (\zeta - 1)^6 &= \zeta^6 - 6\zeta^5 + 15\zeta^4 - 20\zeta^3 + 15\zeta^2 - 6\zeta + 1 \\ &= -7(\zeta^5 - 2\zeta^4 + 3\zeta^3 - 2\zeta^2 + \zeta) \\ &= -7u, \end{aligned}$$

where $u = \zeta^5 - 2\zeta^4 + 3\zeta^3 - 2\zeta^2 + \zeta$. Taking the determinants of both sides,

$$\begin{aligned} \det(\mathrm{RHS}) &= 7^6 \det(u), \\ \det(\mathrm{LHS}) &= \det(\zeta - 1)^6 \\ &= \det(\rho(x) - 1)^6. \end{aligned}$$

From the representation $\rho(x)$, we see that $\det(\rho(x) - 1) = \pm 7$. This implies that $\det(u) = \pm 1$. Hence,

$$\begin{aligned} R &\to P^6, \\ \lambda &\mapsto -7\lambda u \end{aligned}$$

is an isomorphism. □

Thus, *a priori*, we only consider R, P, P^2, P^3, P^4, P^5. By looking at the duality of each module, we establish the distinct modules.

PROPOSITION 3.3. *The following duality relations hold:*

(i) $R \cong P^*$;
(ii) $P^2 \cong (P^2)^*$;
(iii) $P^3 \cong P^*$;

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(iv) $P^4 \cong R^*$;
(v) $P^5 \cong (P^2)^*$.

Proof. For an isomorphism $A : R \to P^*$, the following relations need to hold:

$$A\rho_0(g) = \rho_1^*(g)A,$$
$$\text{i.e.} \quad A\rho_0(g) = \rho_1(g^{-1})^T A.$$

This will need to be checked for the two generators x, y. Hence, an invertible matrix $A \in \mathrm{GL}_6(\mathbb{Z})$ is needed, for which the following conditions hold:

(I) $A\rho(x) = \rho(x^{-1})^T A$;
(II) $A\rho_0(y) = \rho_1(y^{-1})^T A$;
(III) $\det(A) = \pm 1$.

Throughout each of the proofs, the representation for x will be the same:

$$\rho(x) = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

As a result of $\rho(x)$ being the same throughout, a general matrix A satisfying (I) looks like

$$A = \begin{pmatrix} a & b & c & d & e & f \\ \Sigma & a & b & c & d & e \\ f & \Sigma & a & b & c & d \\ e & f & \Sigma & a & b & c \\ d & e & f & \Sigma & a & b \\ c & d & e & f & \Sigma & a \end{pmatrix},$$

where $\Sigma = -(a + b + c + d + e + f)$. In order to obtain A, the representations for y in each P^e are needed. They are described by $\rho_e(y)$:

$$\rho_0(y) = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \quad \rho_1(y) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix},$$

$$\rho_2(y) = \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & -1 & -1 & 2 & 0 \\ 0 & 2 & -1 & -1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix}, \quad \rho_3(y) = \begin{pmatrix} 1 & 0 & -2 & 2 & 0 & -1 \\ 3 & 0 & -3 & 2 & 1 & -1 \\ 3 & 1 & -3 & 0 & 3 & -1 \\ 1 & 3 & -3 & -1 & 3 & 0 \\ 0 & 3 & -2 & -1 & 1 & 2 \\ 0 & 1 & 0 & -1 & 0 & 2 \end{pmatrix},$$

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$$\rho_4(y) = \begin{pmatrix} 1 & -1 & -2 & 3 & 0 & -3 \\ 4 & -1 & -5 & 5 & 1 & -4 \\ 6 & 0 & -6 & 3 & 4 & -4 \\ 4 & 3 & -6 & 0 & 6 & -3 \\ 1 & 5 & -5 & -1 & 4 & 0 \\ 0 & 3 & -2 & -1 & 1 & 2 \end{pmatrix},$$

$$\rho_5(y) = \begin{pmatrix} 1 & -4 & 0 & 4 & -1 & -5 \\ 5 & -5 & -5 & 9 & 0 & -9 \\ 10 & -4 & -9 & 9 & 4 & -10 \\ 10 & 0 & -10 & 4 & 9 & -9 \\ 5 & 5 & -9 & 0 & 9 & -5 \\ 1 & 5 & -5 & -1 & 4 & 0 \end{pmatrix}.$$

For each P^e, it can be checked that $\rho(x^7) = \rho_e(y^3) = 1$. Also, $\rho_e(y)\rho(x) = \rho(x^2)\rho_e(y)$. For each duality relation it is now possible to obtain A subject to conditions (II) and (III):

(i) $R \cong P^*$

$$A\rho_0(y) = \rho_1(y^{-1})^T A, \qquad A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

(ii) $P^2 \cong (P^2)^*$

$$A\rho_2(y) = \rho_2(y^{-1})^T A, \qquad A = \begin{pmatrix} 0 & 0 & -1 & 2 & -2 & 1 \\ 0 & 0 & 0 & -1 & 2 & -2 \\ 1 & 0 & 0 & 0 & -1 & 2 \\ -2 & 1 & 0 & 0 & 0 & -1 \\ 2 & -2 & 1 & 0 & 0 & 0 \\ -1 & 2 & -2 & 1 & 0 & 0 \end{pmatrix};$$

(iii) $P^3 \cong P^*$

$$A\rho_3(y) = \rho_1(y^{-1})^T A, \qquad A = \begin{pmatrix} 0 & 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & 1 & -2 & 2 & -1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ -1 & 0 & 0 & 0 & 1 & -2 \\ 2 & -1 & 0 & 0 & 0 & 1 \\ -2 & 2 & -1 & 0 & 0 & 0 \end{pmatrix};$$

