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On finitely many base q expansions

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Abstract. Given some integer $m \geq 3$, we find the first explicit collection of countably many intervals in $(1, 2)$ such that for any q in one of these intervals, the set of points with exactly m base q expansions is non-empty and, moreover, has positive Hausdorff dimension. Our method relies on an application of a theorem proved by Falconer and Yavicoli [Intersections of thick compact sets in $\mathbb{R}^d$. *Math. Z.* **301**(3) (2022), 2291–2315, Theorem 6], which guarantees that the intersection of a family of compact subsets of $\mathbb{R}^d$ has positive Hausdorff dimension under certain conditions.

Key words: expansions in non-integer bases, Hausdorff dimension, thickness
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1. *Introduction*

Let $q \in (1, 2)$ and let $I_q = [0, 1/(q-1)]$. We say that a sequence $(\epsilon_j)_{j=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ is a base q expansion of $x \in I_q$ if

$$x = \sum_{j=1}^{\infty} \epsilon_j q^{-j}.$$

It is a straightforward exercise to show that x has a base q expansion if and only if $x \in I_q$. In order to put our work in context, we observe where the research in base q expansions started, how it developed, and what current open questions are receiving attention. Initially, base q expansions were studied via the dynamical system $T(x) = qx \pmod 1$ on $[0, 1]$. Each point $x \in [0, 1]$ has a unique orbit in this system which corresponds to what is known as the *greedy expansion* of x. In an early paper by Rényi [16] it is shown that there exists a T-invariant probability measure, equivalent to Lebesgue, which is also ergodic. Following this, Parry [15] gave sufficient conditions on a sequence $(\epsilon_j)_{j=1}^{\infty}$ in order for it to appear as a greedy base q expansion. We note that these results were actually proved in the more general setting where q is some real number greater than 1, where instead of $\{0, 1\}^{\mathbb{N}}$ the sequences $(\epsilon_j)_{j=1}^{\infty}$ are elements of $\{0, 1, \ldots, \lfloor q \rfloor\}^{\mathbb{N}}$ when $q \notin \mathbb{N}$, and are elements of $\{0, 1, \ldots, q-1\}^{\mathbb{N}}$ when $q \in \mathbb{N}$. In the early 1990s, more attention was given

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to the broader system in which all possible base q expansions of $x \in I_q$ are considered. This is the system in which we are interested in this paper.

We define the set of base q expansions of $x \in I_q$ by

$$\Sigma_q(x) = \left\{(\epsilon_j)_{j=1}^{\infty} \in \{0, 1\}^{\mathbb{N}} : x = \sum_{j=1}^{\infty} \epsilon_j q^{-j}\right\},$$

and the projection map $\pi_q : \{0, 1\}^{\mathbb{N}} \to I_q$ by

$$\pi_q((\epsilon_j)_{j=1}^{\infty}) = \sum_{j=1}^{\infty} \epsilon_j q^{-j}.$$

The foundational results in base q expansions demonstrate that they behave very differently than the well-understood integer base expansions. Recall the familiar example of binary expansions, where every point $x \in [0, 1]$ has a unique expansion apart from a countable set of points which have exactly two expansions, by virtue of the fact that $\pi_2(10^{\infty}) = \pi_2(01^{\infty})$. The same is true for all integer base expansions. By comparison, Sidorov proved that for every $q \in (1, 2)$, Lebesgue almost every $x \in I_q$ will have uncountably many base q expansions [18]. Moreover, given any $m \in \mathbb{N} \cup \{\aleph_0\}$, we can choose $q \in (1, 2)$ and $x \in I_q$ such that x has exactly m base q expansions [2, 4, 19]. Define

$$\mathcal{U}_q = \{x \in I_q : |\Sigma_q(x)| = 1\},$$

and for $m \geq 2$,

$$\mathcal{U}_q^{(m)} = \{x \in I_q : |\Sigma_q(x)| = m\}.$$

In this paper, our results concern the sets $\mathcal{B}_m$, where

$$\mathcal{B}_m = \{q \in (1, 2) : \mathcal{U}_q^{(m)} \neq \emptyset\}.$$

For all $k \geq 2$, we define the k-Bonacci number q_k to be the unique number in $(1, 2)$ which satisfies

$$q_k^k - q_k^{k-1} - \cdots - q_k - 1 = 0.$$

For $k = 2$, we denote $q_2 = (1 + \sqrt{5})/2$, the golden ratio, by G. This family of numbers will play an important role in our results and in the discussion below.

Particular attention was given to the number of expansions of $x = 1$ for different values of q. In particular, it was shown by Erdős *et al.* [8, Theorem 1] that the set of q for which 1 has a unique base q expansion has uncountably many elements, while Daróczy and Kátai [7, Theorem 3] showed that it has zero Lebesgue measure, and Hausdorff dimension 1. Moreover, in [9] Erdős and Joó showed that there are uncountably many bases q for which $1 \in \mathcal{U}_q^{\aleph_0}$. Komornik and Loreti [14] found an algebraic construction for the value $q_{\mathrm{KL}} \approx 1.787\,23$, now known as the Komornik–Loreti constant, which is the smallest base for which 1 has a unique expansion. Later, it was found that q_{KL} plays an important role in our understanding of $\mathcal{U}_q$ [12].

Generalizing the investigation of the number of expansions of $x = 1$ for different bases q, we can ask for which values of q there exists at least one $x \in I_q$ that has a

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unique expansion in base q. It was shown by Erdős *et al.* [10] that if $q \in (1, G)$ then all $x \in (0, 1/(q-1))$ have uncountably many base q expansions. (It is obvious that the endpoints of I_q, $0 = \pi_q(0^\infty)$ and $1/(q-1) = \pi_q(1^\infty)$, have unique expansions.) Sidorov and Vershik [20] showed that if $q = G$ then all $x \in (0, 1/(q-1))$ have uncountably many expansions unless $x = nG \pmod 1$ for some $n \in \mathbb{Z}$, in which case $x \in \mathcal{U}_G^{\aleph_0}$. In [12] Glendinning and Sidorov proved the following dichotomy. If $q \in (G, q_{\mathrm{KL}})$ then $\mathcal{U}_q$ is countably infinite and every unique expansion is eventually periodic, but if $q \in (q_{\mathrm{KL}}, 2)$ then $\mathcal{U}_q$ contains a Cantor set with positive Hausdorff dimension. For $q = q_{\mathrm{KL}}$, it is shown that $\mathcal{U}_q$ is uncountably infinite and has zero Hausdorff dimension. We will see in more detail later (Lemma 2.1) that if $q \in (G, 2)$ then $\dim_{\mathrm{H}}(\mathcal{U}_q) \nearrow 1$ as $q \nearrow 2$ [12]. Another avenue of research concerns the function $q \mapsto \dim_{\mathrm{H}}(\mathcal{U}_q)$ which is shown by Komornik *et al.* in [13] to be continuous, have bounded variation, and resemble the Cantor function. Further work was carried out on the properties of this function by the Barrera *et al.* in [6]. Together these results provide us with a reasonably complete understanding of $\mathcal{U}_q$ for all $q \in (1, 2)$.

Following on from the study of $\mathcal{U}_q$ it is natural to ask, given $m \in \mathbb{N}$, for which bases q is the set $\mathcal{U}_q^{(m)}$ non-empty and what is its Hausdorff dimension? In [19] Sidorov shows that the smallest element of $\mathcal{B}_2$ is $q_2 \approx 1.710\,64$, which is the root in $(1, 2)$ of

$$x^4 = 2x^2 + x + 1.$$

The next smallest element of $\mathcal{B}_2$ is $q_f \approx 1.754\,88$, which is the root in $(1, 2)$ of

$$x^3 = 2x^2 - x + 1,$$

and it is shown by Baker and Sidorov in [4] that q_f is also the smallest element of $\mathcal{B}_m$ for all $m \in \mathbb{N}_{\geq 3}$. Sidorov [19] proved the following key results concerning $\mathcal{B}_2$, and it is these results which we build on directly in this paper. After proving

$$q \in \mathcal{B}_2 \iff 1 \in \mathcal{U}_q - \mathcal{U}_q, \tag{1}$$

it is a straightforward application of Newhouse's theorem [17] (see §2.1) to prove that $[q_3, 2) \subset \mathcal{B}_2$, where $q_3 \approx 1.839\,29$. Sidorov generalizes this theorem, claiming that there exists some $\gamma_m > 0$ such that $[2 - \gamma_m, 2) \subset \mathcal{B}_m$ for all $m \geq 3$. Unfortunately, the authors believe this proof contains a mistake which cannot be fixed. If this is true, and this result is invalid, then the question of whether the Lebesgue measure of $\mathcal{B}_m$ is positive is still open for $m \geq 3$. The main theorem of this paper (Theorem A) is a stronger version of Sidorov's general result in the sense that we are able to find the first explicit intervals contained in $\mathcal{B}_m$ for $m \geq 3$. More recently it was proved by Baker and Zou [5] that for Lebesgue almost every $q \in (q_{\mathrm{KL}}, M + 1]$ we have $\dim_{\mathrm{H}} \mathcal{U}_q^{(m)} \leq \max\{0, 2 \dim_{\mathrm{H}} \mathcal{U}_q - 1\}$ for all $m \in \{2, 3, \ldots\}$. This result improves upon the bound of $\dim_{\mathrm{H}} \mathcal{U}_q^{(m)} \leq \dim_{\mathrm{H}} \mathcal{U}_q$ for all $m \in \{2, 3, \ldots\}$ which can be seen by a simple branching argument (see [19] or [2]). The equivalence (1) illustrates how information about $\mathcal{U}_q$ can be used to study $\mathcal{B}_2$. More generally, we will see in §2.2 that we can guarantee that $q \in \mathcal{B}_m$ if a certain intersection of affine images of the set $\mathcal{U}_q$ is non-empty. Moreover, via an application of a theorem of Falconer and Yavicoli [11, Theorem 6], we are able to bound the Hausdorff dimension of

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TABLE 1. For small m, this table shows the leftmost interval contained in $\mathcal{B}_{m+2}$ and the lower bound on $\dim_{\mathrm{H}}(\mathcal{U}_{q_{K_m}}^{(m+2)})$, provided by Theorem A. We emphasize that these are the leftmost intervals guaranteed to exist by Theorem A; there may be other intervals in $\mathcal{B}_{m+2}$ contained to the left of these.

m	K_m	q_{K_m} (approx.)	Interval in q-space contained in $\mathcal{B}_{m+2}$	$\dim_{\mathrm{H}}(\mathcal{U}_{q_{K_m}}^{(m+2)})$
1	31	1.999 999 999 534 342	$[q_{31} - 1.262\,18 \times 10^{-29}, q_{31} + 1.262\,18 \times 10^{-29}]$	$\geq 0.999\,967\,173\ldots$
2	32	1.999 999 999 767 168	$[q_{32} - 3.673\,42 \times 10^{-40}, q_{32} + 3.673\,42 \times 10^{-40}]$	$\geq 0.999\,983\,586\ldots$
3	32	1.999 999 999 767 168	$[q_{32} - 8.552\,85 \times 10^{-50}, q_{32} + 8.552\,85 \times 10^{-50}]$	$\geq 0.999\,979\,240\ldots$
4	32	1.999 999 999 767 168	$[q_{32} - 1.991\,36 \times 10^{-59}, q_{32} + 1.991\,36 \times 10^{-59}]$	$\geq 0.999\,974\,848\ldots$
5	33	1.999 999 999 883 594	$[q_{33} - 3.622\,27 \times 10^{-71}, q_{33} + 3.622\,27 \times 10^{-71}]$	$\geq 0.999\,985\,209\ldots$

TABLE 2. Neighbourhoods of q_k contained in $\mathcal{B}_3$ for small values of k, provided by Theorem B.

k	q_k (approx.)	Interval in q-space contained in $\mathcal{B}_3$
9	1.998 029 470 262 29	$[q_9, q_9 + 6.103\,16 \times 10^{-8}]$
10	1.999 018 632 710 10	$[q_{10} - 1.509\,25 \times 10^{-8}, q_{10} + 1.509\,25 \times 10^{-8}]$
11	1.999 510 401 978 29	$[q_{11} - 3.750\,92 \times 10^{-9}, q_{11} + 3.750\,92 \times 10^{-9}]$
12	1.999 755 500 937 32	$[q_{12} - 9.347\,45 \times 10^{-10}, q_{12} + 9.347\,45 \times 10^{-10}]$
13	1.999 877 832 711 55	$[q_{13} - 2.332\,86 \times 10^{-10}, q_{13} + 2.332\,86 \times 10^{-10}]$

the corresponding $\mathcal{U}_q^{(m)}$ set from below. The version of this theorem that we use appears as Theorem F&Y in §2.1.

Our most general result is Theorem A, which locates an explicit collection of countably many intervals in $\mathcal{B}_m$ for all $m \geq 3$. More precisely, for any $m \geq 3$, Theorem A tells us that there is a neighbourhood of q_k contained in $\mathcal{B}_m$ whenever k is sufficiently large. In the case of $\mathcal{B}_3$ we are able to obtain a stronger result which is the content of Theorem B.

THEOREM A. *Let $m \in \mathbb{N}$ and let $k \geq K_m$ where*

$$K_m = \lceil \tfrac{20}{19}(\log_{1.999}(m+2) + 24) + 4 \rceil.$$

If $|q - q_k| < q_k^{-(m+2)k-3}$, then $q \in \mathcal{B}_{m+2}$ and $\dim_{\mathrm{H}}(\mathcal{U}_q^{(m+2)}) \geq 1 - 1024(m+2)^{20/19} q^{4-k} > 0$.

THEOREM B.

