*Combinatorics, Probability and Computing* (2003) 12, 457–465. © 2003 Cambridge University Press
DOI: 10.1017/S0963548302005497 Printed in the United Kingdom

# Contractible Elements in Graphs and Matroids

HAIDONG WU†

Department of Mathematics, The University of Mississippi, University, MS 38677, USA
(e-mail: hwu@olemiss.edu)

*Received 23 April 2002; revised 12 July 2002*

Let $G$ be a simple 3-connected graph with at least five vertices. Tutte [13] showed that $G$ has at least one contractible edge. Thomassen [11] gave a simple proof of this fact and showed that contractible edges have many applications. In this paper, we show that there are at most $\frac{|V(G)|}{5}$ vertices that are not incident to contractible edges in a 3-connected graph $G$. This bound is best-possible. We also show that if a vertex $v$ is not incident to any contractible edge in $G$, then $v$ has at least four neighbours having degree three, and each such neighbour is incident to exactly two contractible edges. We give short proofs of several results on contractible edges in 3-connected graphs as well. We also study the contractible elements for $k$-connected matroids. We partially solve an open problem for regular matroids.

## 1. Introduction

All graphs considered in this paper are simple. An edge $e$ in a 3-connected graph $G$ is contractible if the contraction $G/e$ is still 3-connected (although it may no longer be simple). In graph theory, one of the most important tools is induction, that is, reducing the size of the graph while maintaining the connectivity. Contractible edges allow one to reduce the size of the graph while maintaining the connectivity. Tutte [13] showed that a 3-connected graph with at least five vertices has at least one contractible edge. The existence of contractible edges provides a powerful tool in graph theory. For example, Thomassen [11] used it to give a very short proof of Kuratowski's well-known planarity theorem. Therefore it is very natural and important to study contractible edges. Indeed, contractible edges have been studied by numerous authors (see, for example, [1, 3, 4, 6, 8, 11–13]). The corresponding problem is studied for matroids in [14]. The following is the main result of [1].

† Research partially supported by the Faculty Summer Research Program of the University of Mississippi in 2001 as well as an ONR grant under grant number N00014-01-1-0917.

https://doi.org/10.1017/S0963548302005497 Published online by Cambridge University Press

**Theorem 1.1. ([1])** *Let $G$ be a simple 3-connected graph with at least five vertices. Then $G$ has at least $\frac{|V(G)|}{2}$ contractible edges.*

Covering contractible edges by some vertices has been studied in [6, 8]. In [8], the following result is obtained.

**Theorem 1.2. ([8])** *Let $G$ be a simple 3-connected graph with at least six vertices. Then the set of contractible edges $E_c(G)$ cannot be covered by two vertices.*

In this paper, we study the contractible elements for graphs and matroids. For matroid terminology not given in this paper, we follow Oxley [9]. We first show that a 3-connected graph $G$ has at most $\frac{|V(G)|}{5}$ vertices that are not incident to contractible edges. This bound is best-possible. Then we give short proofs of Theorem 1.1 and 1.2. We also study the contractible elements for $k$-connected matroids. We partially solve an open problem for regular matroids. The following is our first main result.

**Theorem 1.3.** *Let $G$ be a simple 3-connected graph with at least five vertices. Then $G$ has at most $\frac{|V(G)|}{5}$ vertices that are not incident to contractible edges. If a vertex $v$ is not incident to any contractible edge, then $v$ has at least four degree-three neighbours, each of which is incident to exactly two contractible edges.*

A well-known open problem in matroid theory is the following [9, Problem 14.4.9].

**Problem A.** *Let $M$ be a minimally $k$-connected matroid having at least $2(k-1)$ elements, where $k \geqslant 4$. Does $M$ have a $k$-element cocircuit?*

When $M$ is graphic, the answer to the above problem is affirmative and is a well-known result of Halin [5]. By studying contractible elements in matroids, we prove our next main result, which solves Problem A for regular matroids.