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(iv) $P^4 \cong R^*$

$$A\rho_4(y) = \rho_0(y^{-1})^T A, \qquad A = \begin{pmatrix} 1 & -2 & 2 & -1 & 0 & 0 \\ 0 & 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & 1 & -2 & 2 & -1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ -1 & 0 & 0 & 0 & 1 & -2 \\ 2 & -1 & 0 & 0 & 0 & 1 \end{pmatrix};$$

(v) $P^5 \cong (P^2)^*$

$$A\rho_5(y) = \rho_2(y^{-1})^T A, \qquad A = \begin{pmatrix} 3 & -1 & 0 & 1 & -3 & 5 \\ -5 & 3 & -1 & 0 & 1 & -3 \\ 5 & -5 & 3 & -1 & 0 & 1 \\ -3 & 5 & -5 & 3 & -1 & 0 \\ 1 & -3 & 5 & -5 & 3 & -1 \\ 0 & 1 & -3 & 5 & -5 & 3 \end{pmatrix}.$$

□

COROLLARY 3.4. *The duality relations give rise to the following:*

- $R \cong P^3$;
- $P \cong P^4$;
- $P^2 \cong P^5$.

Proof. The results drop out by manipulating the duality relation results. □

Therefore, it is clear that there are only three distinct non-isomorphic P^e modules contained over Λ, namely R, P, P^2. Also, the duality relations show that P^* is the dual of R and that P^2 is self-dual.

§4. *The indecomposable modules.* Following Curtis and Reiner [2, §31], we say that two modules M, N belong to the same *genus* when

$$M_{(p)} \cong N_{(p)}$$

for all primes p.

A $\mathbb{Z}[G]$-module M is said to be indecomposable if $M \neq 0$ and if it is impossible to express M as a direct sum of two non-trivial submodules $M \neq M_1 \oplus M_2$. A $\mathbb{Z}[G]$-module that is finitely generated and free over $\mathbb{Z}$ is called a $\mathbb{Z}[G]$-lattice. All modules in this paper are $\mathbb{Z}[G]$-lattices unless otherwise stated.

In order to obtain a complete description of all the indecomposable modules in Λ, we must first describe the basic indecomposable modules in Λ. We look at the integral commutative square over Λ:

$$\begin{array}{ccc} \Lambda & \longrightarrow & \mathscr{C}_3(R, \theta_{[3]}, 1) \\ \downarrow & & \downarrow \\ \mathbb{Z}[C_3] & \longrightarrow & \mathbb{F}_7[C_3] \end{array}$$

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From this square arise basic indecomposable modules over $\mathbb{Z}[C_3]$ and $\mathscr{C}_3(R, \theta_{[3]}, 1)$. There are three indecomposable modules over $\mathbb{Z}[C_3]$.

(i) The trivial module: $\mathbb{Z}$.
(ii) The augmentation ideal, namely $\mathcal{J} = \mathrm{Ker}(\mathbb{Z}[C_3] \to \mathbb{Z})$.
(iii) The group ring itself $\mathbb{Z}[C_3]$. (This may be obtained by the non-split extension $0 \to \mathcal{J} \to \mathbb{Z}[C_3] \to \mathbb{Z} \to 0$.)

The non-isomorphic distinct modules over $\mathscr{C}_3(R, \theta, 1)$ were established earlier, and are:

(iv) $R = P^0 = \mathbb{Z}[\zeta] = \mathbb{Z}[x]/c_7(x)$, where $c_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$;
(iiv) $P = P^1 = (\zeta - 1)R$;
(iiiv) $P^2 = (\zeta - 1)^2 R$.

To describe the other genera of indecomposable Λ modules, consider the cohomological relations between the two types of "basic" indecomposable modules. These indecomposables are formed by the non-split extensions $0 \to \mathcal{A} \to ? \to \mathcal{B} \to 0$, where $\mathcal{A}$ is a direct sum of a combination of the modules R, P, P^2, and $\mathcal{B} = \mathbb{Z}$, $\mathcal{J}$ or $\mathbb{Z}[C_3]$.

Pu [10] describes half of the cohomological relations. An alternative method of proof is established below. It also includes the duality arguments of the cohomological relations.

We recall the Eckmann–Shapiro lemma, as it will be extensively utilized when describing $\mathrm{Ext}^1_{\mathbb{Z}[G]}(\mathcal{A}, \mathcal{B})$.

LEMMA 4.1 (Eckmann–Shapiro). *Let G be a finite group. Then let H be a subgroup $H \subset G$ and take the inclusion map to be $i : \mathbb{Z}[H] \hookrightarrow \mathbb{Z}[G]$. Then take N as a G-lattice and M as an H-lattice, so that $i^*(N)$ is the restriction of scalars, while $i_*(M) = M \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G]$ is the extension of scalars. Hence, the following isomorphisms exist:*

$$\mathrm{Hom}_{\mathbb{Z}[G]}(i_*(M), N) \cong \mathrm{Hom}_{\mathbb{Z}[H]}(M, i^*(N))$$

and the more difficult case (Shapiro's lemma):

$$\mathrm{Hom}_{\mathbb{Z}[G]}(N, i_*(M)) \cong \mathrm{Hom}_{\mathbb{Z}[H]}(i^*(N), M).$$

We proceed by looking at the sequence $0 \to \mathcal{I} \to \mathbb{Z}[C_7] \to \mathbb{Z} \to 0$. Here $\mathcal{I}$ is the augmentation ideal and, over $\mathbb{Z}[C_7]$, $\mathcal{I} = R$ because R is the dual of the augmentation ideal and, as stated earlier, $\mathbb{Z}[C_7]$ has cohomological period 2, leading to R being the augmentation ideal.