(a) *For $k \geq 10$, if $|q - q_k| \leq q_k^{-2k-6}$ then $q \in \mathcal{B}_3$.*
(b) *If $0 < q - q_9 \leq q_9^{-24}$ then $q \in \mathcal{B}_3$.*

Together Theorems A and B provide the first explicit intervals contained in $\mathcal{B}_m$ for $m \geq 3$. Since $K_1 = 31$, Theorem B proves the existence of intervals in $\mathcal{B}_3$ where Theorem A does not apply, namely, when $m = 1$ and $9 \leq k < K_1$. The reason for this discrepancy in the values of k where the theorems apply is due to the number of sets in the intersection which we require to be non-empty. To be precise, the argument for the proof of Theorem B boils down to proving that the intersection of two compact subsets of $\mathbb{R}$ is non-empty; this allows us to apply Theorem N [**17**]. However, to prove Theorem A for some $m \in \mathbb{N}$, we require a collection of $(m+2)$ sets to be non-empty, hence we cannot directly apply Theorem N [**17**].

In §2 we outline the main theorems and propositions used for Theorems A and B, whose proofs are contained in §§3 and 4, respectively.

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2. *Preliminaries*

In this section we introduce the results required for the proofs of Theorems A and B. We draw special attention to Theorem F&Y—a special case of a theorem of Falconer and Yavicoli [11, Theorem 6]—on which Theorem A relies.

2.1. *Thickness and interleaving.*

Let $|X|$ and $\mathrm{conv}(X)$ denote the diameter and convex hull respectively of a set $X \subset \mathbb{R}$. Given some compact set $C \subset \mathbb{R}$, Newhouse [17] defines its *thickness* $\tau(C)$ via its *gaps* and *bridges*. A *gap* G of C is a connected component of $\mathbb{R} \setminus C$ (including both the unbounded components to the left and right of $\mathrm{conv}(C)$), and since C is compact, all such gaps are open intervals. Any compact set $C \subset \mathbb{R}$ admits a stepwise construction removing gaps in order of decreasing diameter. Specifically, we start by removing the unbounded gaps from $\mathbb{R}$ to leave $\mathrm{conv}(C)$. From here we remove from $\mathrm{conv}(C)$ the next largest gap G_1 (if there is a tie, take any). This leaves two closed intervals, called *bridges*, B_1^L and B_1^R with the gap G_1 in between. We continue in this way, removing the next largest gap from the remaining union of bridges. In general, at step n, gap G_n, with $|G_{n-1}| \geq |G_n| \geq |G_{n+1}|$, is removed from a closed interval, leaving bridges B_n^L and B_n^R either side of G_n. We define the thickness of C by

$$\tau(C) = \inf_n \left\{ \min\left(\frac{|B_n^L|}{|G_n|}, \frac{|B_n^R|}{|G_n|} \right) \right\}.$$

Note that the initial step removing the unbounded gaps of C is not involved in the thickness calculation. Moreover, it can be checked that the thickness is independent of the choice of gap when two or more gaps have the same diameter [11]. If B is a bridge of C then all gaps contained within B have diameter no larger than the gaps on either side of B. Note also that any bridge B of C has the property that all gaps of C are either contained in B or contained in the complement of B.

We will be interested in the thickness of certain compact subsets of $\mathcal{U}_q$ for $q \in (1, 2)$. To study these effectively we will need to define some notation. Let $\{0, 1\}^*$ be the set of all finite words in $\{0, 1\}$ and let $|\delta|$ denote the length of a finite word $\delta \in \{0, 1\}^*$. A sequence $(\epsilon_j)_{j=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ is said to *avoid* a word (finite or infinite) $(\delta_j)_{j=1}^{m} \in \{0, 1\}^m$ if there is no index $i \in \mathbb{N}$ such that $(\epsilon_j)_{j=i}^{i+m-1} = (\delta_j)_{j=1}^{m}$. For any $k \in \mathbb{N}$, define the set $S_k \subset \{0, 1\}^{\mathbb{N}}$ by

$$S_k = \{(\epsilon_j)_{j=1}^{\infty} \in \{0, 1\}^{\mathbb{N}} : (\epsilon_j)_{j=1}^{\infty} \text{ avoids } (01^k) \text{ and } (10^k)\}.$$

The following key lemma is a consequence of the work in [12, Lemma 4].

LEMMA 2.1. *Let* $k \geq 2$.

(a) *If* $q \in (q_k, 2)$ *then* $\pi_q(S_k) \subset \mathcal{U}_q$.

(b) *If* $q \in (q_k, 2)$ *then* $\pi_q(S_k)$ *is a Cantor set and, moreover, every gap of* $\pi_q(S_k)$ *is an open interval of the form*

$$(\pi_q(\delta(01^{k-1})^{\infty}), \pi_q(\delta(10^{k-1})^{\infty})),$$

where $\delta \in \{0, 1\}^*$ *avoids* (01^k) *and* (10^k).

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It is also shown in [12] that $\dim_{\mathrm{H}}(\pi_q(S_k)) \to 1$ as $q \to 2$, but more important for us is the following result which is a consequence of the estimates in the proof of [19, Theorem 4.4].

LEMMA 2.2. *We have* $\tau(\pi_q(S_k)) > q^{k-3}$ *whenever* $k \geq 4$ *and* $q > q_k$.

We define interleaving along with a stronger version we call ϵ-*strong interleaving*. Two compact sets $C_1, C_2 \subset \mathbb{R}$ are *interleaved* if neither set is contained in a gap of the other. The Hausdorff metric, defined on compact subsets of $\mathbb{R}$, is given by

$$d_{\mathrm{H}}(X, Y) = \max\left\{\sup_{x\in X} d(x, Y), \sup_{y\in Y} d(y, X)\right\},$$

where d is the normal Euclidean distance on $\mathbb{R}$. Let $\epsilon > 0$ and let A and B be compact subsets of $\mathbb{R}$. We say that the sets A and B are ϵ-*strongly interleaved* if the sets A' and B' are interleaved whenever A' and B' are compact subsets of $\mathbb{R}$ with the property that $d_{\mathrm{H}}(A, A') \leq \epsilon$ and $d_{\mathrm{H}}(B, B') \leq \epsilon$. Observe that if A and B are ϵ-strongly interleaved for some $\epsilon > 0$ then A and B are ϵ'-strongly interleaved for every $0 < \epsilon' < \epsilon$. It is clear that for any $\epsilon > 0$, if A and B are ϵ-strongly interleaved then they are interleaved. The following lemma allows us to declare that two compact sets in $\mathbb{R}$ are ϵ-strongly interleaved if they contain points which are sufficiently well separated. This will be useful for the proof of Theorem B.

LEMMA 2.3. *Suppose A and B are compact subsets of* $\mathbb{R}$ *and there exist* $a_1, a_2 \in A$ *and* $b_1, b_2 \in B$ *with the property that*

$$a_1 \leq b_1 \leq a_2 \leq b_2;$$

then A and B are interleaved. Moreover, if

$$\min\{(b_1 - a_1), (a_2 - b_1), (b_2 - a_2)\} \geq 2\epsilon,$$

for some $\epsilon > 0$ *then A and B are* ϵ*-strongly interleaved.*

Proof. Let A and B be compact subsets of $\mathbb{R}$ with $a_1, a_2 \in A$ and $b_1, b_2 \in B$ satisfying $a_1 \leq b_1 \leq a_2 \leq b_2$. By definition, A and B are interleaved if neither is contained in a gap of the other. Since $a_2 \in A$ we can partition the gaps of A into those that lie to the left of a_2 and those that lie to the right of a_2. B cannot be contained in a gap to the left of a_2 because $b_2 \in B$ and $b_2 \geq a_2$. Similarly, B cannot be contained in a gap to the right of a_2 because $b_1 \in B$ and $b_1 \leq a_2$. A cannot be contained in a gap of B by a similar argument partitioning the gaps of B into those to the left and right of b_1.

Suppose $\epsilon > 0$ and $\min\{(b_1 - a_1), (a_2 - b_1), (b_2 - a_2)\} \geq 2\epsilon$. Let A' and B' be compact subsets of $\mathbb{R}$ satisfying $d_{\mathrm{H}}(A, A'), d_{\mathrm{H}}(B, B') \leq \epsilon$. Therefore, there exist $a_1', a_2' \in A'$ and $b_1', b_2' \in B'$ satisfying

$$|a_1' - a_1|, |a_2' - a_2|, |b_1' - b_1|, |b_2' - b_2| \leq \epsilon.$$

Hence,

$$b_1' - a_1' \geq (b_1 - a_1) - |b_1' - b_1| - |a_1' - a_1| \geq 2\epsilon - \epsilon - \epsilon = 0,$$

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$$a_2' - b_1' \geq (a_2 - b_1) - |a_2' - a_2| - |b_1' - b_1| \geq 2\epsilon - \epsilon - \epsilon = 0,$$
$$b_2' - a_2' \geq (b_2 - a_2) - |b_2' - b_2| - |a_2' - a_2| \geq 2\epsilon - \epsilon - \epsilon = 0.$$

So $a_1' \leq b_1' \leq a_2' \leq b_2'$ and we conclude that A' and B' are interleaved by the first part of the lemma. □

Newhouse [17] proved that information on the thickness and interleaving of two compact sets in $\mathbb{R}$ can be sufficient to imply a non-empty intersection of the sets. This is the content of Theorem N and will be used to prove Theorem B.

THEOREM N. *Let C_1 and C_2 be interleaved compact subsets of $\mathbb{R}$ with thicknesses τ_1 and τ_2, respectively. If $\tau_1\tau_2 \geq 1$ then $C_1 \cap C_2 \neq \emptyset$.*

In our proof of Theorem A, we apply Theorem F&Y, which is a special case of a more general theorem proved by Falconer and Yavicoli [11, Theorem 6].

THEOREM F&Y. *For any $m \in \mathbb{N}$, let $(C_j)_{j=1}^{m+2}$ be compact subsets of $\mathbb{R}$ where each C_j has thickness at least τ. Assume that*

(i) $B = \bigcap_j \operatorname{conv}(C_j)$ *is non-empty*

and

(ii) *there exists $c \in (0, 1)$ such that*

$$(m + 2)\tau^{-c} \leq \tfrac{1}{(432)^2}\beta^c(1 - \beta^{1-c}), \tag{2}$$

where $\beta = \min\{\frac{1}{4}, |B|/\max_j |C_j|\}$.

Then $\dim_{\mathrm{H}}(\bigcap_j C_j) \geq 1 - 1024(m + 2)^{1/c}\tau^{-1} > 0$.

2.2. *A sufficient condition for $q \in \mathcal{B}_m$.* We define the maps f_0 and f_1 by

$$f_0 : \left[0, \frac{1}{q(q-1)}\right] \to I_q, \quad f_0(x) = qx,$$
$$f_1 : \left[\frac{1}{q}, \frac{1}{q-1}\right] \to I_q, \qquad f_1(x) = qx - 1,$$

and set $E_q = \{f_0, f_1\}$ (Figure 1). For any map h, we denote its inverse by h^{-1} and the i-fold composition $\underbrace{h \circ \cdots \circ h}_{i \text{ times}}$ by h^i.

Note that f_0 and f_1 depend on q implicitly, but in all cases the value of q will be clear from the context, so we suppress this in the notation. The following lemma is a routine check from the definitions of the maps f_0, f_1 and π_q, and builds the picture of how the maps of E_q 'generate' the base q expansions.

LEMMA 2.4. *For any $q \in (1, 2)$, $(\delta_j)_{j=1}^m \in \{0, 1\}^*$, $(\epsilon_j)_{j=1}^\infty \in \{0, 1\}^{\mathbb{N}}$ we have*

$$f_{\delta_1}^{-1} \circ \cdots \circ f_{\delta_m}^{-1}(\pi_q((\epsilon_j)_{j=1}^\infty)) = \pi_q((\delta_j)_{j=1}^m(\epsilon_j)_{j=1}^\infty).$$

The map π_q also applies to the extended alphabet $\{-1, 0, 1\}^{\mathbb{N}}$. Define $I_q^* = [-1/(q-1), 1/(q-1)]$. If $(\epsilon_j)_{j=1}^\infty \in \{-1, 0, 1\}^{\mathbb{N}}$ then $\pi_q((\epsilon_j)_{j=1}^\infty) \in I_q^*$. Moreover, if

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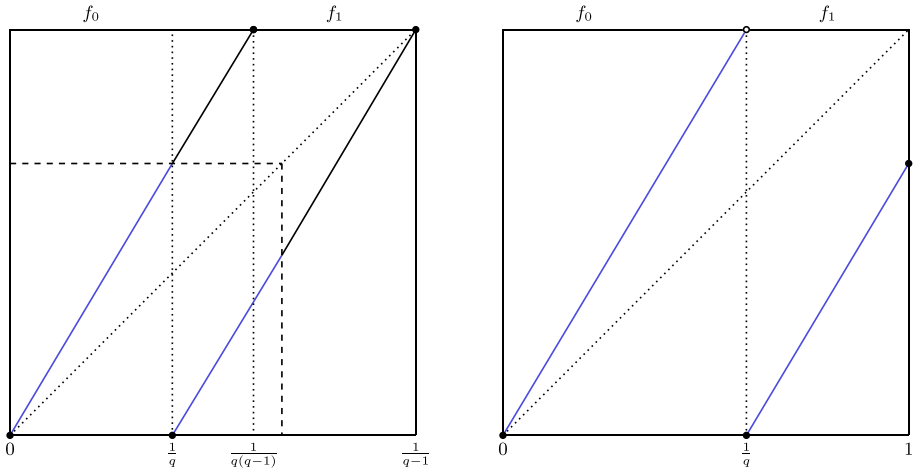


FIGURE 1. (a) The maps generating base q expansions. (b) The special case of the greedy expansion given by $f(x) = qx \pmod 1$ on $[0, 1]$.

we define the set of maps E_q^* by $\{f_{-1}^*, f_0^*, f_1^*\}$, where

$$f_{-1}^* : \left[\frac{-1}{q-1}, \frac{2-q}{q(q-1)}\right] \to I_q^*, \qquad f_{-1}^*(x) = qx + 1,$$
$$f_0^* : \left[\frac{-1}{q(q-1)}, \frac{1}{q(q-1)}\right] \to I_q^*, \qquad f_0^*(x) = qx,$$
$$f_1^* : \left[\frac{-(2-q)}{q(q-1)}, \frac{1}{q-1}\right] \to I_q^*, \qquad f_1^*(x) = qx - 1,$$

then Lemma 2.4 has the following natural generalization. However, we will not need this result until §4.2.