**Theorem 1.4.** *Let $M$ be a minimally $k$-connected regular matroid where $k \geqslant 4$.*

(i) *If $M$ has at least $2(k-1)$ elements, then $M$ has a $k$-element cocircuit.*
(ii) *If $M$ has at least $r(M)+3k-1$ elements, then $M$ has at least*

$$\min\left\{\frac{(k-1)r(M)+3k-1}{2k-1}, \frac{2|E(M)|-2r(M)+3k^2-6k+2}{2k}\right\}$$

*cocircuits with size $k$.*

We will give proofs of Theorems 1.3 and 1.4 in Sections 2 and 3, respectively. Let $e$ be an edge of a simple 3-connected graph $G$. An edge $e$ is *essential* [13] if and only if neither the deletion $G\backslash e$ nor the contraction $G/e$ is both simple and 3-connected. A *triad* of $G$ is a set of three edges of $G$ which meet a vertex of degree three. The following result is due to Tutte [13]. Part (b) is Tutte's Triangle Lemma.

https://doi.org/10.1017/S0963548302005497 Published online by Cambridge University Press

**Lemma 1.5.** *Let $G$ be a simple 3-connected graph.*

(a) *If $e$ is essential in $G$, then $e$ is in a triangle or a triad.*
(b) *If $T = \{e, f, g\}$ is a triangle and neither $G\backslash e$ nor $G\backslash f$ is 3-connected, then $G$ has a triad containing $e$.*

The next lemma is essentially a restatement of [1, Lemma 1].

**Lemma 1.6.** *Let $x$ be a vertex of degree three with neighbours $u$, $v$, $w$ in a 3-connected graph $G$ with at least five vertices. If neither $xu$ nor $xv$ is contractible, then:*

(a) *both $u$ and $v$ have degree three, and $uv \in E(G)$, and*
(b) *each of $x$, $u$, $v$ is incident to exactly one contractible edge, the edges forming a matching of $G$.*

The fact that these contractible edges form a matching of $G$ in the last lemma follows easily from the fact that $G$ is 3-connected. An immediate consequence of Lemma 1.6 is a result of Halin [5] that each degree-three vertex in a 3-connected graph is incident to at least one contractible edge. Next we introduce some notation for the rest of the paper. Let $U$ denote the set of vertices of $G$ that are not incident to contractible edges. We use $C(G)$ to denote the subgraph induced by the contractible edges and $E_c(G)$ to denote the set of contractible edges in $G$. For each vertex $v$ and vertex set $S \subseteq V(G)$, we use $d_C(v)$ to denote the degree of $v$ in $C(G)$ and let $d_C(S) = \sum_{x \in S} d_C(x)$.

## 2. Contractible edges in 3-connected graphs

Theorem 1.1 gives a best-possible bound on the number of contractible edges in a 3-connected graph. Sometimes it is very important to know where these contractible edges are. Therefore it is useful to study the distribution of contractible edges. For example, Dean, Hemminger and Ota [3] proved that each longest cycle of a 3-connected graph with at least seven vertices has at least three contractible edges. Theorem 1.3 gives interesting information on the distribution of contractible edges in 3-connected graphs. It gives a best-possible upper bound on $|U|$, the number of vertices that are not incident to contractible edges in a 3-connected graph. Our result can have applications such as induction tools related to vertices.

**Proof of Theorem 1.3.** Suppose that $v$ is not incident to any contractible edge of $G$. Then $d_G(v) > 3$. Let $E_G(v)$ be the set of edges incident to $v$ in $G$ and $T(v)$ be the set of degree-three neighbours of $v$. Delete a subset $S$ of $E_G(v)$ from $G$ to get a new graph $G_1$ such that $G\backslash S$ is 3-connected, and choose $S$ such that $S$ is maximal. Now $v$ has degree at least four in $G_1$, since otherwise $v$ is incident to a contractible edge of $G_1$ (and therefore also of $G$). For each edge $e$ in $E_{G_1}(v)$, $G_1\backslash e$ is not 3-connected by the choice of $S$. Moreover $G_1/e$ is not 3-connected, since otherwise $e$ is contractible in $G$. Thus $e$ is essential in $G_1$. By Lemma 1.5(a), $e$ must be in a triangle or a triad. If $e$ is in a triangle, by Lemma 1.5(b), $e$ must be in a triad of $G_1$. Thus each vertex in $N_{G_1}(v)$ has degree three in $G_1$ and clearly has degree three in $G$ too. Therefore $T(v) = N_{G_1}(v)$ has at least four elements. For any vertex

https://doi.org/10.1017/S0963548302005497 Published online by Cambridge University Press

$x \in T(v)$ with neighbours $u, v, w$ in $G$, if $x$ is incident to only one contractible edge in $G$, say $xu$, by Lemma 1.6, as $xv$ is not contractible, we conclude that $d_G(v) = 3$, a contradiction. Therefore, if a vertex $v$ is not incident to any contractible edge, then $v$ has at least four degree-three neighbours, and each degree-three neighbour is incident to exactly two contractible edges.