PROPOSITION 4.2. *Over Λ, the augmentation ideal $\mathcal{I} = P$.*

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Proof. Take the sequence $0 \to \mathcal{I} \to \mathbb{Z}[C_7] \to \mathbb{Z} \to 0$ over Λ. One gets

$$0 \to P \to \overline{\mathbb{Z}[C_7]} \to \mathbb{Z} \to 0,$$

where $\overline{\mathbb{Z}[C_7]}$ is $\mathbb{Z}[C_7]$ by conjugation. Over Λ, R is still the dual of the augmentation ideal, but, to obtain the augmentation ideal $\mathcal{I}$ this time, we can take the dual of $\mathcal{I}^* = R$, giving R^*. From earlier we have seen that over Λ, $R^* = P$. □

From this, it can be established that the above sequences are non-split extensions, a fact which will be needed when describing $\mathrm{Ext}^1_\Lambda(\mathbb{Z}, P)$.

PROPOSITION 4.3. *Let* $i : \mathbb{Z}[C_7] \to \Lambda$. *Then* $i_*(\mathcal{I}) \cong R \oplus P \oplus P^2$.

Proof. Take $i_*(M) = M \otimes_{\mathbb{Z}[C_7]} \Lambda$. Then the non-split extension defines the group ring Λ:

$$\begin{array}{lccccccc} \varepsilon : & (0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathbb{Z}[C_7] & \longrightarrow & \mathbb{Z} \longrightarrow 0) \\ & & & & & \big\downarrow i_* & & \\ 1_*(\varepsilon) : & (0 & \longrightarrow & 1_*(\mathcal{I}) & \longrightarrow & \Lambda & \longrightarrow & \mathbb{Z}[C_3] \longrightarrow 0) \end{array}$$

where $i_*(\mathcal{I}) = R \oplus P \oplus P^2$. This is the only option as any other possibility for $i_*(\mathcal{I})$ splits. □

PROPOSITION 4.4. $\mathrm{Ext}^1_\Lambda(\mathbb{Z}, P) \neq 0$.

Proof.

$$0 \to P \to \overline{\mathbb{Z}[C_7]} \to \mathbb{Z} \to 0.$$

This does not split over Λ. So, this defines a non-trivial class in $\mathrm{Ext}^1_\Lambda(\mathbb{Z}, P)$. □

Knowing the above, it is possible to obtain the following result.

COROLLARY 4.5.

$$\mathrm{Ext}^1_\Lambda(\mathbb{Z}, P^e) \cong \begin{cases} \mathbb{Z}/7, & e = 1, \\ 0, & e = 0, 2. \end{cases}$$

Proof. Using the Eckmann–Shapiro lemma,

$$\mathrm{Ext}^1_\Lambda(\mathbb{Z}, i_*(\mathcal{I})) \cong \mathrm{Ext}^1_{\mathbb{Z}[C_7]}(i^*(\mathbb{Z}), \mathcal{I}) \cong \mathrm{Ext}^1_{\mathbb{Z}[C_7]}(\mathbb{Z}, \mathcal{I}) \cong \mathbb{Z}/7.$$

As $i_*(\mathcal{I}) \cong R \oplus P \oplus P^2$, $\mathrm{Ext}^1_\Lambda(\mathbb{Z}, i_*(\mathcal{I}))$ can be split up:

$$\mathrm{Ext}^1_\Lambda(\mathbb{Z}, i_*(\mathcal{I})) \cong \mathrm{Ext}^1_\Lambda(\mathbb{Z}, R) \oplus \mathrm{Ext}^1_\Lambda(\mathbb{Z}, P) \oplus \mathrm{Ext}^1_\Lambda(\mathbb{Z}, P^2) \cong \mathbb{Z}/7.$$

However, we know that $\mathrm{Ext}^1_\Lambda(\mathbb{Z}, P) \neq 0$. As $\mathbb{Z}/7$ has no proper non-trivial subgroup, $\mathrm{Ext}^1_\Lambda(\mathbb{Z}, P) \cong \mathbb{Z}/7$. In turn, $\mathrm{Ext}^1_\Lambda(\mathbb{Z}, R) = \mathrm{Ext}^1_\Lambda(\mathbb{Z}, P^2) = 0$. □

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PROPOSITION 4.6. $\mathrm{Ext}^1_\Lambda(\mathbb{Z}[C_3], P^e) \cong \mathbb{Z}/7$ *for all* $e \geqslant 0$.

Proof. Again using the Eckmann–Shapiro lemma,

$$\begin{aligned}\mathrm{Ext}^1_\Lambda(\mathbb{Z}[C_3], P^e) &\cong \mathrm{Ext}^1_\Lambda(\mathbb{Z} \otimes_{\mathbb{Z}[C_7]} \mathbb{Z}[C_3], P^e)\\ &\cong \mathrm{Ext}^1_{\mathbb{Z}[C_7]}(\mathbb{Z}, P^e).\end{aligned}$$

Now it is enough to check that $\mathrm{Ext}^1_\Lambda(\mathbb{Z}[C_3], R) \cong \mathbb{Z}/7$ (as $R \cong P \cong P^2$ over $\mathbb{Z}[C_7]$). Over $\mathbb{Z}[C_7]$, it is also known that $R = \mathcal{I}$, the dual augmentation ideal, so $R \cong \Omega^{C_7}_{-1}(\mathbb{Z})$. So, within the derived module category,

$$\begin{aligned}\mathrm{Ext}^1_{\mathbb{Z}[C_7]}(\mathbb{Z}, R) &\cong \mathrm{Ext}^1_{\mathbb{Z}[C_7]}(\mathbb{Z}, \Omega^{C_7}_{-1}(\mathbb{Z}))\\ &\cong \mathrm{Hom}_{\mathrm{Der}}(\mathbb{Z}, \Omega^{C_7}_1\Omega^{C_7}_{-1}(\mathbb{Z}))\\ &\cong \mathrm{Hom}_{\mathrm{Der}}(\mathbb{Z}, \mathbb{Z})\\ &\cong \mathbb{Z}/7.\end{aligned}$$

□

With all this information, it is now possible to solve the last set of cohomological relations.