LEMMA 2.5. *For any* $q \in (1, 2)$, $(\delta_j)_{j=1}^m \in \{-1, 0, 1\}^*$, $(\epsilon_j)_{j=1}^\infty \in \{-1, 0, 1\}^{\mathbb{N}}$ *we have*

$$f_{\delta_1}^{*-1} \circ \cdots \circ f_{\delta_m}^{*-1}(\pi_q((\epsilon_j)_{j=1}^\infty)) = \pi_q((\delta_j)_{j=1}^m (\epsilon_j)_{j=1}^\infty).$$

For any $k \in \mathbb{N}_{\geq 2}$, $q \in (1, 2)$, define $g_{q,k} = f_1^{-(k-1)} \circ f_0^{-1}$, where the functions f_i are implicitly dependent on q. For brevity, whenever $m \geq 0$, we define

$$N_k^m(q) = \bigcap_{i=0}^{m} g_{q,k}^i(\mathcal{U}_q + 1) \cap g_{q,k}^m(\mathcal{U}_q).$$

Proposition 2.6 shows that to prove $q \in \mathcal{B}_{m+2}$ it is sufficient to prove that $N_k^m(q)$ is non-empty.

PROPOSITION 2.6. *Let* $k \geq 2$, $q \in (G, 2)$ *and* $m \geq 0$. *If* x *is such that* $f_0(x) \in N_k^m(q)$, *then* $x \in \mathcal{U}_q^{(m+2)} \cap J_q$, *and hence* $q \in \mathcal{B}_{m+2}$.

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To prove this proposition we will need the following definitions and lemmas. Let $q \in (1, 2)$, $x \in I_q$ and recall that $\Sigma_q(x)$ is the set of base q expansions of x. We define the orbit space of x by

$$\Omega_q(x) = \{(f_{\epsilon_j})_{j=1}^{\infty} \in \{f_0, f_1\}^{\mathbb{N}} : f_{\epsilon_k} \circ \cdots \circ f_{\epsilon_1}(x) \in I_q \text{ for all } k \in \mathbb{N}\}.$$

The following lemma was proved by the first author in [1, Lemma 2.4].

LEMMA 2.7. *For any $x \in I_q$, $\Sigma_q(x)$ is in bijection with $\Omega_q(x)$ via the map $(\epsilon_j)_{j=1}^{\infty} \to (f_{\epsilon_j})_{j=1}^{\infty}$.*

This bijection allows one to see clearly how some $x \in I_q$ may have multiple base q expansions. The region $J_q = [1/q, 1/q(q-1)]$, known as the *switch region*, has the property that $x \in J_q$ if and only if $f_0(x), f_1(x) \in I_q$. Clearly, such an x satisfies $|\Omega_q(x)| \geq 2$ and hence $|\Sigma_q(x)| \geq 2$. Let $x \in I_q \setminus J_q$, $y \in I_q$. We say that *x maps uniquely to y* if there is a finite sequence of maps $(f_{\epsilon_j})_{j=1}^{m} \in \{f_0, f_1\}^*$ such that

$$f_{\epsilon_m} \circ \cdots \circ f_{\epsilon_1}(x) = y$$

and

$$f_{\epsilon_k} \circ \cdots \circ f_{\epsilon_1}(x) \in I_q \setminus J_q, \tag{3}$$

for all $1 \leq k \leq m-1$. Note that we allow $y \in J_q$.

LEMMA 2.8. *Let $q \in (1, 2)$ and suppose $x \in I_q \setminus J_q$ maps uniquely to $y \in I_q$. Then $|\Sigma_q(x)| = |\Sigma_q(y)|$.*

Proof. Assuming the hypotheses of the lemma, we have $f_{\epsilon_k} \circ \cdots \circ f_{\epsilon_1}(x) \in I_q \setminus J_q$ for all $1 \leq k \leq m-1$, and $x \in I_q \setminus J_q$, where $(f_{\epsilon_j})_{j=1}^{m}$ exists by the assumption that x maps uniquely to y. For any $z \in I_q \setminus J_q$, we can see by inspection of the domains of f_0 and f_1 that there is a unique element $\epsilon \in \{0, 1\}$ such that $f_\epsilon(z) \in I_q$. Applying this to $x \in I_q \setminus J_q$ and to $f_{\epsilon_k} \circ \cdots \circ f_{\epsilon_1}(x)$ for every $1 \leq k \leq m-1$, we see that the sequence $(f_{\epsilon_j})_{j=1}^{m}$ is unique among those sequences which satisfy (3) for all $1 \leq k \leq m-1$. Hence, $(f_{\delta_j})_{j=1}^{\infty} \in \Omega_q(x)$ if and only if $(f_{\delta_j})_{j=1}^{\infty}$ is of the form $(f_{\epsilon_j})_{j=1}^{m}(f_{\delta_j})_{j=m+1}^{\infty}$, where $(f_{\delta_j})_{j=m+1}^{\infty} \in \Omega_q(f_{\epsilon_m} \circ \cdots \circ f_{\epsilon_1}(x))$. Since $f_{\epsilon_m} \circ \cdots \circ f_{\epsilon_1}(x) = y$, we know $(f_{\delta_j})_{j=m+1}^{\infty} \in \Omega_q(y)$. Therefore, the elements of $\Omega_q(x)$ are generated by taking the elements of $\Omega_q(y)$ and adding a unique prefix of $(f_{\epsilon_j})_{j=1}^{m}$. Hence, $|\Omega_q(x)| = |\Omega_q(y)|$, and by Lemma 2.7, $|\Sigma_q(x)| = |\Sigma_q(y)|$. □

The first part of the following lemma is precisely stating the heuristic that if $q \in (G, 2)$, then any orbit of a point under E_q cannot remain in J_q for more than one 'step'. The second part is a result touched upon in the introduction and is a consequence of Sidorov's work in [19, Lemma 2.2], but for completion we prove it here anyway.

LEMMA 2.9. *Let $q \in (G, 2)$.*

(a) *If $x \in J_q$ then for all $l \in \mathbb{N}$, the points $f_0^{-l}(x)$, $f_1^{-l}(x)$ map uniquely to x via the sequences of maps given by $(f_0)_{j=1}^{l}$ and $(f_1)_{j=1}^{l}$, respectively.*

(b) *If $x \in I_q$ is such that $f_0(x) \in (\mathcal{U}_q + 1) \cap \mathcal{U}_q$ then $x \in \mathcal{U}_q^{(2)} \cap J_q$ and hence $q \in \mathcal{B}_2$.*

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Proof. (a) Notice that $f_0^{-1}(x) = x/q$ is linear and strictly increasing for all $x \in I_q$ whenever $q \in (1, 2)$. If $q \in (G, 2)$ then $q^2 - q - 1 > 0$, so $q^{-2}/(q-1) < q^{-1}$, which is equivalent to $f_0^{-1}(1/q(q-1)) < 1/q$. This means that the image of the right endpoint of J_q under f_0^{-1} is less than the left endpoint of J_q. Hence, $f_0^{-1}(J_q) \cap J_q = \emptyset$. Extending this we can see that $f_0^{-i}(1/q(q-1)) = 1/(q^{i+1}(q-1)) < 1/q$ for all $i \geq 1$ and hence $f_0^{-i}(J_q) \cap J_q = \emptyset$. Therefore, if $l \in \mathbb{N}$ and $x \in J_q$ is arbitrary, then $f_0^i \circ f_0^{-l}(x) \in f_0^{i-l}(J_q)$, hence $f_0^i \circ f_0^{-l}(x) \in I_q \setminus J_q$ for all $i = 0, \ldots, l-1$. The case for f_1 is similar.

(b) Notice that for any $x \in J_q$, we have that $f_0(x) - f_1(x) = 1$, so $f_0(x) - 1 = f_1(x)$. Therefore, if $f_0(x) \in (\mathcal{U}_q + 1)$ then $f_1(x) \in \mathcal{U}_q$, and since both $f_0(x), f_1(x) \in I_q$, we know that $x \in J_q$. If we also know that $f_0(x) \in \mathcal{U}_q$, then $x \in \mathcal{U}_q^{(2)}$ by Lemma 2.7, and hence the claim. □

Equipped with Lemmas 2.7, 2.8 and 2.9, we are now able to prove Proposition 2.6.

Proof of Proposition 2.6. Fix any $q \in (G, 2)$, $k \geq 2$, and proceed by induction on m. For the base case, let $m = 0$ and suppose that $N_k^0(q) = (\mathcal{U}_q + 1) \cap \mathcal{U}_q \neq \emptyset$. Let $x \in I_q$ be such that $f_0(x) \in N_k^0(q)$. Then by Lemma 2.9(b), $x \in \mathcal{U}_q^{(2)} \cap J_q$ and $q \in \mathcal{B}_2$.

For the inductive step, notice that $N_k^{m+1}(q) = (\mathcal{U}_q + 1) \cap g_{q,k}(N_k^m(q))$. Assume the conclusion holds for m, namely that if $f_0(x) \in N_k^m(q)$ then $x \in \mathcal{U}_q^{(m+2)} \cap J_q$. Suppose that $x \in I_q$ is such that $f_0(x) \in N_k^{m+1}(q)$. Then $f_1(x) \in \mathcal{U}_q$ because $f_0(x) \in (\mathcal{U}_q + 1)$. Since $f_0(x) \in g_{q,k}(N_k^m(q))$, we know that $f_0 \circ f_1^{k-1} \circ f_0(x) \in N_k^m(q)$. Therefore, by assumption, $f_1^{k-1} \circ f_0(x) \in \mathcal{U}_q^{(m+2)} \cap J_q$. Since $f_1^{k-1} \circ f_0(x) \in J_q$, Lemma 2.9(a) implies that $f_0(x)$ maps uniquely to $f_1^{k-1} \circ f_0(x)$ and hence, by Lemma 2.8, $f_0(x) \in \mathcal{U}_q^{(m+2)}$. We can conclude by Lemma 2.7 that $x \in \mathcal{U}_q^{(m+3)}$ and since $f_0(x), f_1(x) \in I_q$, we know $x \in J_q$. □

Since $q > q_{k-1}$ implies that $\pi_q(S_{k-1}) \subset \mathcal{U}_q$ by Lemma 2.1(a), we have the following lemma as an immediate corollary to Proposition 2.6.

LEMMA 2.10. *If $k \geq 2$, $m \geq 1$, $q > q_{k-1}$ and the intersection*

$$\bigcap_{i=0}^{m} g_{q,k}^i(\pi_q(S_{k-1}) + 1) \cap g_{q,k}^m(\pi_q(S_{k-1})) \tag{4}$$

is non-empty, then $q \in \mathcal{B}_{m+2}$.

We will see that it will be useful to consider subsets of the sets in (4) rather than the sets in $N_k^m(q)$ because of our existing knowledge of the structure of the sets $\pi_q(S_{k-1})$.

3. *Proof of Theorem A*

Section 3.1 serves to introduce our approach to proving Theorem A.