Next we show that $G$ has at most $\frac{|V(G)|}{5}$ vertices that are not incident to contractible edges. By the above argument, $T(v) \cup \{v\}$ has at least five vertices. Suppose that $u \neq v$, where $u, v \in U$. If $T(u) \cap T(v)$ has an element $y$, then $y$ is incident to two non-contractible edges, a contradiction. Now it is clear that $(T(u)) \cup \{u\}) \cap (T(v) \cup \{v\}) = \emptyset$. Thus $|V(G)| \geqslant 5|U|$. Thus $G$ has at most $\frac{|V(G)|}{5}$ vertices which are not incident to contractible edges. □

Our result improves the following result of [1, Corollary 4].

**Corollary 2.1.** *Let $G$ be a simple 3-connected graph with at least five vertices. If a vertex $v$ is not incident to any contractible edge, then $v$ has at least three degree-three neighbours, each of which is incident to exactly two contractible edges.*

The next class of graphs $\mathcal{G}$ shows that the bound in our result is best-possible. Start with a cycle with $4m$ vertices $a_1, b_1, a_2, b_2, \ldots, a_{2m}, b_{2m}$, where $m \geqslant 2$. Add $m$ vertices $c_1, c_2, \ldots, c_m$ such that $c_\ell$ is incident to $a_\ell, b_\ell, a_{m+\ell}, b_{m+\ell}$, for $\ell = 1, \ldots, m$. The resulting graph $G$ is clearly 3-connected and has exactly $m = \frac{|V(G)|}{5}$ vertices $c_1, c_2, \ldots, c_m$, which are not incident to any contractible edges.

Next we give short proofs of Theorems 1.1 and 1.2.

**Proof of Theorem 1.1.** Partition the vertex set $V(G)$ into disjoint classes as follows. For each vertex $v$ that is not incident to any contractible edge, let the class containing $v$ be the degree-three neighbour set of $v$, $T(v)$, union $\{v\}$. For each vertex $x$ not in the above classes, let the class containing $x$ be the single vertex set $\{x\}$ (thus $x$ must be incident to at least one contractible edge). Note that $(T(u)) \cup \{u\}) \cap (T(v) \cup \{v\}) = \emptyset$ for each $u$, $v \in U$, so indeed we have a partition of $V(G)$. For each class $P$ with more than one element, we have $d_C(P) = 2(|P| - 1) \geqslant |P|$. For each class $P$ with exactly one element, clearly $d_C(P) \geqslant 1 = |P|$. Summing the vertex degree of the subgraph $C(G)$, we get $2 \cdot |E_c(G)| = \sum d_C(P) \geqslant \sum |P| = |V(G)|$. Thus $G$ has at least $\frac{|V(G)|}{2}$ contractible edges. □

By a similar argument to that in Theorem 1.3, one can get the following result of [1, Corollary 3].

**Proposition 2.2.** *Let $G$ be a 3-connected graph with at least five vertices, and let $v$ be a vertex with degree at least four. If $v$ is incident to exactly one contractible edge $vw$, then $v$ has at least two degree-three neighbours different from $w$, each of which is incident to exactly two contractible edges.*

Let $U_2$ denote the set of degree-three vertices incident to exactly two contractible edges, and let $W_i$ be the set of degree exceeding three vertices which are incident to exactly $i$

https://doi.org/10.1017/S0963548302005497 Published online by Cambridge University Press

contractible edges [1]. An immediate consequence of Theorem 1.3 and Proposition 2.2 is the following generalization of [1, Theorem 5].

**Corollary 2.3.** *Let $G$ be a 3-connected graph of order at least five. Then* $|U_2| \geqslant 2|W_1| + 4|W_0|$.

This bound is also attained by the graphs in the class $\mathcal{G}$.

**Proof of Theorem 1.2.** Let $S = \{u, v\}$ be a set which covers all contractible edges of $G$. Then clearly $S$ covers all contractible edges of a minimally 3-connected spanning subgraph as well. We show next that this is impossible. For convenience, assume that $G$ is minimally 3-connected. For any edge $xy \in E(G)$ which is incident to neither $u$ nor $v$, the graph $G/xy$ is not 3-connected. Thus $xy$ must be essential, hence it is in a triad, by Lemma 1.5(a) and (b). Assume that $d(y) = 3$. If $y$ is incident to only one contractible edge, then, by Lemma 1.6(b), $G$ has three contractible edges which form a matching, a contradiction, as $S$ covers all contractible edges. Hence $y$ must be incident to two contractible edges. Therefore $y$ is incident to both $u$ and $v$. If $d(x) = 3$, then similarly $x$ must be incident to both $u$ and $v$. As $G$ has at least five vertices, $G - S$ is not connected, a contradiction to the fact that $G$ is 3-connected. Thus $d(x) \geqslant 4$. Let $T = \{e_1, e_2, \ldots, e_k\}$ be the set of edges not incident to $S$. Then, by the above argument, we deduce that there are vertices $x_i$, $y_i$ of $G$, $x_i, y_i \notin S$, where $i = 1, 2, \ldots, k$, satisfying the following.