PROPOSITION 4.7.

$$\mathrm{Ext}^1_\Lambda(\mathcal{J}, P^e) \cong \begin{cases} 0, & e = 1,\\ \mathbb{Z}/7, & e = 0, 2.\end{cases}$$

Proof. Take the following exact sequence: $0 \to \mathcal{J} \to \mathbb{Z}[C_3] \to \mathbb{Z} \to 0$. Suppose that N is one of R, P, P^2. Taking a long exact sequence in cohomology, one gets

$$\begin{aligned}\cdots \mathrm{Hom}_{\mathbb{Z}[\mathbb{G}]}(\mathcal{J}, N) &\to \mathrm{Ext}^1_\Lambda(\mathbb{Z}, N) \to \mathrm{Ext}^1_\Lambda(\mathbb{Z}[C_3], N)\\ &\to \mathrm{Ext}^1_\Lambda(\mathcal{J}, N) \to \mathrm{Ext}^2_\Lambda(\mathbb{Z}, N) \cdots .\end{aligned}$$

Over $\mathbb{Z}[C_7]$, $\mathcal{J}$ is trivial. So, $\mathrm{Hom}_{\mathbb{Z}[C_7]}(\mathcal{J}, N) = 0$. As a result, $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{J}, N) = 0$. One can also see that

$$\mathrm{Ext}^2_\Lambda(\mathbb{Z}, i_*(N)) \cong \mathrm{Ext}^2_{\mathbb{Z}[C_7]}(\mathbb{Z}, N) = 0.$$

This leaves $\mathrm{Ext}^2_\Lambda(\mathbb{Z}, N) = 0$. Eliminating these factors from the long exact sequence, one is left with an exact sequence:

$$0 \to \mathrm{Ext}^1_\Lambda(\mathbb{Z}, N) \to \mathrm{Ext}^1_\Lambda(\mathbb{Z}[C_3], N) \to \mathrm{Ext}^1_\Lambda(\mathcal{J}, N) \to 0.$$

It has been proven in the previous proposition that $\mathrm{Ext}^1_\Lambda(\mathbb{Z}[C_3], N) \cong \mathbb{Z}/7$. So, all that is left to do is take each N in turn, put in the value known for $\mathrm{Ext}^1_\Lambda(\mathbb{Z}, N)$ and the value for $\mathrm{Ext}^1_\Lambda(\mathcal{J}, N)$ falls out. For example, when $N{=}R$, $\mathrm{Ext}^1_\Lambda(\mathbb{Z}, R) = 0$, so that $\mathrm{Ext}^1_\Lambda(\mathcal{J}, R) \cong \mathrm{Ext}^1_\Lambda(\mathbb{Z}[C_3], R) \cong \mathbb{Z}/7$. □

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COROLLARY 4.8. *The duality arguments are as follows.*

- $\operatorname{Ext}^1_\Lambda(P^e, \mathbb{Z}) \cong \begin{cases} 0, & e = 1, \\ \mathbb{Z}/7, & e = 0, 2. \end{cases}$
- $\operatorname{Ext}^1_\Lambda(P^e, \mathcal{J}) \cong \begin{cases} \mathbb{Z}/7, & e = 1, \\ 0, & e = 0, 2. \end{cases}$
- $\operatorname{Ext}^1_\Lambda(P^e, \mathbb{Z}[C_q]) \cong \mathbb{Z}/7$ *for all* $e \geqslant 0$.

Proof. Use the same principles as above. □

Thus, there are a total of 21 indecomposable modules over Λ. There exist a total of 17 distinct non-isomorphic genera of indecomposable modules. The remaining four indecomposable modules (Swan-like modules) arise from elements of the projective class group $\widetilde{K}_0(\Lambda)$. We recall the basic modules.

(I) There are three distinct modules over $\mathbb{Z}[C_3]$:
  (i) the trivial module: $\mathbb{Z}$ (rank = 1);
  (ii) the augmentation ideal, namely $\mathcal{J} = \operatorname{Ker}(\mathbb{Z}[C_3] \to \mathbb{Z})$ (rank = 2);
  (iii) the group ring itself: $\mathbb{Z}[C_3]$ (rank = 3).

  These are modules over Λ via the quotient map $G \to C_3$.

(II) There are three distinct non-isomorphic modules over $\mathscr{C}_3(R, \theta, 1)$. They are all of rank = 6:
  (iv) $R = P^0 = \mathbb{Z}[\zeta] = \mathbb{Z}[x]/c_7(x)$, where $c_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$;
  (iiv) $P = P^1 = (\zeta - 1)R$;
  (iiiv) $P^2 = (\zeta - 1)^2 R$.

This completes all the basic indecomposable modules over Λ. We shall consider the non-split extensions.

PROPOSITION 4.9 [2, pp. 750–751]. *Take the non-split extensions of indecomposable modules as*

$$0 \to \mathcal{A} \to ? \to \mathcal{B} \to 0.$$

Then $\mathcal{A}$ is a direct sum of modules of particular combinations of $R, P, \ldots, P^{q-1}$, where each of the modules $R, P, \ldots, P^{q-1}$ can be used at most once in any extension, and $\mathcal{B} = \mathbb{Z}, \mathcal{J}$ or $\mathbb{Z}[C_q]$, where only one of the three modules can be used in any extension.