3.1. *Proof outline.*

By Proposition 2.6, for all $m \in \mathbb{N}$, if $N_k^m(q) \neq \emptyset$ then $q \in \mathcal{B}_{m+2}$. Moreover, if $f_0(x) \in N_k^m(q)$ then $x \in \mathcal{U}_q^{(m+2)} \cap J_q$. So a lower bound on the Hausdorff dimension of $N_k^m(q)$ provides us with a lower bound on the Hausdorff dimension of $\mathcal{U}_q^{(m+2)}$. We seek to modify the sets in $N_k^m(q)$ in order to apply Theorem F&Y. More

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precisely, for $i = 0, \ldots, m$, we seek subsets $P_i(q) \subset g^i_{q,k}(\mathcal{U}_q + 1)$ and $Q_m(q) \subset g^m_{q,k}(\mathcal{U}_q)$ such that Theorem F&Y can be applied to the collection of compact subsets of $\mathbb{R}$ given by $\{P_0(q), \ldots, P_m(q), Q_m(q)\}$. For any given finite collection of sets $A_i \subset \mathbb{R}$, with arbitrary subsets $B_i \subset A_i$, it is obvious that if $\bigcap_i B_i \neq \emptyset$ then $\bigcap_i A_i \neq \emptyset$. Hence, if

$$M^m_k(q) = \bigcap_{i=0}^{m} P_i(q) \cap Q_m(q)$$

is non-empty, then $N^m_k(q) \neq \emptyset$ and $q \in \mathcal{B}_{m+2}$. Similarly, if $f_0(x) \in M^m_k(q)$ then $x \in \mathcal{U}^{(m+2)}_q \cap J_q$, so $\dim_{\mathrm{H}}(M^m_k(q)) \leq \dim_{\mathrm{H}}(\mathcal{U}^{(m+2)}_q)$. For k sufficiently large, and $q > q_{k-1}$, each of the sets $P_0(q), \ldots, P_m(q), Q_m(q)$ is shown to have thickness at least q^{k-4}. Let $B(q) = \bigcap_{i=0}^m \mathrm{conv}(P_i(q)) \cap \mathrm{conv}(Q_m(q))$ (see Figure 2), which we show is non-empty, and let

$$\beta_q = \min\left\{\frac{1}{4}, \frac{|B(q)|}{\max\{|P_0(q)|, \ldots, |P_m(q)|, |Q_m(q)|\}}\right\}.$$

β_q corresponds to β in Theorem F&Y except that we have dependency on q here because the sets to which we are applying Theorem F&Y depend on q. To conclude Theorem A from Theorem F&Y it remains to find some $c \in (0, 1)$ such that

$$(m + 2)(q^{k-4})^{-c} \leq \tfrac{1}{(432)^2}\beta^c_q(1 - \beta^{1-c}_q) \tag{5}$$

holds for all $m \in \mathbb{N}$, $k \geq K_m$ and $|q - q_k| < q_k^{-(m+2)k-3}$. Note that c could depend on q because the sets to which we are applying Theorem F&Y depend on q. However, since we eventually show that a constant value of c satisfies (5) under the required conditions, the standalone c notation reflects the independence on q. By requiring $|q - q_k| < q_k^{-(m+2)k-3}$ we achieve a uniform lower bound on β_q when $k \geq K_m$. This bound on β_q allows us to choose $c \in (0, 1)$ to satisfy (5) as described above. From here, Theorem F&Y tells us, under our assumptions $m \in \mathbb{N}$, $k \geq K_m$ and $|q - q_k| < q_k^{-(m+2)k-3}$, that $\dim_{\mathrm{H}}(\mathcal{U}^{(m+2)}_q) \geq \dim_{\mathrm{H}}(M^m_k(q)) \geq 1 - 1024(m + 2)^{1/c}q^{k-4} > 0$. Moreover, an immediate implication of this is that $\mathcal{U}^{(m+2)}_q$ is non-empty, so $q \in \mathcal{B}_{m+2}$. The majority of the work of the proof is in showing that the sets $P_0(q), \ldots, P_m(q), Q_m(q)$ satisfy the hypotheses of Theorem F&Y. We will see that it is simpler to consider $P_i(q)$ as a subset of $g^i_{q,k}(\pi_q(S_{k-1}) + 1)$ for all $i = 0, \ldots, m$, and $Q_m(q)$ as a subset of $g^m_{q,k}(\pi_q(S_{k-1}))$. This is justified by Lemma 2.10. In the following section we construct the subsets $P_i(q)$ of $g^i_k(\pi_q(S_{k-1}) + 1)$ for all $0 \leq i \leq m$, and $Q_m(q)$ of $g^m_k(\pi_q(S_{k-1}))$.

3.2. *Construction of $P_i(q)$ and $Q_m(q)$.* Fix $m \in \mathbb{N}$, $k \geq 3$ and $q > q_{k-1}$. Recall from §2.1 that any compact subset of $\mathbb{R}$ admits a construction as a complement of at most countably many gaps. Let $G(q)$ be the gap of $\pi_q(S_{k-1})$ given by the open interval

$$G(q) = (\pi_q(0^{k-3}(01^{k-2})^\infty), \pi_q(0^{k-3}(10^{k-2})^\infty)).$$

(Recall that we know this is a gap of $\pi_q(S_{k-1})$ by Lemma 2.1(b).) Similarly, let $H(q)$ be the gap of $\pi_q(S_{k-1})$ given by the open interval

$$H(q) = (\pi_q(1^{k-3}(01^{k-2})^\infty), \pi_q(1^{k-3}(10^{k-2})^\infty)).$$

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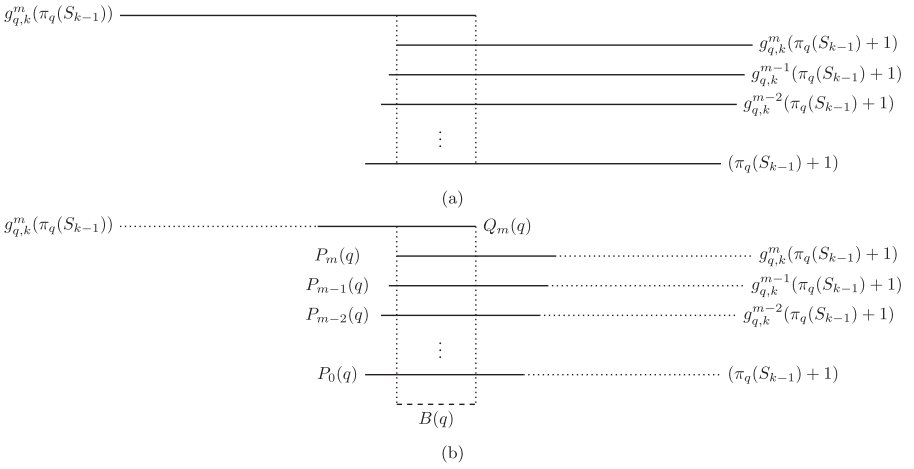


FIGURE 2. The process of taking subsets of affine images of $\pi_q(S_{k-1})$ in order to bound β_q. For $|q-q_k| < q_k^{-2k-6}$, the relative structure of the sets in all cases where $q < q_k$, $q = q_k$ and $q > q_k$ is shown in Figure 3. (a) The relative structure of the convex hulls of the sets in (4) in the case when $|q-q_k| < q_k^{-2k-6}$ and $q < q_k$. (b) For q as in (a), the figure shows the process of taking subsets given by $\{P_0(q), \ldots, P_m(q), Q_m(q)\}$, and their overlap given by $B(q)$. For clarity, only the convex hulls of the $P_0(q), \ldots, P_m(q), Q_m(q)$ sets are shown.

Let $0 \le i \le m$ and define $P_i(q)$ to be the subset of $g^i_{q,k}(\pi_q(S_{k-1})+1)$ to the left of the gap $G_i(q)$ where

$$G_i(q) = g^i_{q,k}(q^{-(m-i)k}G(q)+1),$$

that is,

$$P_i(q) = g^i_{q,k}(\pi_q(S_{k-1})+1) \cap [g^i_{q,k}(1),\, g^i_{q,k}(1+q^{-(m-i)k}\pi_q(0^{k-3}(10^{k-2})^\infty))].$$

Similarly, define $Q_m(q)$ to be the subset of $g^m_{q,k}(\pi_q(S_{k-1}))$ to the right of the gap $H_m(q)$ where

$$H_m(q) = g^m_{q,k}(H(q)),$$

that is,

$$Q_m(q) = g^m_{q,k}(\pi_q(S_{k-1})) \cap [g^m_{q,k}(\pi_q(1^{k-3}(10^{k-2})^\infty)),\, g^m_{q,k}(\pi_q(1^\infty))].$$

Since

$$(\pi_q(0^{(m-i)k}0^{k-3}(01^{k-2})^\infty),\, \pi_q(0^{(m-i)k}0^{k-3}(10^{k-2})^\infty))$$

is also a gap of $\pi_q(S_{k-1})$ for all $0 \le i \le m$ by Lemma 2.1(b), and we can write the gap $G_i(q)$ as

$$g^i_{q,k}(\pi_q(0^{(m-i)k}0^{k-3}(01^{k-2})^\infty)+1,\, \pi_q(0^{(m-i)k}0^{k-3}(10^{k-2})^\infty)+1),$$

we know that $G_i(q)$ is a gap of $g^i_{q,k}(\pi_q(S_{k-1})+1)$. Similarly, $H_m(q)$ is a gap of $g^m_{q,k}(\pi_q(S_{k-1}))$. By Lemma 2.2, we know that $\tau(\pi_q(S_{k-1})) > q^{k-4}$. To apply Theorem F&Y we need to make sure the process of taking the subsets $P_0(q), \ldots, P_m(q), Q_m(q)$ described above preserves this lower bound on the thickness. This is the content of Lemma 3.2. To prove this, we will need the following definition and a lemma which is another

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consequence of [12, Lemma 4]. We say that $(\epsilon_j)_{j=1}^{\infty}$ is *lexicographically less than* $(\epsilon'_j)_{j=1}^{\infty}$ if $\epsilon_i < \epsilon'_i$ where $i = \min\{j : \epsilon_j \neq \epsilon'_j\}$. In this case we write $(\epsilon_j)_{j=1}^{\infty} \prec (\epsilon'_j)_{j=1}^{\infty}$.

LEMMA 3.1. *Let $k \geq 2$, $q \in (q_k, 2)$, and let $\delta_1, \delta_2 \in \{0,1\}^*$ be finite sequences which avoid (01^k) and (10^k). For $i \in \{1,2\}$, let $G_i(q)$ be the gap of $\pi_q(S_k)$ given by the open interval*

$$(\pi_q(\delta_i(01^{k-1})^{\infty}), \pi_q(\delta_i(10^{k-1})^{\infty})).$$

(a) *If $|\delta_1| < |\delta_2|$ then $|G_1| > |G_2|$.*
(b) *If $|\delta_1| = |\delta_2|$ and $\delta_1 \prec \delta_2$ then the right endpoint of G_1 is less than the left endpoint of G_2.*

LEMMA 3.2. *For $k \geq 3$ and any $0 \leq i \leq m$, $\tau(P_i(q)) \geq \tau(\pi_q(S_{k-1}))$ and $\tau(Q_m(q)) \geq \tau(\pi_q(S_{k-1}))$.*

Proof. Let $k \geq 3$, $q > q_{k-1}$, and let $P^*(q)$ be the subset of $\pi_q(S_{k-1})$ to the left of the gap given by $G(q)$. That is,

$$P^*(q) = \pi_q(S_{k-1}) \cap [0, \pi_q(0^{k-3}(01^{k-2})^{\infty})].$$

We require $k \geq 3$ in order for the gap $G(q)$ to be well defined, and we require $q > q_{k-1}$ to guarantee that $\pi_q(S_{k-1})$ is a Cantor set by Lemma 2.1(b). Thickness is invariant under affine maps, and for each $0 \leq i \leq m$, $P_i(q)$ is the image under an affine map of $P^*(q)$. Hence, to prove the first part it is sufficient to show that $\tau(P^*(q)) \geq \tau(\pi_q(S_{k-1}))$. The result for $\tau(Q_m(q))$ can be proved in a similar way, and the lemma follows.

Using Lemma 3.1, since 0^{k-3} is the lexicographically smallest sequence of length $k-3$, we know that all gaps of $\pi_q(S_{k-1})$ contained in the interval $[0, \pi_q(0^{k-3}(01^{k-2})^{\infty})]$ must be smaller than $G(q)$. Hence, $\text{conv}(P^*(q))$ is a bridge of $\pi_q(S_{k-1})$ so all bounded gaps of $\pi_q(S_{k-1})$ are either contained in $\text{conv}(P^*(q))$ or contained in the complement of $\text{conv}(P^*(q))$. When we evaluate the thickness of $P^*(q)$, we can observe that the set of bounded gaps over which the infimum is taken is a subset of the set of bounded gaps of $\pi_q(S_{k-1})$. Moreover, for any gap G' contained in $\text{conv}(P^*(q))$, the left and right bridges of G' will be the same regardless of whether we consider G' to be a gap of $P^*(q)$ or a gap of $\pi_q(S_{k-1})$. Let $\mathcal{G}$ be the set of bounded gaps of $\pi_q(S_{k-1})$ and let $\mathcal{G}_{P^*}$ be the set of bounded gaps of $P^*(q)$. With the above observation in mind, and the fact that $\mathcal{G}_{P^*} \subset \mathcal{G}$, we see that

$$\begin{aligned}\tau(P^*(q)) &= \inf\left\{ \min\left\{\frac{|L_n|}{|G_n|}, \frac{|R_n|}{|G_n|}\right\} : G_n \in \mathcal{G}_{P^*}\right\} \\ &\geq \inf\left\{ \min\left\{\frac{|L_n|}{|G_n|}, \frac{|R_n|}{|G_n|}\right\} : G_n \in \mathcal{G}\right\} \\ &= \tau(\pi_q(S_{k-1})).\end{aligned}$$

□

We will only need to apply Lemma 3.2 in the case $k \geq 5$ because this is where the bound $\tau(\pi_q(S_{k-1})) \geq q^{k-4}$ provided by Lemma 2.2 applies.

3.3. *The relative structure of the sets $P_i(q)$ and $Q_m(q)$.* Recall that our plan is to apply Theorem F&Y to the collection $\{P_0(q), \ldots, P_m(q), Q_m(q)\}$ where $B(q)$

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is the intersection of their convex hulls. In order for the convex hulls of the sets $P_0(q), \dots, P_m(q), Q_m(q)$ to overlap in a sufficiently large interval, that is, for $|B(q)|$ to be sufficiently large, we need to impose stronger conditions on q. This is done in the following lemmas. Note that the projection of any sequence—besides 0^∞—decreases as q increases. This is the content of the following lemma.

LEMMA 3.3. *For any sequence* $(\delta_j)_{j=1}^\infty \in \{0,1\}^{\mathbb{N}} \setminus \{0^\infty\}$, *and any* $q, q' \in (1,2)$,

$$q < q' \iff \pi_{q'}((\delta_j)_{j=1}^\infty) < \pi_q((\delta_j)_{j=1}^\infty).$$

Given some $q \in (1,2)$, $k \geq 2$, define $\epsilon_q \in \mathbb{R}$ by

$$1 = \pi_q((1^{k-1}0)^\infty) + \epsilon_q. \tag{6}$$

We introduce ϵ_q as a means to bound $|B(q)|$ and hence also β_q from below. Recall from §3.1 that a lower bound on β_q allows us to prove that (5) holds for some $c \in (0,1)$ and ultimately to apply Theorem F&Y to prove Theorem A. We will always be dealing with q in a small neighbourhood of q_k for some fixed k, so the implicit dependence of ϵ_q on k is suppressed in the notation. Since $1 = \pi_{q_k}((1^{k-1}0)^\infty)$, Lemma 3.3 tells us that $\epsilon_q < 0$ if $q < q_k$, $\epsilon_q = 0$ if $q = q_k$, and $\epsilon_q > 0$ if $q > q_k$. The purpose of the following two lemmas is to find an upper bound on $|\epsilon_q|$, given an upper bound on $|q - q_k|$.