(1) $e_i = x_i y_i$, where $d(x_i) \geqslant 4$, $d(y_i) = 3$ and $y_i$ is incident to $x_i$, $u$, $v$. Thus $y_i \neq y_j$ for all distinct $i, j \in \{1, 2, \ldots, k\}$.
(2) $x_i x_j$ is not an edge of $G$ for any $i, j \in \{1, 2, \ldots, k\}$ (otherwise one of $x_i$ and $x_j$ has degree three, which is impossible).

If $G$ has another vertex $w \notin \{x_1, y_1, x_2, y_2, \ldots, x_k, y_k\} \cup S$, then, as $wt \notin E(G)$ for all $t \in V(G) \backslash S$, we conclude that $d(w) \leqslant 2$, a contradiction. Thus

$$V(G) = \{x_1, y_1, x_2, y_2, \ldots, x_k, y_k\} \cup S.$$

Now $G \backslash S$ is disconnected unless $x_1 = x_2 = \cdots = x_k$. Thus $x_1 = x_2 = \cdots = x_k$. The only possible other edges of $G$ are $uv$, $x_1 u$, and $x_1 v$. As $G$ has at least six vertices, $k \geqslant 3$. Thus $G$ is either $K_{3,k}$, or can be obtained by $K_{3,k}$ with one bipartite class $\{u, v, x_1\}$ by adding possible edges in the set $\{uv, x_1 u, x_1 v\}$. This is a contradiction as the contractible edges of $G$ now cannot be covered by $S$. □

## 3. Contractible elements in $k$-connected matroids and $k$-element cocircuits

Let $M$ be a matroid and let $k$ be a positive integer. A $k$-separation of $M$ is a partition $\{X, Y\}$ of $E(M)$ such that

$$\min\{|X|, |Y|\} \geqslant k \tag{3.1}$$

and

$$r(X) + r(Y) - r(M) \leqslant k - 1. \tag{3.2}$$

https://doi.org/10.1017/S0963548302005497 Published online by Cambridge University Press

For all $n \geqslant 2$, $M$ is $n$-connected if, for all $k$ in $\{1, 2, \ldots, n-1\}$, $M$ has no $k$-separation. Hence $M$ is 2-connected if and only if $M$ is connected. If $M$ has no $k$-separation for all $k > 0$, then we say that $M$ has infinite connectivity.

We will follow the definition of Cunningham [2] for *vertical connectivity*. Note that there is only a very minor difference between Cunningham's definition and the definition in [9]. For a positive integer $k$, a *vertical k-separation* of $M$ is a partition $\{X, Y\}$of $E(M)$ that satisfies (3.2) and the following strengthened form of (3.1):

$$\min\{r(X), r(Y)\} \geqslant k.$$

For an integer $n$ exceeding one, a matroid $M$ is *vertically n-connected* if $M$ has no vertical $k$-separation for any $0 < k < n$. If $M$ has no vertical $k$-separation for all $k > 0$, then we say that $M$ has infinite vertical connectivity. From the definition, a $k$-connected matroid is also vertically $k$-connected.

Let $M$ be a $k$-connected matroid where $k \geqslant 2$. An element $e$ in $M$ is *contractible* if the contraction $M/e$ is vertically $k$-connected. We will show that Problem A in Section 1 is closely related to the existence of contractible elements in $k$-connected matroids. Numerous authors have studied the existence and distribution of the contractible edges in 3-connected graphs. In [14], we have shown that a 3-connected matroid with at least three elements has at least three contractible elements. Very little is known about $k$-connected matroids for $k \geqslant 4$. We make the following two conjectures.

**Conjecture B.** Let $M$ be a $k$-connected matroid with at least $2(k-1)$ elements where $k \geqslant 4$. Suppose that $M$ has no $k$-element circuit. Then $M$ has at least one contractible element.

The next conjecture is stronger.