Using the cohomological relations established earlier for $0 \to \mathcal{A} \to ? \to \mathcal{B} \to 0$, we describe all the remaining genera of indecomposable modules that exist.

(III) There is only one extension for when $\mathcal{B} = \mathbb{Z}$. This is a result of the cohomological properties, where when $M = R, P^2$ the extension splits.

  (vii) $0 \to P \to \overline{\mathbb{Z}[C_7]} \to \mathbb{Z} \to 0$, where $\overline{\mathbb{Z}[C_7]}$ is $\mathbb{Z}[C_7]$ acting by conjugation.

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(IV) rank $= 8$:

(viii) $0 \to R \to V_1 \to \mathcal{J} \to 0$;
(ix) $0 \to P \to V_2 \to \mathcal{J} \to 0$.

(V) rank $= 9$:

(x) $0 \to R \to Y_1 \to \mathbb{Z}[C_3] \to 0$;
(xi) $0 \to P \to Y_2 \to \mathbb{Z}[C_3] \to 0$;
(xii) $0 \to P^2 \to Y_3 \to \mathbb{Z}[C_3] \to 0$.

(VI) rank $= 14$:

(xiii) $0 \to R \oplus P \to X \to \mathcal{J} \to 0$.

(VII) rank $= 15$:

(xiv) $0 \to R \oplus P \to \mathcal{Q}(P^2) \to \mathbb{Z}[C_3] \to 0$;
(xv) $0 \to R \oplus P^2 \to \mathcal{Q}(P) \to \mathbb{Z}[C_3] \to 0$;
(xvi) $0 \to P \oplus P^2 \to \mathcal{Q}(R) \to \mathbb{Z}[C_3] \to 0$.

(VIII) rank $= 21$:

(xvii) $0 \to R \oplus P \oplus P^2 \to \Lambda \to \mathbb{Z}[C_3] \to 0$. This module was seen earlier, and is the non-split extension that gives the group ring.

On completion of the 17 genera of indecomposable modules, there are still four more modules that need to be considered. Looking at the extension for the module of rank 7, it can be seen there are in fact more non-isomorphic indecomposable modules of rank 7:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \longrightarrow & \varprojlim(\pi, \alpha) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & P & \longrightarrow & \overline{\mathbb{Z}[C_7]} & \xrightarrow{\pi} & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

As a result of $\mathrm{Ext}^1(\mathbb{Z}, P) \cong \mathbb{Z}/7$, there seem to be six extensions corresponding to $(\mathbb{Z}/7)^*$. However, as a result of conjugation, there are in fact three indecomposable modules of this form:

$\mathcal{Q}_0 = \overline{\mathbb{Z}[C_7]} = \varprojlim(\pi, 1) \cong \varprojlim(\pi, 6)$;

$\mathcal{Q}_1 = \varprojlim(\pi, 2) \cong \varprojlim(\pi, 5)$;

$\mathcal{Q}_2 = \varprojlim(\pi, 3) \cong \varprojlim(\pi, 4)$.

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The remaining indecomposable modules are obtained by calculating $\widetilde{K}_0(\Lambda)$. From the results of Cassou-Nogues [3, pp. 265–266], we have the following sequence for this group:

$$0 \to D_0(\Lambda) \to D(\Lambda) \to D(\mathbb{Z}[C_3]) \to 0,$$

where $D(\mathbb{Z}[C_3])$ is a well-defined subgroup of $K_0(\mathbb{Z}[C_3])$. But $\widetilde{K}_0(\mathbb{Z}[C_3]) = 0$. Next, as a result of $D(\mathbb{Z}[C_7]) = 0$, $D(\Lambda) \cong C_3$. So, from the sequence we are left with $D_0(\Lambda) \cong D(\Lambda) \cong C_3$. Thus, $\widetilde{K}_0(\Lambda) \cong C_3$, implying that there are three distinct stable equivalence classes of projective modules. This gives three indecomposable modules of rank 21.

In fact, the three modules of rank 21 can be obtained by tensoring the three modules of rank 7 by $\mathbb{Z}[C_3]$. So, the three projective modules are the group ring itself $\overline{\mathbb{Z}[C_7]} \otimes \mathbb{Z}[C_3] = \Lambda$ and the two remaining projective modules, $\mathcal{Q}_1 \otimes \mathbb{Z}[C_3]$ and $\mathcal{Q}_2 \otimes \mathbb{Z}[C_3]$, which are all of rank 21. Pu explains in the case of Λ that any other potential indecomposable modules arising from the non-trivial elements of $\widetilde{K}_0(\Lambda)$ are not distinct and are actually isomorphic to indecomposable modules from the trivial element of $\widetilde{K}_0(\Lambda)$ using unit arguments. Thus, there are 21 distinct non-isomorphic indecomposable modules for Λ. More detail on the projective class group for metacyclic groups is given in a paper by Galovich *et al* [5].

§5. *Syzygy decompositions and a free resolution.* The general condition for a finite group G to have a periodic resolution depends on the Sylow subgroup structure of G.