LEMMA 3.4. *Let* $m \in \mathbb{N}$, $k \geq 4$. *If* $|q - q_k| < q_k^{-(m+1)k-3}$ *then there is some sequence* $(c_j)_{j=1}^\infty \in \{0,1\}^{\mathbb{N}}$ *such that* $1 = \pi_q((c_j)_{j=1}^\infty)$ *and* $(c_j)_{j=1}^\infty = (1^{k-1}0)^m(c_j)_{j=mk+1}^\infty$.

Proof. Let $m \in \mathbb{N}$, $k \geq 4$. Let $q_- \in (1,2)$ be such that $1 = \pi_{q_-}((1^{k-1}0)^m 0^\infty)$ and let $q_+ \in (1,2)$ be such that $1 = \pi_{q_+}((1^{k-1}0)^m 1^\infty)$. By Lemma 3.3, $q \in (q_-, q_+)$ is equivalent to

$$\pi_q((1^{k-1}0)^m 0^\infty) < \pi_{q_-}((1^{k-1}0)^m 0^\infty) = 1 = \pi_{q_+}((1^{k-1}0)^m 1^\infty) < \pi_q((1^{k-1}0)^m 1^\infty). \tag{7}$$

This means that

$$1 \in (\pi_q((1^{k-1}0)^m 0^\infty), \pi_q((1^{k-1}0)^m 1^\infty)),$$

that is, there must exist some sequence $(c_j)_{j=1}^\infty \in \{0,1\}^{\mathbb{N}}$ such that $1 = \pi_q((c_j)_{j=1}^\infty)$ where $(c_j)_{j=1}^\infty = (1^{k-1}0)^m(c_j)_{j=km+1}^\infty$. Therefore, to prove the lemma, it suffices to show that if $|q - q_k| < q_k^{-(m+1)k-3}$ then $q \in (q_-, q_+)$.

If $q = q_k$ then we know that $1 = \pi_q((1^{k-1}0)^\infty)$, which immediately gives $(c_j)_{j=1}^\infty = (1^{k-1}0)^\infty$ and also $q_k \in (q_-, q_+)$. Assume that $m \in \mathbb{N}$, $k \geq 4$ and $|q - q_k| < q_k^{-(m+1)k-3}$. Since $1 = \pi_{q_k}((1^{k-1}0)^\infty)$, expression (7) becomes

$$\pi_q((1^{k-1}0)^m 0^\infty) < \pi_{q_k}((1^{k-1}0)^\infty) < \pi_q((1^{k-1}0)^m 1^\infty), \tag{8}$$

and so this is also equivalent to $q \in (q_-, q_+)$. If $q < q_k$ then, by exchanging $\pi_q((1^{k-1}0)^m 1^\infty)$ for the smaller expression $\pi_q((1^{k-1}0)^\infty)$, the second inequality of (8) is implied by

$$\pi_{q_k}((1^{k-1}0)^\infty) < \pi_q((1^{k-1}0)^\infty),$$

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which is itself a direct consequence of Lemma 3.3. If $q > q_k$ then the first inequality of (8) holds by similar reasoning. Therefore, it suffices to show

(1) if $q < q_k$ then $\pi_q((1^{k-1}0)^m0^\infty) < \pi_{q_k}((1^{k-1}0)^\infty)$,

and

(2) if $q > q_k$ then $\pi_{q_k}((1^{k-1}0)^\infty) < \pi_q((1^{k-1}0)^m1^\infty)$.

Case 1. Let $q < q_k$. Using the fact that $\pi_q((1^{k-1}0)^m0^\infty) = (1-q^{-km})\pi_q((1^{k-1}0)^\infty)$, the inequality $\pi_q((1^{k-1}0)^m0^\infty) < \pi_{q_k}((1^{k-1}0)^\infty)$ becomes

$$\begin{aligned} q^{-km}\pi_q((1^{k-1}0)^\infty) &> \pi_q((1^{k-1}0)^\infty) - \pi_{q_k}((1^{k-1}0)^\infty), \\ &= \left(\frac{1}{q-1} - \frac{1}{q^k-1}\right) - \left(\frac{1}{q_k-1} - \frac{1}{q_k^k-1}\right) \\ &= \left(\frac{1}{q-1} - \frac{1}{q_k-1}\right) - \left(\frac{1}{q^k-1} - \frac{1}{q_k^k-1}\right). \end{aligned} \tag{9}$$

We show that this inequality is a consequence of our assumption that $|q-q_k| < q_k^{-(m+1)k-3}$. The second term in parentheses in (9) is positive and the first term in parentheses equals $(q_k-q)/((q-1)(q_k-1))$. Using Lemma 3.3 again, $\pi_q((1^{k-1}0)^\infty) > \pi_{q_k}((1^{k-1}0)^\infty) = 1$, so (9) is implied by

$$q^{-km} > \frac{q_k-q}{(q-1)(q_k-1)}. \tag{10}$$

With $k \geq 4$, we know that $q > 15/8$ (since $q_4 \approx 1.9276$ (by direct computation), the smallest possible value of q is $q_4 - q_4^{-11}$, so this is a very loose lower bound), so $(q-1)(q_k-1) > 49/64 > q^{-1}$, and hence (10) is implied by

$$q_k - q < q_k^{-km-1},$$

which is definitely true if $q_k - q < q_k^{-(m+1)k-3}$. This completes the first case.

Case 2. Let $q > q_k$. We have the following chain of equivalences from the inequality we set out to prove:

$$\begin{aligned} & \pi_{q_k}((1^{k-1}0)^\infty) < \pi_q((1^{k-1}0)^m1^\infty) \\ \Leftrightarrow \quad & \pi_{q_k}((1^{k-1}0)^\infty) - \pi_q((1^{k-1}0)^\infty) < q^{-km}\pi_q((0^{k-1}1)^\infty) \\ \Leftrightarrow \quad & \left(\frac{1}{q_k-1} - \frac{1}{q-1}\right) - \left(\frac{1}{q_k^k-1} - \frac{1}{q^k-1}\right) < q^{-km}\frac{1}{q^k-1}. \end{aligned} \tag{11}$$

The first equivalence makes use of the equalities

$$\begin{aligned} \pi_q((1^{k-1}0)^m1^\infty) &= \pi_q((1^{k-1}0)^\infty) + \pi_q(0^{km}(0^{k-1}1)^\infty) \\ &= \pi_q((1^{k-1}0)^\infty) + q^{-km}\pi_q((0^{k-1}1)^\infty), \end{aligned}$$

while the second is simply rewriting the π_q expressions using the standard formulas for geometric series, and rearranging them. As in the previous case, the second term in parentheses on the left-hand side of (11) is positive, which means this inequality is implied

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by

$$\frac{q-q_k}{(q_k-1)(q-1)} < \frac{q^{-km}}{q^k-1},$$

which we rearrange to

$$q-q_k < q^{-km}\frac{(q_k-1)(q-1)}{q^k-1}.$$

Again, we aim to show this is a consequence of our assumptions on q. Using $k \geq 4$, it can be checked that $q^{-km} > q_k^{-km-1}$ and $((q_k-1)(q-1))/(q^k-1) > q_k^{-k-2}$, so the above inequality is implied by

$$q-q_k < q^{-(m+1)k-3},$$

which is obviously a restriction of $|q-q_k| < q^{-(m+1)k-3}$ to the case $q > q_k$. Therefore, the proof of the second case is complete. Hence, we have proved the conclusion in both cases and the lemma holds. □

Recall the definition of ϵ_q from (6) above.

LEMMA 3.5. *Let $m \in \mathbb{N}$, $k \geq 4$ and $|q-q_k| < q_k^{-(m+2)k-3}$. Then $-q_k^{-(m+1)k+1} < \epsilon_q < q_k^{-(m+2)k+1}$.*

Proof. Assuming the hypotheses of the lemma, let $(c_j)_{j=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ be of the form $(1^{k-1}0)^{m+1}(c_j)_{j=k(m+1)+1}^{\infty}$ where $\pi_q((c_j)_{j=1}^{\infty}) = 1$, which we know exists by Lemma 3.4. Then,

$$\pi_q((1^{k-1}0)^{m+1}0^{\infty}) \leq \pi_q((1^{k-1}0)^{\infty}) + \epsilon_q \leq \pi_q((1^{k-1}0)^{m+1}1^{\infty}). \tag{12}$$

If $q = q_k$ then $\epsilon_q = 0$ so we consider the remaining two cases separately.

Case 1: $q < q_k$. If $q < q_k$ then $\epsilon_q < 0$ and the first part of (12) is equivalent to

$$\epsilon_q \geq (1-q^{-k(m+1)})\pi_q((1^{k-1}0)^{\infty}) - \pi_q((1^{k-1}0)^{\infty}) = -q^{-k(m+1)}(1-\epsilon_q),$$

so

$$\epsilon_q \geq -\frac{q^{-k(m+1)}}{1-q^{-k(m+1)}}.$$

From here, using $k \geq 4$, it can be checked that $\epsilon_q > -q_k^{-k(m+1)+1}$. This is intuitive given the factor $1/(1-q^{-km})$ is very close to 1 and q is very close to q_k.

Case 2: $q > q_k$. If $q > q_k$ then $\epsilon_q > 0$ and the second part of (12) is equivalent to

$$\pi_q((1^{k-1}0)^{\infty}) + \epsilon_q \leq \pi_q((1^{k-1}0)^{m+1}1^{\infty}),$$

which is equivalent to

$$\epsilon_q \leq \frac{q^{-k(m+1)}}{q^k-1}.$$

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It can be checked that this implies that

$$\epsilon_q < q_k^{-(m+2)k+1}.$$

Combining the two cases, we see that $-q_k^{-(m+1)k+1} < \epsilon_q < q_k^{-(m+2)k+1}$. □

For $q \in (1, 2)$ and $k \geq 2$, the map $g_{q,k} = f_1^{-(k-1)} \circ f_0^{-1}$ has a natural symbolic interpretation. Let $(\delta_j)_{j=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ be a base q expansion for some point $y \in I_q$. Then an application of Lemma 2.4 is that

$$g_{q,k}(\pi_q((\delta_j)_{j=1}^{\infty})) = \pi_q((1^{k-1}0)(\delta_j)_{j=1}^{\infty}). \tag{13}$$

That is, the point $g_{q,k}(y)$ has a base q expansion given by $(1^{k-1}0)(\delta_j)_{j=1}^{\infty}$ where $(\delta_j)_{j=1}^{\infty}$ is a base q expansion of y. In other words, $g_{q,k}$ prefixes any base q expansion of y by $(1^{k-1}0)$. By inspection $g_{q,k}$ is an affine transformation whose linear part is a contraction by q^k and whose translation is a shift of $\pi_q(1^{k-1}0^{\infty})$. It is straightforward to check that the fixed point of $g_{q,k}$ is $\pi_q((1^{k-1}0)^{\infty})$. For any $k \geq 2$ and any x such that $\pi_q((1^{k-1}0)^{\infty}) + x \in I_q$,

$$g_{q,k}(\pi_q((1^{k-1}0)^{\infty}) + x) = \pi_q((1^{k-1}0)^{\infty}) + q^{-k}x. \tag{14}$$

We will also use the fact that, since $g_{q,k}$ is an affine transformation,

$$g_{q,k}(x) - g_{q,k}(z) = q^{-k}(x - z) \tag{15}$$

whenever $x, z \in I_q$.

The following lemmas allow us to determine the interval $B(q)$ as q varies relative to q_k. The argument provides useful bounds for the diameters of the $P_i(q)$ and $Q_m(q)$ sets which are needed in order to bound β_q below in §3.4. We denote by $L(X)$ and $R(X)$ the left and right endpoints of a compact subset $X \subset \mathbb{R}$. Figure 3 provides a useful reference for Lemma 3.7.