**Conjecture C.** Let $M$ be a $k$-connected matroid with at least $2(k-1)$ elements where $k \geqslant 4$. Then $M$ has at least one contractible element.

Contractible elements have been studied for special matroids, namely, graphs and 3-connected matroids. For $k \geqslant 4$, contractible edges in $k$-connected triangle-free graphs have been studied in [4, 12].

**Theorem 3.1. ([12])** *Every k-connected triangle-free graph, where $k \geqslant 2$, contains at least one contractible edge.*

**Theorem 3.2. ([4])** *Every k-connected triangle-free graph of order $n \geqslant 3k$, where $k \geqslant 2$, has at least $n + \frac{3}{2}k^2 - 3k$ contractible edges.*

**Theorem 3.3. ([7])** *Every minimally k-connected graph contains at least $\frac{(k-1)n+2k}{2k-1}$ vertices of degree k.*

The following well-known result can be found in [9].

https://doi.org/10.1017/S0963548302005497 Published online by Cambridge University Press

**Lemma 3.4.** *A $k$-connected matroid with at least $2(k-1)$ elements has no circuits or cocircuits of size less than $k$.*

From the definition, if a matroid $M$ is $k$-connected, then $M$ is vertically $k$-connected. The next result is a slight variation of [9, Theorem 8.2.6].

**Lemma 3.5.** *Let $M$ be a vertically $k$-connected with girth at least $k$. Then $M$ is $k$-connected.*

Our next result shows that Problem A and Conjecture B are closely related.

**Theorem 3.6.** *The answer to Problem A is affirmative if and only if Conjecture B is true.*

**Proof.** Suppose that Conjecture B is true. Let $M$ be a minimally $k$-connected matroid with at least $2(k-1)$ elements. Then, for each $e \in E(M)$, the matroid $M^*/e = (M\backslash e)^*$ is not $k$-connected, and $M^*$ is $k$-connected. If $M^*$ has a $k$-element circuit then the result holds. Thus we assume that $M^*$ has no $k$-element circuit. Then, as Conjecture B is true, $M^*$ has an element $e$ such that $M^*/e$ is vertically $k$-connected. Since $M^*$ has at least $2(k-1)$ elements, by Lemma 3.4, each circuit and cocircuit of $M^*$ has at least $k$ elements. As $M^*$ has no $k$-element circuit, the girth of $M^*/e$ is at least $k$. We conclude that $M^*/e$ is $k$-connected by Lemma 3.5. Hence $M\backslash e$ is $k$-connected, a contradiction.

Suppose that the answer to Problem A is affirmative. Let $M$ be a $k$-connected matroid with at least $2(k-1)$ elements, where $k \geqslant 4$. Suppose that $M$ has no $k$-element circuit. Now we show that Conjecture B must be true too. Suppose not. Then, for each $e$ of $E(M)$, the matroid $M/e$ is not vertically $k$-connected, hence not $k$-connected. Thus $M^*$ is a minimally $k$-connected matroid. We deduce that $M^*$ has a $k$-element cocircuit. Hence $M$ has a circuit of size $k$; a contradiction. □

**Proof of Theorem 1.4.** Let $v_k$ be the number of $k$-element cocircuits of $M$. We will prove (i) and (ii) at the same time. As $k \geqslant 4$, by Seymour's Splitter Theorem [10], $M$ is graphic, cographic, or isomorphic to $R_{10}$. If $M$ is $R_{10}$, then $k = 4$ and it is easy to verify that both (i) and (ii) hold. Suppose that $M$ is graphic and $M \cong M(G)$ for some graph $G$ with no isolated vertices. Then $G$ is a minimally $k$-connected graph. By Theorem 3.3, $G$ has at least

$$\frac{(k-1)n + 2k}{2k-1} = \frac{(k-1)r(M) + 3k - 1}{2k-1}$$

vertices of degree of $k$. Hence

$$v_k \geqslant \frac{(k-1)r(M) + 3k - 1}{2k-1}. \tag{3.3}$$

If $M$ is cographic, then $M \cong M^*(G)$ for some graph $G$, where $G$ is a $k$-connected graph. As $M$ has at least $2(k-1)$ elements, each circuit and cocircuit of $M$ has size at least $k$. Hence each circuit of $G$ has size at least $k$. As $M$ is minimally $k$-connected, for each edge $e$ of $E(G)$, $M^*(G)\backslash e$ is not $k$-connected. If $G$ has a circuit of size $k$, then $M$ has a cocircuit

https://doi.org/10.1017/S0963548302005497 Published online by Cambridge University Press

of size $k$ and therefore (i) holds. Otherwise, each circuit of $G$ has size at least $k+1$. By Theorem 3.1, $G$ has a contractible edge $e$ such that $G/e$ is $k$-connected and has no circuit of size less than $k$. Thus $M^*(G)\backslash e = M^*(G/e)$ is $k$-connected. This is a contradiction. We conclude that (i) holds.