THEOREM 5.1 (Artin–Tate–Zassenhaus theorem [1, Ch. XII, p. 262]). *Let G be a finite group and let $p_1, \ldots, p_N$ be the distinct primes dividing $|G|$. Then a resolution $\mathbb{Z}$ over the group ring $\mathbb{Z}[G]$ has a finite cohomological period if and only if:*

(i) *for every odd prime p_i, every p_i-subgroup is cyclic;*
(ii) *if $p_i = 2$, then every Sylow 2-subgroup is either cyclic or a generalized quaternion.*

This theorem implies that all groups $G(pq)$ have finite cohomological period, and so have periodic free resolutions. However, groups that contain a $C_p \times C_p$ subgroup, where p is prime, do not have periodic resolutions. Groups such as the dihedral groups D_{4n} have this subgroup property (i.e. D_8 contains a $C_2 \times C_2$ subgroup), and so do not have periodic resolutions. In addition, such groups have infinite families of indecomposable modules, complicating the integral representation theory.

PROPOSITION 5.2 [7, p. 163]. *The group $G(21)$ has cohomological period* 6.

An important property for periodic resolutions is that if $M \in \mathcal{F}(\Lambda)$ has period 6, then for $0 \leqslant m \leqslant 6$ there exists a duality between syzygies:

$$\Omega_m^*(\mathbb{Z}) \cong \Omega_{6-m}(\mathbb{Z}),$$

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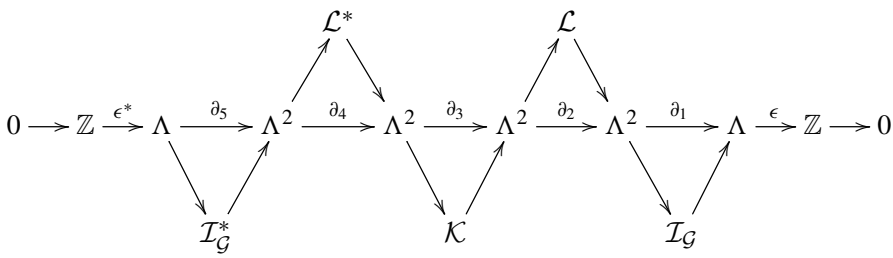
where $\Omega_m^*(\mathbb{Z})$ denotes the dual stable class of $\Omega_m(\mathbb{Z})$. We are able obtain the rank of the first three Λ^{n_i} terms using the formal method of *free differential calculus* given by Fox [4]. Evidently, as we have two generators and two relations, the free resolution begins as follows:

$$\ldots \Lambda^{n_3} \xrightarrow{\partial_3} \Lambda^2 \xrightarrow{\partial_2} \Lambda^2 \xrightarrow{\partial_1} \Lambda \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

Furthermore, as there exists a duality between syzygies,

$$\begin{aligned}\Lambda^{n_0} &= \Lambda^{n_{2q-1}} = \Lambda,\\ \Lambda^{n_1} &= \Lambda^{n_{2q-2}} = \Lambda^2,\\ \Lambda^{n_2} &= \Lambda^{n_{2q-3}} = \Lambda^2.\end{aligned}$$

So, for $G(21)$, we know the structure of the free resolution straight away:



By describing $\mathcal{I}_{\mathcal{G}}$ in terms of indecomposable modules, we will be able to establish the minimal representatives for the other syzygies by using certain numerical syzygy ideas and the elimination of modules.

For any module M, write $\mathcal{E}(M) = \mathrm{End}_{\mathcal{D}\mathrm{er}}(M)$. Then, if M is any representative module for $\Omega_1(\mathbb{Z})$, we recall:

(i) $\mathcal{E}(M) \cong \mathbb{Z}/21 \cong \mathbb{Z}/7 \oplus \mathbb{Z}/3$;
(ii) $\mathcal{E}(N) \cong 0 \iff N$ is projective.

PROPOSITION 5.3. *If $K(1)$ is a minimal representative of $\Omega_1(\mathbb{Z})$, then either:*

(a) *$K(1)$ is indecomposable; or*
(b) *$K(1) \cong K(1)_1 \oplus K(1)_2$, where each $K(1)_j$ is indecomposable.*

Proof. Suppose that $K(1) \cong M_1 \oplus M_2 \oplus M_3$, where each M_j is non-zero. Then $\mathcal{E}(K(1))$ will contain $\mathcal{E}(M_1) \oplus \mathcal{E}(M_2) \oplus \mathcal{E}(M_3)$. However, we know from (i) that

$$\mathcal{E}(K(1)) \cong \mathbb{Z}/7 \oplus \mathbb{Z}/3$$

and so for some j, $\mathcal{E}(M_j) = 0$. Thus, using (ii), we have a non-zero projective module for some j. Hence, $\mathrm{rank}(K(1)) \geqslant 21$. However, this is a contradiction since $\mathrm{rank}(K(1)) = 20$. The result now follows. □

For the next result to hold, recall Swan's local freeness theorem that over a finite group $\mathbb{Z}[G]$ projective modules necessarily have the same $\mathbb{Z}$ ranks as free modules [2, Theorem 32.11, p. 676].

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COROLLARY 5.4. *If $K(1)$ is a minimal representative of $\Omega_1(\mathbb{Z})$, then $K(1)$ must split into two indecomposable modules,*

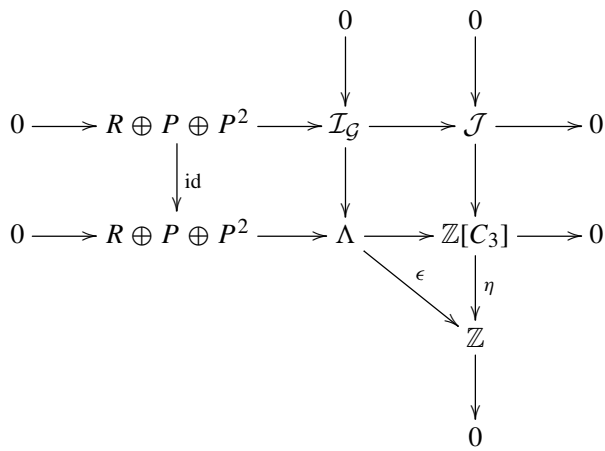
$$K(1) \cong K(1)_1 \oplus K(1)_2.$$

Proof. From our complete list of indecomposable modules over Λ, there does not exist an indecomposable module of rank 20. Hence, $K(1)$ must split into two indecomposable modules. □

It is clear that $K(3)$ follows the exact same argument and so also has rank 20. $K(2)$ also follows the same argument above, but this time $\text{rank}_{\mathbb{Z}}(K(2)) = 22$. All the steps remain the same and so $K(2)$ splits into two indecomposable modules, as there does not exist an indecomposable module of rank 22.