LEMMA 3.6. *For any $m \in \mathbb{N}$, $k \geq 3$ and $q > q_{k-1}$,*

$$|P_0(q)| = \cdots = |P_m(q)| = |Q_m(q)| = q^{-(m+1)k+2}\pi_q((1^{k-2}0)^{\infty}).$$

Proof. Recall that for $k \geq 3$, $q > q_{k-1}$,

$$G_i(q) = g_{q,k}^i(q^{-(m-i)k}G(q) + 1),$$

where

$$G(q) = (\pi_q(0^{k-3}(01^{k-2})^{\infty}), \pi_q(0^{k-3}(10^{k-2})^{\infty})),$$

and that $P_i(q)$ is the subset of $g_{q,k}^i(\pi_q(S_{k-1}) + 1)$ to the left of $G_i(q)$. So $P_i(q)$ has the same left endpoint as $g_{q,k}^i(\pi_q(S_{k-1}) + 1)$, namely $g_{q,k}^i(\pi_q(0^{\infty}) + 1) = g_{q,k}^i(1)$. Observe that, using (15), for any $0 \leq i \leq m$,

$$\begin{aligned}
|P_i(q)| &= L(G_i(q)) - L(P_i(q)) \\
&= g_{q,k}^i(\pi_q(0^{(m-i)k}0^{k-3}(01^{k-2})^{\infty}) + 1) - g_{q,k}^i(1) \\
&= q^{-km}(\pi_q(0^{k-3}(01^{k-2})^{\infty})) \\
&= q^{-(m+1)k+2}\pi_q((1^{k-2}0)^{\infty}),
\end{aligned}$$

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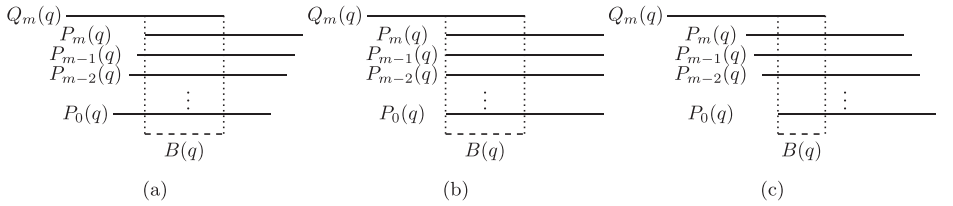


FIGURE 3. The relative structure of the convex hulls of the sets $P_0(q), \dots P_m(q)$, $Q_m(q)$ and $B(q)$ when $|q - q_k| < q_k^{-(m+2)k-3}$, as is proved in Lemma 3.7. (a) $q < q_k$. (b) $q = q_k$. (c) $q > q_k$.

which we note is independent of i. Recall also that

$$H_m(q) = g_{q,k}^m(H(q)),$$

where

$$H(q) = (\pi_q(1^{k-3}(01^{k-2})^\infty), \pi_q(1^{k-3}(10^{k-2})^\infty)),$$

and that $Q_m(q)$ is the subset of $g_{q,k}^m(\pi_q(S_{k-1}))$ to the right of the gap $H_m(q)$. Using (15) again, we see that

$$\begin{aligned} |Q_m(q)| &= R(Q_m(q)) - R(H_m(q)) \\ &= g_{q,k}^m(\pi_q(1^\infty)) - g_{q,k}^m(\pi_q(1^{k-3}(10^{k-2})^\infty)) \\ &= q^{-km}(\pi_q(0^{k-3}(01^{k-2})^\infty)) \\ &= |P_i(q)|, \end{aligned}$$

for all $0 \le i \le m$. □

By Lemma 3.6, we know that $|P_0(q)| = \cdots = |P_m(q)| = |Q_m(q)|$. We denote this value by D_q.

Recall that $L(X)$ and $R(X)$ denote the left and right endpoints respectively of a compact set $X \subset \mathbb{R}$.

LEMMA 3.7. *Let $m \in \mathbb{N}$, $k \ge 4$, and let $|q - q_k| < q_k^{-(m+2)k-3}$.*

(a) *If $q < q_k$ then*

$$\begin{aligned} &L(Q_m(q)) < L(P_0(q)) < \cdots < L(P_m(q)) \\ &\quad < R(Q_m(q)) < R(P_0(q)) < \cdots < R(P_m(q)). \end{aligned}$$

(b) *If $q = q_k$ then*

$$\begin{aligned} &L(Q_m(q)) < L(P_0(q)) = \cdots = L(P_m(q)) \\ &\quad < R(Q_m(q)) < R(P_0(q)) = \cdots = R(P_m(q)). \end{aligned}$$

(c) *If $q > q_k$ then*

$$\begin{aligned} &L(Q_m(q)) < L(P_m(q)) < \cdots < L(P_0(q)) \\ &\quad < R(Q_m(q)) < R(P_m(q)) < \cdots < R(P_0(q)). \end{aligned}$$

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Proof. The proof of this lemma is calculation heavy so we take a moment to separate the proof into the stages the argument takes. Each step of the argument handles a different part of the long inequality to be proven, but will handle all three of the cases $q < q_k$, $q = q_k$ and $q > q_k$ together. Suppose $m \in \mathbb{N}$, $k \geq 4$, $|q - q_k| < q_k^{-(m+2)k-3}$. The inequalities we aim to prove in the cases $q < q_k$, $q = q_k$ and $q > q_k$ are respectively

$$\begin{aligned} &L(Q_m(q)) \underbrace{<}_{\text{A}} L(P_0(q)) \underbrace{< \cdots <}_{\text{B}} L(P_m(q)) \\ &\underbrace{< R(Q_m(q)) <}_{\text{C}} R(P_0(q)) \underbrace{< \cdots <}_{\text{D}} R(P_m(q)), \end{aligned}$$

$$\begin{aligned} &L(Q_m(q)) \underbrace{<}_{\text{A}} L(P_0(q)) \underbrace{= \cdots =}_{\text{B}} L(P_m(q)) \\ &\underbrace{< R(Q_m(q)) <}_{\text{C}} R(P_0(q)) \underbrace{= \cdots =}_{\text{D}} R(P_m(q)), \end{aligned}$$

$$\begin{aligned} &L(Q_m(q)) \underbrace{<}_{\text{A}} L(P_m(q)) \underbrace{< \cdots <}_{\text{B}} L(P_0(q)) \\ &\underbrace{< R(Q_m(q)) <}_{\text{C}} R(P_m(q)) \underbrace{< \cdots <}_{\text{D}} R(P_0(q)). \end{aligned}$$

We start by proving all relations (B) by finding a general expression for $L(P_i(q))$. Lemma 3.6 tells us that $|P_i(q)|$ takes the common value D_q for all i and so the relations (D) are implied by the relations (B). It also tells us that $|Q_m(q)|$ shares this common value, so the relations (A) follow from the relations (C). Therefore, it suffices to prove the relations (B) and (C).

Relations (B). As in the proof of Lemma 3.6, $L(P_i(q))$ is the left endpoint of $g_{q,k}^i(\pi_q(S_{k-1}) + 1)$. Since $0 \in \pi_q(S_{k-1})$ is the smallest element of this set, we see, using (14), that

$$L(P_i(q)) = g_{q,k}^i(1) = g_{q,k}^i(\pi_q((1^{k-1}0)^\infty) + \epsilon_q) = \pi_q((1^{k-1}0)^\infty) + q^{-ki}\epsilon_q. \tag{16}$$

By (16) we have the following implications. If $q < q_k$ then $\epsilon_q < 0$ and

$$L(P_0(q)) < \cdots < L(P_m(q)).$$

If $q = q_k$ then $\epsilon_q = 0$ and

$$L(P_0(q)) = \cdots = L(P_m(q)).$$

If $q > q_k$ then $\epsilon_q > 0$ and

$$L(P_m(q)) < \cdots < L(P_0(q)).$$

Relations (C). Given $|q - q_k| < q_k^{-(m+2)k-3}$ we show that

(1) if $q < q_k$ then $L(P_m(q)) < R(Q_m(q)) < R(P_0(q))$,

and

(2) if $q \geq q_k$ then $L(P_0(q)) < R(Q_m(q)) < R(P_m(q))$.

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$R(Q_m(q))$ is the right endpoint of the set $g_{q,k}^m(\pi_q(S_{k-1}))$, so

$$\begin{aligned} R(Q_m(q)) &= g_{q,k}^m(\pi_q(1^\infty)) \\ &= g_{q,k}^m(\pi_q((1^{k-1}0)^\infty) + \pi_q((0^{k-1}1)^\infty)) \\ &= \pi_q((1^{k-1}0)^\infty) + q^{-km}\pi_q((0^{k-1}1)^\infty) \\ &= \pi_q((1^{k-1}0)^\infty) + \frac{q^{-km}}{q^k-1}. \end{aligned} \tag{17}$$

To find values for $R(P_i(q))$ for $0 \le i \le m$, we use $R(P_i(q)) = L(P_i(q)) + D_q$. By (16) and Lemma 3.6 this gives

$$R(P_i(q)) = \pi_q((1^{k-1}0)^\infty) + q^{-ki}\epsilon_q + q^{-(m+1)k+2}\left(\frac{1}{q-1} - \frac{1}{q^{k-1}-1}\right). \tag{18}$$

In the first case, when $q < q_k$ and $\epsilon_q < 0$, by (16) and (17),

$$R(Q_m(q)) - L(P_m(q)) = \frac{q^{-km}}{q^k-1} - q^{-km}\epsilon_q > q^{-(m+1)k} > 0. \tag{19}$$

The second inequality is only valid when $q < q_k$, but we only use this bound under this premise. Also, by (17) and (18),

$$\begin{aligned} R(P_0(q)) - R(Q_m(q)) &= \left(\epsilon_q + q^{-(m+1)k+2}\left(\frac{1}{q-1} - \frac{1}{q^{k-1}-1}\right)\right) - q^{-km}\left(\frac{1}{q^k-1}\right) \\ &= \epsilon_q + q^{-km}\left(q^{-k+2}\left(\frac{1}{q-1} - \frac{1}{q^{k-1}-1}\right) - \frac{1}{q^k-1}\right). \end{aligned}$$

Since $\epsilon_q > -q_k^{-(m+1)k+1}$ whenever $|q - q_k| < q_k^{-(m+2)k-3}$, according to Lemma 3.5, by factoring out q^{-km}, the above expression is positive if

$$\left(q^{-k+2}\left(\frac{1}{q-1} - \frac{1}{q^{k-1}-1}\right) - \frac{1}{q^k-1}\right) - q^{-k+1} > 0. \tag{20}$$

This inequality is made clear by interpreting the terms as projections of sequences in base q. In this way, the left-hand side of (20) is equal to

$$\begin{aligned} &\pi_q(0^{k-2}(1^{k-2}0)^\infty) - \pi_q((0^{k-1}1)^\infty) - \pi_q(0^{k-2}10^\infty) \\ &\quad = \pi_q(0^{k-1}1^{k-3}0(1^{k-2}0)^\infty) - \pi_q((0^{k-1}1)^\infty). \end{aligned}$$

Thus, to verify (20) it suffices to show that

$$\pi_q(0^{k-1}1^{k-3}0(1^{k-2}0)^\infty) - \pi_q((0^{k-1}1)^\infty) > 0,$$

or equivalently

$$\pi_q(1^{k-3}0(1^{k-2}0)^\infty) - \pi_q((10^{k-1})^\infty) > 0. \tag{21}$$

When $k = 4$, inequality (21) is a consequence of the inequality

$$q^{-1} + q^{-3} + q^{-4} > q^{-1} + \sum_{l=5}^{\infty} q^{-l},$$

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which holds whenever $q > q_2$ and in particular when $|q - q_4| < q_4^{-(m+2)4-3}$. When $k \geq 5$, inequality (21) is a consequence of the inequality

$$q^{-1} + q^{-2} > q^{-1} + \sum_{l=6}^{\infty} q^{-l},$$

which holds whenever $q > q_2$ and in particular when $|q - q_k| < q_k^{-(m+2)k-3}$.

In the second case, when $q \geq q_k$ and $\epsilon_q \geq 0$, using Lemma 3.5 again, we know $\epsilon_q < q_k^{-(m+2)k+1}$, and by (16) and (17) we have

$$\begin{aligned} R(Q_m(q)) - L(P_0(q)) &= \pi_q((1^{k-1}0)^\infty) + q^{-km}\left(\frac{1}{q^k - 1}\right) - (\pi_q((1^{k-1}0)^\infty) + \epsilon_q) \\ &= \frac{q^{-km}}{q^k - 1} - \epsilon_q \\ &> q^{-(m+1)k} - q_k^{-(m+2)k+1} \\ &> q_k^{-(m+1)k-1} \\ &> 0. \end{aligned} \tag{22}$$

The lower bound of $q_k^{-(m+1)k-1}$ from step (22) follows from a routine calculation. Also, by (18) and (17),

$$\begin{aligned} R(P_m(q)) - R(Q_m(q)) &= \left(q^{-km}\epsilon_q + q^{-(m+1)k+2}\left(\frac{1}{q-1} - \frac{1}{q^k - 1}\right)\right) - q^{-km}\left(\frac{1}{q^{k-1} - 1}\right) \\ &> q^{-km}\epsilon_q + q^{-km}\left(q^{-k+2}\left(\frac{1}{q-1} - \frac{1}{q^{k-1} - 1}\right) - \frac{1}{q^k - 1}\right), \end{aligned}$$

and it follows from our analysis of (20) that this is positive. This completes the proof. □

3.4. *Bounding β_q below.* Recall that

$$B(q) = \bigcap_i \operatorname{conv}(P_i(q)) \cap \operatorname{conv}(Q_m(q))$$

and

$$\beta_q = \min\left\{\frac{1}{4}, \frac{|B(q)|}{\max\{|P_0(q)|, \ldots, |P_m(q)|, |Q_m(q)|\}}\right\}.$$

LEMMA 3.8. *If $m \in \mathbb{N}$, $k \geq 2$ and $|q - q_k| < q_k^{-(m+2)k-3}$, then $\beta_q > 1/8$.*

Proof. Let $m \in \mathbb{N}$, $k \geq 2$ and $|q - q_k| < q_k^{-(m+2)k-3}$. Using Lemma 3.7, we know that if $q < q_k$, then $B(q)$ is given by the interval $[L(P_m(q)), R(Q_m(q))]$, and if $q \geq q_k$, then $B(q)$ is given by the interval $[L(P_0(q)), R(Q_m(q))]$. By Lemma 3.6, $\max\{|P_0(q)|, \ldots, |P_m(q)|, |Q_m(q)|\} = D_q$ (their common value), so $\beta_q = \min\{\frac{1}{4}, |B(q)|/D_q\}$.

If $q < q_k$ then $q^{-1} > q_k^{-1}$ and $|B(q)| = R(Q_m(q)) - L(P_m(q))$. Recall that $|P_i(q)| = D_q$ for all $i = 0, \ldots, m$, and it can be checked using Lemma 3.6 that $D_q < q_k^{-(m+1)k+3}$.