Now suppose that $M$ has at least $r(M)+3k-1$ elements. Then $r^*(M) \geqslant 3k-1$ and therefore $G$ has at least $3k$ vertices. For each contractible edge $e$ of $G$, we know that $G/e$ is $k$-connected, but $M^*(G/e) = M^*(G)\backslash e = M\backslash e$ is not $k$-connected. Thus $G/e$ must have a circuit of size less than $k$. As each circuit of $G$ has size at least $k$, we conclude that $e$ must be in a circuit of $G$ of size $k$. Hence $e$ is in a cocircuit of $M$ of size $k$. Note that $v_k$ is the number of $k$-circuits in $G$. Let $c$ denote the number of contractible edges of $G$. As each contractible edge is in a $k$-circuit of $G$, we conclude that

$$kv_k \geqslant c.$$

By Theorem 3.2, we have

$$c \geqslant n + \frac{3}{2}k^2 - 3k.$$

Thus

$$\begin{aligned} v_k &\geqslant \frac{1}{k}\left(n + \frac{3}{2}k^2 - 3k\right) \\ &= \left(r^*(M) + \frac{3}{2}k^2 - 3k + 1\right) \\ &\geqslant \frac{2|E(M)| - 2r(M) + 3k^2 - 6k + 2}{2k}. \end{aligned} \tag{3.4}$$

The theorem follows from (3.3) and (3.4). □

## Note added in proof

After the acceptance of this paper, the author discovered that M. Kriesell also proved the second part of Theorem 1.3 (using a different method) in his paper 'Contractible subgraphs in 3-connected graphs', *J. Combin. Theory Ser. B* **80** (2000) 32–48. He also showed that if a vertex in a 3-connected graph with at least five vertices is not incident to any contractible edge, then the vertex has at least four degree-3 neighbours, each of which is incident to exactly two contractible edges.

## References

[1] Ando, K., Enomoto, H. and Saito, A. (1987) Contractible edges in 3-connected graphs. *J. Combin. Theory Ser. B* **42** 87–93.
[2] Cunningham, W. H. (1981) On matroid connectivity. *J. Combin. Theory Ser. B* **30** 94–99.
[3] Dean, N., Hemminger, R. E. and Ota, K. (1989) Longest cycles in 3-connected graphs contain three contractible edges. *J. Graph Theory* **12** 17–21.
[4] Egawa, Y., Enomoto, H. and Saito, A. (1986) Contractible edges in triangle-free graphs. *Combinatorica* **6** 269–274.

https://doi.org/10.1017/S0963548302005497 Published online by Cambridge University Press

[5] Halin, R. (1969) Untersuchungen über minimale $n$-fach zusammenhängende graphen. *Math. Ann.* **182** 175–188.
[6] Hemminger, R. E. and Yu, X. (1993) Covering contractible edges in 3-connected graphs II: Characterizing those with covers of size three. *J. Graph Theory* **17** 247–252.
[7] Mader, W. (1972) Ecken vom Grad $n$ in minimalen $n$-fach zusammenhängenden Graphen. *Arch. Math.* **23** 219–224.
[8] Ota, K. and Saito, A. (1988) Non-separating induced cycles in 3-connected graphs. *Scientia Ser. A* **2** 101–105.
[9] Oxley, J. G. (1992) *Matroid Theory*, Oxford University Press, New York.
[10] Seymour, P. D. (1980) Decomposition of regular matroids. *J. Combin. Theory Ser. B* **28** 305–359.
[11] Thomassen, C. (1980) Planarity and duality of finite and infinite graphs. *J. Combin. Theory Ser. B* **29** 244–271.
[12] Thomassen, C. (1981) Non-separating cycles in $k$-connected graphs. *J. Graph Theory* **5** 351–354.
[13] Tutte, W. T. (1961) A theory of 3-connected graphs. *Nederl. Akad. Wetensch. Proc. Ser. A* **64** 441–455.
[14] Wu, H. (1998) On contractible and vertically contractible elements in 3-connected matroids and graphs. *Discrete Math.* **179** 185–203.

https://doi.org/10.1017/S0963548302005497 Published online by Cambridge University Press