THEOREM 5.5. *At the minimal level, the first syzygy $\Omega_1(\mathbb{Z})$ can be described as $\mathcal{I}_{\mathcal{G}} = P \oplus X$.*

Proof. Take the following commutative diagram:



We established that $\text{Ext}^1_{\Lambda}(\mathcal{J}, P) = 0$, so this term can split off from the extension. This means that the quotient $\mathcal{I}_{\mathcal{G}}/P$ is of dimension 14 and is contained in $\mathcal{I}_{\mathcal{G}}$. As X is the only indecomposable of rank 14, we must therefore have $\mathcal{I}_{\mathcal{G}}/P = X$. Hence, from this, one can see that $\mathcal{I}_{\mathcal{G}} = X \oplus P$. Also, $\mathcal{I}_{\mathcal{G}}$ is minimal in its stability class because $\text{rank}(\mathcal{I}_{\mathcal{G}}) = 20 < 21 = \text{rank}(\Lambda)$. □

Now that $\mathcal{I}_{\mathcal{G}}$ is known, it is possible to describe minimal representatives of the other syzygies. We will start by looking at $\mathcal{I}^*_{\mathcal{G}}$ and $\mathcal{K}$.

PROPOSITION 5.6.

(i) $\mathcal{I}^*_{\mathcal{G}} = R \oplus X$.

(ii) $\mathcal{K} = P^2 \oplus X$.

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Proof. The free resolution for $G(21)$ is periodic, and so we know that both $\mathcal{I}_{\mathcal{G}}^*$ and $\mathcal{K}$ are also of rank = 20 < rank(Λ). This in turn means that they must also be minimal representatives of their respective syzygies. From the free resolution being periodic, it is possible to go a step further and see that both $\mathcal{I}_{\mathcal{G}}^*, \mathcal{K} = \langle 6\rangle \oplus \langle 14\rangle$, where $\langle \mathcal{N}\rangle$ are indecomposable modules of rank $\mathcal{N}$. There is only one indecomposable of rank 14, and hence is self-dual, so in both cases we can take $\langle 14\rangle = X$.

(i) $\mathcal{I}_{\mathcal{G}}^*$ can be written in this case as $\Omega_1(\mathbb{Z})^*$. Looking at it like this, we can see that $\Omega_1(\mathbb{Z})^* = P^* \oplus X^*$ as X is self-dual and $P^* \cong R$. So, $\mathcal{I}_{\mathcal{G}}^* = R \oplus X$.

(ii) $\mathcal{K}$ is self-dual. This is obvious from the free resolution. This in turn implies that both the indecomposable modules are self-dual. The module X we know is self-dual. The only rank-6 module over $G(21)$ that is self-dual is P^2. By uniqueness, this must be the module of rank 6. □

Hence, the odd syzygy representatives are obtained. However the even syzygy representatives take a little more manipulation to obtain.

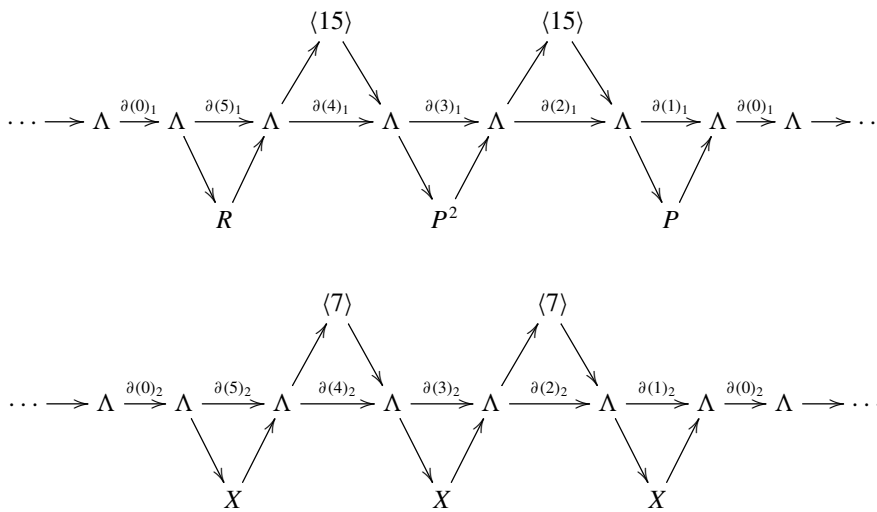
PROPOSITION 5.7. *The even syzygies are described by:*

(i) $\mathcal{L} = \overline{\mathbb{Z}[C_7]} \oplus \mathcal{Q}(P^2)$;

(ii) $\mathcal{L}^* = \overline{\mathbb{Z}[C_7]} \oplus \mathcal{Q}(R)$.

Proof. Directly from the free resolution, both $\mathcal{L}$ and $\mathcal{L}^*$ have rank 22. They also must be minimal in their stable classes, otherwise they would have to split $\mathcal{L} \cong \langle 1\rangle \oplus \Lambda$. However, the only possibility for $\langle 1\rangle$ is $\mathbb{Z}$, which would imply that G has cohomological period 2, which would mean that G is cyclic. We know that G is not cyclic, so $\mathcal{L}$ and $\mathcal{L}^*$ are both minimal in their respective syzygies.