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It follows from (19) and the fact that $q^{-(m+1)k} > q_k^{-(m+1)k}$ that $R(Q_m(q)) - L(P_m(q)) > q_k^{-(m+1)k}$ and hence $\beta_q > q_k^{-3} > \frac{1}{8}$.

If $q \geq q_k$ then $q^{-1} \leq q_k^{-1}$ and $|B(q)| = R(Q_m(q)) - L(P_0(q))$. We know from (22) that $R(Q_m(q)) - L(P_0(q)) > q_k^{-(m+1)k-1}$. Using Lemma 3.3, we have $\pi_q((1^{k-2}0)^\infty) \leq \pi_{q_k}((1^{k-2}0)^\infty) < 1$, so Lemma 3.6 implies that $D_q < q_k^{-(m+1)k+2}$, so $\beta_q > q_k^{-3} > \frac{1}{8}$. Hence, if $|q - q_k| < q_k^{-(m+2)k-3}$ then $\beta_q > \frac{1}{8}$. □

3.5. *Proof of Theorem A.* The above results allow us to prove Theorem A.

Proof of Theorem A. Let $m \in \mathbb{N}$, $k \geq K_m$ and $|q - q_k| < q_k^{-(m+2)k-3}$. As discussed in §3.1, to prove Theorem A it suffices to show that the collection of compact subsets of $\mathbb{R}$ given by $\{P_0(q), \ldots P_m(q), Q_m(q)\}$ satisfy the assumptions of Theorem F&Y. We know that $B(q) = \bigcap_{i=1}^m \operatorname{conv}(P_i(q)) \cap Q_m(q)$ is non-empty by Lemma 3.7, and we know that each of $P_0(q), \ldots, P_m(q), Q_m(q)$ has thickness at least q^{k-4} by Lemmas 2.2 and 3.2. By Lemma 3.8 it remains to show that there exists some $c \in (0, 1)$ such that the following inequality holds:

$$(m+2)(q^{k-4})^{-c} \leq \tfrac{1}{(432)^2} 8^{-c}(1 - 8^{c-1}). \tag{23}$$

Given any $c \in (0, 1)$, since $\lim_{k\to\infty}(q^{k-4})^{-c} = 0$, and since the right-hand side of (23) is independent of k, we know there is some value of k for which (23) holds. Hence, we can arbitrarily fix $c = \frac{19}{20}$ and find the smallest value of k in terms of m for which (23) holds. (Some numerical testing showed that $c = \frac{19}{20}$ is a reasonable choice, and the authors do not believe that we can strengthen Theorem A by choosing a different value of c.) Notice that K_m is non-decreasing with m and $K_1 = \lceil \frac{19}{20}(\log_{1.999}(3) + 24) + 4 \rceil = 31$. Therefore, $K_m \geq 31$ for all $m \in \mathbb{N}$ and because $q_{31} > q_{10} > 1.999$ (by direct computation), if $k \geq K_m$, then (23) is implied by

$$(m+2)(1.999)^{(4-k)\frac{19}{20}} \leq \tfrac{1}{(432)^2}(8^{-\frac{19}{20}})(1 - 8^{-\frac{1}{20}}). \tag{24}$$

The right-hand side of (24) is bounded below by 7.3389×10^{-8}. Using $\log_{1.999}(7.3389 \times 10^{-8}) > -24$, a sufficient condition on k to satisfy (24) is

$$k \geq \tfrac{20}{19}(\log_{1.999}(m+2) + 24) + 4,$$

which is obviously true when $k \geq K_m$. This completes the proof. □

4. *Proof of Theorem B*

We first outline the proof of Theorem B.

4.1. *Proof outline.* Using Proposition 2.6, we know that for any $q \in (G, 2)$, if

$$(\mathcal{U}_q + 1) \cap g_{q,k}((\mathcal{U}_q + 1) \cap (\mathcal{U}_q)) \neq \emptyset,$$

then $q \in \mathcal{B}_3$. Let $k \geq 9$. If $|q - q_k| \leq q_k^{-2k-6}$ then it can be checked that $q > q_{k-1}$ and hence $\pi_q(S_{k-1}) \subset \mathcal{U}_q$ by Lemma 2.1(a). For each q such that $|q - q_k| < q_k^{-2k-6}$, we construct a family of sets $A_q \subset \{0, 1\}^{\mathbb{N}}$ such that $\pi_q(A_q) \subset (\mathcal{U}_q + 1) \cap \mathcal{U}_q$ whenever

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$q > q_9$. This means that if

$$(\pi_q(S_{k-1}) + 1) \cap g_{q,k}(\pi_q(A_q)) \neq \emptyset, \tag{25}$$

then $q \in \mathcal{B}_3$. Hence, to prove Theorem B it suffices to prove that (25) holds when either

(a) $k \geq 10$ and $|q - q_k| \leq q_k^{-2k-6}$

or

(b) $0 < q - q_9 \leq q_9^{-24}$.

Since (25) concerns an intersection of only two sets, we can employ Theorem N to conclude their intersection is non-empty. To do this, we require the sets in question to be interleaved and for the product of their thicknesses to be at least 1. By Lemma 2.2 we know that $\tau(\pi_q(S_{k-1})) > q^{k-4}$ when $k \geq 4$ and $q > q_{k-1}$, and it has been shown in previous work by the authors [3, Proposition 3.13] that the set A_q we will construct satisfies

$$\tau(\pi_q(A_q)) > q^{-5}, \tag{26}$$

when $q > q_9$. Recall by the definition of $g_{q,k} = f_1^{-(k-1)} \circ f_0^{-1}$ that $g_{q,k}$ is an affine map and hence preserves thickness, so

$$\tau(\pi_q(S_{k-1}) + 1) \times \tau(g_{q,k}(\pi_q(A_q))) > q^{k-4} \times q^{-5} \geq 1,$$

whenever $k \geq 9$, $q > q_9$. Therefore, the thickness condition of Theorem N is satisfied for both cases (a) and (b) of Theorem B, and it only remains to prove that the sets in (25) are interleaved for q in the required range. We do this using the notion of ϵ-strong interleaving, introduced in §2.1.

Let $k \geq 9$, $|q - q_k| \leq q_k^{-2k-6}$. We show the following assertions.

(1) The sets $(\pi_{q_k}(S_{k-1}) + 1)$ and $g_{q_k,k}(\pi_{q_k}(A_{q_k}))$ are q_k^{-2k-4}-strongly interleaved.

(2) Both $d_{\mathrm{H}}((\pi_{q_k}(S_{k-1}) + 1), (\pi_q(S_{k-1}) + 1))$ and $d_{\mathrm{H}}(g_{q_k,k}(\pi_{q_k}(A_{q_k})), g_{q,k}(\pi_q(A_q)))$ are less than q_k^{-2k-4}.

By the definition of ϵ-strong interleaving, these results tell us that $(\pi_q(S_{k-1}) + 1)$ and $g_{q,k}(\pi_q(A_q))$ are interleaved. To prove the first item, we will find four points in $(\pi_{q_k}(S_{k-1}) + 1) \cap g_{q_k,k}(\pi_{q_k}(A_{q_k}))$ that are sufficiently far apart and then apply Lemma 2.3. Since the thickness condition (26) only holds when $q > q_9$, the complete argument only applies in the smaller domain where either (a) or (b) is true. Hence, Theorem N applies to (25) when either (a) or (b) is true and the proof is complete.

4.2. *Construction of A_q.* In this subsection we construct the set A_q which will have the important property that for $q > q_9$, $\pi_q(A_q) \subset \mathcal{U}_q \cap (\mathcal{U}_q + 1)$ and $\tau(\pi_q(A_q)) > q^{-5}$. Define the set

$$W_2 = \{(-10), (0\,{-1}), (00), (01), (10)\},$$

and write $W_2^{\mathbb{N}}$ for the set of infinite concatenations of elements of W_2. For any $k \in \mathbb{N}$ and $(\delta_j)_{j=1}^k \in \{-1, 0, 1\}^k$, define the notation

$$(\delta_j)_{j=1}^k W_2^{\mathbb{N}} = \{(\epsilon_j)_{j=1}^\infty \in \{-1, 0, 1\}^{\mathbb{N}} : (\epsilon_j)_{j=1}^k = (\delta_j)_{j=1}^k \text{ and } (\epsilon_j)_{j=k+1}^\infty \in W_2^{\mathbb{N}}\}.$$

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LEMMA 4.1. *If $k \geq 9$ and $|q - q_k| \leq q_k^{-2k-6}$ then there exists a sequence $(c_j)_{j=1}^{\infty} \in 1^k 0^{k+4} W_2^{\mathbb{N}}$ such that $1 = \pi_q((c_j)_{j=1}^{\infty})$.*

In words, this lemma is claiming the existence of an expansion of 1 in base q which agrees with the expansion of 1 in base q_k, given by $1^k 0^{\infty}$, for the first $2k+4$ digits, and, moreover, this expansion has a tail in $W_2^{\mathbb{N}}$. We will see that this lemma is required to show that the sets $\pi_q(A_q)$ and $\pi_{q_k}(A_{q_k})$ remain sufficiently close in the Hausdorff metric when $|q - q_k| < q_k^{-2k-6}$. The intuition here is that since the sets A_q will be shown to depend upon a given fixed expansion of 1 in base q, we require the expansions of 1 themselves to be nearby in the sense that they agree on a large initial segment.

Proof. For all $q \in (1, 2)$, define H_q to be the interval $[\pi_q((-10)^{\infty}), \pi_q((10)^{\infty})]$. We want to show that if $k \geq 9$ and $|q - q_k| < q_k^{-2k-6}$ then 1 satisfies

$$f_0^{k+4} \circ f_1^k(1) \in H_q. \tag{27}$$

It was shown in [**3**, Lemma 3.8] that if $x \in H_q$ then x admits a base q expansion in $W_2^{\mathbb{N}}$. Therefore, if (27) holds, there is a sequence $(\epsilon_j)_{j=1}^{\infty} \in W_2^{\mathbb{N}}$ such that $f_0^{k+4} \circ f_1^k(1) = \pi_q((\epsilon_j)_{j=1}^{\infty})$. By Lemma 2.5, this tells us that $1 = \pi_q(1^k 0^{k+4}(\epsilon_j)_{j=1}^{\infty})$. Hence, we know there is a sequence $(c_j)_{j=1}^{\infty} \in 1^k 0^{k+4} W_2^{\mathbb{N}}$ with $1 = \pi_q((c_j)_{j=1}^{\infty})$, and so the lemma holds. Setting $(c_j)_{j=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ to be any sequence with $1 = \pi_q((c_j)_{j=1}^{\infty})$, we can observe, using Lemma 2.5 again, that (27) is equivalent to

$$\pi_q((c_j)_{j=1}^{\infty} \in [\pi_q(1^k 0^{k+4}(-10)^{\infty}), \pi_q(1^k 0^{k+4}(10)^{\infty})],$$

that is,

$$\pi_q(1^k 0^{k+4}(-10)^{\infty}) \leq \pi_q((c_j)_{j=1}^{\infty}) \leq \pi_q(1^k 0^{k+4}(10)^{\infty}),$$

and we have the equivalent inequalities

$$\pi_q(0^{2k+4}(-10)^{\infty}) \leq \pi_q((c_j)_{j=1}^{\infty}) - \pi_q(1^k 0^{\infty}) \leq \pi_q(0^{2k+4}(10)^{\infty}),$$

$$|\pi_{q_k}(1^k 0^{\infty}) - \pi_q(1^k 0^{\infty})| \leq \pi_q(0^{2k+4}(10)^{\infty}), \tag{28}$$

where we have replaced $\pi_q((c_j)_{j=1}^{\infty})$ with $\pi_{q_k}(1^k 0^{\infty})$ in the last step since both are equal to 1. It can be checked that (28) is equivalent to

$$\left| \left(\frac{1}{q_k - 1} - \frac{1}{q - 1} \right) - \left(\frac{q_k^{-k}}{q_k - 1} - \frac{q^{-k}}{q - 1} \right) \right| \leq q^{-2k-4} \frac{q}{q^2 - 1}. \tag{29}$$

Both terms in parentheses always have the same sign and the latter has smaller magnitude, so we can ignore it and deduce that inequality (29) is implied by

$$\frac{|q - q_k|}{(q - 1)(q_k - 1)} \leq q^{-2k-4} \frac{q}{q^2 - 1}.$$

Since $(q^2/q^2 - 1) > 1$ for all $q \in (1, 2)$, expression (27) is true if

$$|q - q_k| < q^{-2k-5}(q_k - 1)(q - 1).$$

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Using $|q - q_k| < q_k^{-2k-6}$, this is implied by

$$1 < q_k \left(\frac{q_k}{q}\right)^{2k+5} (q_k - 1)(q - 1), \tag{30}$$

and this is what we aim to prove.