Knowing that $\mathcal{L}$ and $\mathcal{L}^*$ are of rank 22, it is actually possible to see that they also split up into two indecomposable modules; $\mathcal{L}, \mathcal{L}^* = \langle 7\rangle \oplus \langle 15\rangle$. This observation is made clearer by noticing that the free resolution for $G(21)$ can be "untwisted" to form two separate monogenic infinite resolutions:



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where $\partial(m) = \begin{pmatrix} \partial(m)_1 & 0 \\ 0 & \partial(m)_2 \end{pmatrix}$. So, it is now possible to obtain the minimal syzygy representatives using the two resolutions and then summing the terms. The indecomposable module $\langle 7 \rangle$ is interpreted as the module of rank 7 acting by conjugation, $\overline{\mathbb{Z}[C_7]}$. In the second of the infinite resolutions, the odd syzygies are all X, implying that the even syzygies here must all be the same to give the repetition for each syzygy.

All that is left to obtain are the two $\langle 15 \rangle$ indecomposable modules. We look at the following sequence within the resolution, $0 \to P^2 \to \Lambda \to \langle 15 \rangle \to 0$. From this, it is possible to take the quotient of the sequence to get $\langle 15 \rangle = \Lambda / P^2$. This can be taken as the indecomposable module $(0 \to R \oplus P \to \mathcal{Q}(P^2) \to \mathbb{Z}[C_3] \to 0)$. Then $\mathcal{L} = \overline{\mathbb{Z}[C_7]} \oplus \mathcal{Q}(P^2)$.

To obtain $\mathcal{L}^*$, the same as above is applied except with a different short exact sequence from the resolution: $0 \to R \to \Lambda \to \langle 15 \rangle \to 0$ gives $\langle 15 \rangle = \Lambda / P^2$. This can be described by $\mathcal{Q}(R)$. Hence. $\mathcal{L}^* = \overline{\mathbb{Z}[C_7]} \oplus \mathcal{Q}(R)$. □

Now that we have obtained a complete minimal resolution for $G(21)$ in terms of syzygies broken up into indecomposable modules, it is possible to obtain the mappings for the free resolution by combining the mappings of the two "untwisted" resolutions, giving a diagonalized resolution. This is simpler than trying to obtain mappings directly from the full resolution, which had been the original method to obtain free resolutions for $\mathbb{Z}[D_{4n+2}]$. This is a useful result when looking at the $\mathcal{R}(2)$–$\mathcal{D}(2)$ problem, and also helps simplify calculations such as obtaining cohomology groups. Below is a description of each of the indecomposable modules needed in the resolution described in terms of polynomials.

$$X = [\, y^2 - y \,],$$

$$\overline{\mathbb{Z}[C_7]} = \Sigma_y = [\, 1 + y + y^2 \,],$$

$$R = [1 + x - x^4 - x^5 + (1 + x^2 - x^4 - x^6)y + (1 - x)y^2),$$

$$P = [\, -1 + x - x^2 + x^6 + (-x^3 + x^6)y + (x - x^3 - x^5 + x^6)y^2 \,),$$

$$P^2 = [\, 1 - x^4 + (1 - x + x^2 - x^3)y + (1 - x)y^2 \,),$$

$$\mathcal{Q}(P^2) = [\, 1 + x - y^2 \,),$$

$$\mathcal{Q}(R) = [\, 1 + x^4 - y \,).$$

Using the above descriptions for the indecomposable modules, the following is a full free diagonalized resolution for $G(21)$:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\epsilon^*} \Lambda \xrightarrow{\partial_5} \Lambda^2 \xrightarrow{\partial_4} \Lambda^2 \xrightarrow{\partial_3} \Lambda^2 \xrightarrow{\partial_2} \Lambda^2 \xrightarrow{\partial_1} \Lambda \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0,$$

where the mappings can be described as

(a) $\epsilon = \Sigma_x(1 + y + y^2)$,

(b) $\partial_1 = (-1 + x - x^2 + x^6 + (-x^3 + x^6)y + (x - x^3 - x^5 + x^6)y^2, \quad y^2 - y)$,

(c) $\partial_2 = \begin{pmatrix} 1 + x - y^2 & 0 \\ 0 & \Sigma_y \end{pmatrix}$,

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(d) $\partial_3 = \begin{pmatrix} 1 - x^4 + (1 - x + x^2 - x^3)y + (1 - x)y^2 & 0 \\ 0 & y^2 - y \end{pmatrix}$,

(e) $\partial_4 = \begin{pmatrix} 1 + x^4 - y & 0 \\ 0 & \Sigma_y \end{pmatrix}$,

(f) $\partial_5 = \begin{pmatrix} 1 + x - x^4 - x^5 + (1 + x^2 - x^4 - x^6)y + (1 - x)y^2 \\ y^2 - y \end{pmatrix}$,

(g) $\epsilon^* = \Sigma_x(1 + y + y^2)$.

Finally, from this free resolution, the cohomology groups for the non-abelian group of order 21 can be obtained in a straightforward manner. Below are the cohomology groups using trivial coefficients:

$$H^r(\mathbb{Z}, \mathbb{Z}) = \mathrm{Ext}^r_\Lambda(\mathbb{Z}, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & r = 0, \\ 0, & r = \text{odd}, \\ \mathbb{Z}/3, & r = 2, 4 (\bmod 6), \\ \mathbb{Z}/21, & r = 0, 6 (\bmod 6), r > 0. \end{cases}$$

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