Let $k \geq 9$ and $|q - q_k| \leq q_k^{-2k-6}$. If $q_k > q$ then $(q_k/q) > 1$, so (30) is a result of $q_k > 1.9$ and $(q_k - 1)(q - 1) > 0.9$ (which can be easily checked). If $q_k < q$, set $\delta = q - q_k > 0$ and observe that

$$\delta/q < \delta \leq q_k^{-2k-6}. \tag{31}$$

Since $k \geq 9$ we know that $q_k > 1.998$ (by direct computation). Bernoulli's inequality tells us that if $2k + 5 \geq 1$ and $0 \leq \delta/q \leq 1$ (which are true under our assumptions) then

$$\left(\frac{q_k}{q}\right)^{2k+5} = (1 - \delta/q)^{2k+5} > (1 - (2k + 5)q_k^{-2k-6}) > 0.999\,99,$$

where the first inequality makes use of (31) and the final inequality follows from a direct computation estimating $(2k + 5)(1.99)^{-2k-6} > 1.55 \times 10^{-6}$ at $k = 9$, and the fact that $(2k + 5)q_k^{-2k-6}$ is decreasing as k increases. Because $q_k > 1.998$ we also know that

$$(q_k - 1)(q - 1) > (\tfrac{499}{500})^2 > 0.996,$$

so the right-hand side of (30) is bounded below by $1.998 \times 0.999\,99 \times 0.996 > 1$. Therefore, (30) holds when $k \geq 9$ and $|q - q_k| < q_k^{-2k-6}$, which completes the proof. □

Let $k \geq 9$, and let $|q - q_k| \leq q_k^{-2k-6}$. We associate to q a so-called *fixed expansion of* 1, given by $(c_j)_{j=1}^{\infty} \in 1^k 0^{k+4} W_2^{\mathbb{N}}$ and satisfying $\pi_q((c_j)_{j=1}^{\infty}) = 1$, which we know exists by Lemma 4.1. There may be several possible choices for the sequence $(c_j)_{j=1}^{\infty}$; in this case we just pick one arbitrarily. As in [3], we let J be the set of zeros of the sequence $(c_j)_{j=1}^{\infty}$,

$$J = \{j \in \mathbb{N} : c_j = 0\},$$

which we enumerate as follows: $J = \{j_0, j_1, \ldots\}$ where $j_0 < j_1 < \cdots$. Define

$$J_{\text{fixed},1} = \{j_n \in J : n = 4m + 1 \text{ for some } m \in \mathbb{N}\},$$

$$J_{\text{fixed},0} = \{j_n \in J : n = 4m + 3 \text{ for some } m \in \mathbb{N}\}$$

and

$$J_{\text{free}} = \{j_n \in J : n \text{ is even}\}.$$

If $j \in J_{\text{free}}$ then we call j a *free zero* of $(c_j)_{j=1}^{\infty}$. If $j \in J_{\text{fixed},1} \cup J_{\text{fixed},0}$ or if $c_j \in \{-1, 1\}$ then j is called a *fixed index*. Clearly, if j is not a fixed index then it must be a free zero. Let $1^k 0^{k+4} (c_j)_{j=2k+5}^{\infty} \in 1^k 0^{k+4} W_2^{\mathbb{N}}$ be the fixed expansion of 1 associated to q. The set A_q consists of the sequences $(a_j)_{j=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ which satisfy the following properties:

(1) $a_j = 1$ if $c_j = 1$;
(2) $a_j = 0$ if $c_j = -1$;

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(3) $a_j = 1$ if $j \in J_{\text{fixed},1}$;
(4) $a_j = 0$ if $j \in J_{\text{fixed},0}$.

Notice that there are no restrictions on the value of a_j if $j \in J_{\text{free}}$. Notice also that A_q depends on $(c_j)_{j=1}^{\infty}$ and recall that this sequence was itself an arbitrary choice. The following lemma is an immediate consequence of [3].

LEMMA 4.2. *If $q \in (q_9, 2)$ then $\pi_q(A_q) \subset \mathcal{U}_q \cap (\mathcal{U}_q + 1)$.*

Heuristically, this lemma is proved by showing that the strings (01^9) and (10^9) are forbidden in elements of A_q, so $A_q \subset S_9$. Then, if $q \in (q_9, 2)$, we know that $\pi_q(S_9) \subset \mathcal{U}_q$ and hence $\pi_q(A_q) \subset \mathcal{U}_q$. For any $(a_j)_{j=1}^{\infty} \in A_q$, let $(b_j)_{j=1}^{\infty} = (a_j - c_j)_{j=1}^{\infty}$; then $\pi_q((b_j)_{j=1}^{\infty}) = \pi_q((a_j)_{j=1}^{\infty}) - 1$. With this, the containment of $\pi_q(A_q)$ in $(\mathcal{U}_q + 1)$ is a result of $(b_j)_{j=1}^{\infty}$ also being contained in S_9.

As mentioned in the outline in §4.1, the authors previously proved the following lemma [3, Proposition 3.13].

LEMMA 4.3. *If $q > q_9$ then $\tau(\pi_q(A_q)) > q^{-5}$.*

4.3. *Hausdorff distances and interleaving.* Using Lemmas 4.2, and 2.1(a), the following lemma is an immediate corollary to Proposition 2.6.

LEMMA 4.4. *If $k \geq 10$, $q > q_{k-1}$ and the intersection*

$$(\pi_q(S_{k-1}) + 1) \cap g_{q,k}(\pi_q(A_q))$$

is non-empty then $q \in \mathcal{B}_3$.

We prove the following result on ϵ-strong interleaving with the intention of applying Lemma 2.3.

LEMMA 4.5. *If $k \geq 9$ then the sets $(\pi_{q_k}(S_{k-1}) + 1)$ and $g_{q_k,k}(\pi_{q_k}(A_{q_k}))$ are q_k^{-2k-4}-strongly interleaved.*

Proof. Let $k \geq 9$. By the definition of q_k, $1 = \pi_{q_k}(1^k 0^{\infty})$ and we label $(c_{k,j})_{j=1}^{\infty} = 1^k 0^{\infty}$. By the definition of A_{q_k} when associated with the fixed expansion of 1 given by $(c_{k,j})_{j=1}^{\infty}$, we know that for any $m \in \mathbb{N}_{\geq 0}$, the indices $k + (2m + 1)$ are free 0s, the indices $k + (4m + 2)$ are fixed 1s, and the indices $k + (4m + 4)$ are fixed 0s. Using this, we construct four points in $\pi_{q_k}(A_{q_k})$, with the intention of applying Lemma 2.3.

Consider the points $a^{(i)} = \pi_q((a_j^i)_{j=1}^{\infty})$ where $i \in \{1, 2, 3, 4\}$ and $a^{(1)} < a^{(2)} < a^{(3)} < a^{(4)}$ are given by

$$a^{(1)} = \pi_{q_k}(1^k(0100)^{\infty}) = q_k^{-k} \frac{1}{q_k^4 - 1}(q_k^2),$$

$$a^{(2)} = \pi_{q_k}(1^k(0110)^{\infty}) = q_k^{-k} \frac{1}{q_k^4 - 1}(q_k + q_k^2),$$

$$a^{(3)} = \pi_{q_k}(1^k(1100)^{\infty}) = q_k^{-k} \frac{1}{q_k^4 - 1}(q_k^2 + q_k^3),$$

$$a^{(4)} = \pi_{q_k}(1^k(1110)^{\infty}) = q_k^{-k} \frac{1}{q_k^4 - 1}(q_k + q_k^2 + q_k^3).$$

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We check that $g_{q_k,k}(a^{(i)}) \in (\pi_{q_k}(S_{k-1}) + 1) \cap g_{q_k,k}(\pi_{q_k}(A_{q_k}))$ for all $i \in \{1, 2, 3, 4\}$. Since $a^{(i)} \in \pi_{q_k}(A_{q_k})$ by construction, it is obvious that $g_{q_k,k}(a^{(i)}) \in g_{q_k,k}(\pi_{q_k}(A_{q_k}))$ for all $i \in \{1, 2, 3, 4\}$. For the other containment, we show that $g_{q_k,k}(a^{(i)}) - 1 \in \pi_{q_k}(S_{k-1})$. Recall that $1 = \pi_{q_k}(1^k 0^\infty)$, and since $q = q_k$, we can replace 01^k with 10^k at any point in the expansion. (This is a consequence of the definition of q_k, which is equivalent to $q_k^{-1} = q_k^{-2} + \cdots + q_k^{-(k+1)}$.) Recall also that (13) tells us that $g_{q,k}$ acts to prefix base q expansions by $(1^{k-1}0)$. This allows us to write

$$\begin{aligned} g_{q_k,k}(a^{(i)}) - 1 &= \pi_{q_k}(1^{k-1}01^k(a_j^{(i)})_{j=k+1}^\infty) - \pi_{q_k}(1^k 0^\infty) \\ &= \pi_{q_k}(1^k 0^k(a_j^{(i)})_{j=k+1}^\infty) - \pi_{q_k}(1^k 0^\infty) \\ &= \pi_{q_k}(0^{2k}(a_j^{(i)})_{j=k+1}^\infty). \end{aligned}$$

For all $i \in \{1, 2, 3, 4\}$, the tail $(a_j^{(i)})_{j=k+1}^\infty$ avoids 0^4 and 1^4 because they are sequences in A_q. This allows us to declare that $0^{2k}(a_j^{(i)})_{j=k+1}^\infty$ is contained in S_{k-1}. Therefore, $g_{q_k,k}(a^{(i)}) \subset (\pi_{q_k}(S_{k-1}) + 1)$ for all $i \in \{1, 2, 3, 4\}$. The four points $g_{q_k,k}(a^{(i)})$ where $i \in \{1, 2, 3, 4\}$ are given by

$$\begin{aligned} g_{q_k,k}(a^{(1)}) &= \pi_{q_k}(1^{k-1}01^k(0100)^\infty), \\ g_{q_k,k}(a^{(2)}) &= \pi_{q_k}(1^{k-1}01^k(0110)^\infty), \\ g_{q_k,k}(a^{(3)}) &= \pi_{q_k}(1^{k-1}01^k(1100)^\infty), \\ g_{q_k,k}(a^{(4)}) &= \pi_{q_k}(1^{k-1}01^k(1110)^\infty), \end{aligned}$$

and we seek the minimum distance between them. Since $a^{(1)} < a^{(2)} < a^{(3)} < a^{(4)}$ and $g_{q,k}$ is increasing, we need only consider the distances $(g_{q,k}(a^{(i+1)}) - g_{q,k}(a^{(i)}))$ for $i \in \{2, 3, 4\}$. Observe that since $q_k^3 - q_k = q_k(q_k^2 - 1) > q_k$, the smallest of

$$\begin{aligned} (a^{(2)} - a^{(1)}) &= q_k^{-k} \frac{1}{q_k^4 - 1} q_k, \\ (a^{(3)} - a^{(2)}) &= q_k^{-k} \frac{1}{q_k^4 - 1} (q_k^3 - q_k), \\ (a^{(4)} - a^{(3)}) &= q_k^{-k} \frac{1}{q_k^4 - 1} q_k \end{aligned}$$

is given by $(q_k^{-k+1})/(q_k^4 - 1)$. The smallest over $i \in \{1, 2, 3, 4\}$ of $g_{q_k,k}(a^{(i+1)}) - g_{q_k,k}(a^{(i)}) = q_k^{-k}(a^{(i+1)} - a^{(i)})$ is then given by $q_k^{-k}(q_k^{-k+1})/(q_k^4 - 1)$. Therefore, by Lemma 2.3, $(\pi_{q_k}(S_{k-1}) + 1)$ and $g_{q_k,k}(\pi_{q_k}(A_{q_k}))$ are ϵ-strongly interleaved where $2\epsilon = (q_k^{-2k+1})/(q_k^4 - 1)$. So

$$\epsilon = \frac{q_k^{-2k+1}}{2(q_k^4 - 1)} = q_k^{-2k-3} \frac{1}{2(1 - q_k^{-4})}.$$

Since $q_k \geq q_9 \geq 1.998 > 2(1 - 2^{-4}) = 1.875 > 2(1 - q_k^{-4})$, we know that

$$1/(2(1 - q_k^{-4})) > (1/q_k),$$

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so $\epsilon > q_k^{-2k-4}$. Hence, we have shown that $(\pi_{q_k}(S_{k-1}) + 1)$ and $g_{q_k,k}(\pi_{q_k}(A_{q_k}))$ are q_k^{-2k-4}-strongly interleaved. □

We use the following lemma as a tool in the proofs of Lemmas 4.7 and 4.8.

LEMMA 4.6. *Let* $q_1, q_2 \in (1, 2)$ *and* $(\epsilon_j)_{j=1}^{\infty} \in \{-1, 0, 1\}^{\mathbb{N}}$. *Then*

$$|\pi_{q_1}((\epsilon_j)_{j=1}^{\infty}) - \pi_{q_2}((\epsilon_j)_{j=1}^{\infty})| \le \frac{|q_1 - q_2|}{(q_1 - 1)(q_2 - 1)}.$$

Proof. By the definition of the projection maps,

$$\begin{aligned} |\pi_{q_1}((\epsilon_j)_{j=1}^{\infty}) - \pi_{q_2}((\epsilon_j)_{j=1}^{\infty})| &= \left|\sum_{j=1}^{\infty} \epsilon_j (q_1^{-j} - q_2^{-j})\right| \le \left|\sum_{j=1}^{\infty} (q_1^{-j} - q_2^{-j})\right| \\ &= \left|\frac{1}{q_1 - 1} - \frac{1}{q_2 - 1}\right| = \frac{|q_1 - q_2|}{(q_1 - 1)(q_2 - 1)}. \end{aligned}$$

□

The following two lemmas provide us with upper bounds on the Hausdorff distances between the sets in question when q is in the required range.

LEMMA 4.7. *If* $k \ge 9$ *and* $|q - q_k| < q_k^{-2k-6}$ *then* $d_{\mathrm{H}}(\pi_q(S_{k-1}), \pi_{q_k}(S_{k-1})) < q_k^{-2k-4}$.

Proof. Let $k \ge 9$, $|q - q_k| < q_k^{-2k-6}$, and let $(s_j)_{j=1}^{\infty} \in S_{k-1}$ be arbitrary so $\pi_{q_k}((s_j)_{j=1}^{\infty}) \in \pi_{q_k}(S_{k-1})$ is arbitrary. Then by Lemma 4.6,

$$\begin{aligned} |\pi_{q_k}((s_j)_{j=1}^{\infty}) - \pi_q((s_j)_{j=1}^{\infty})| &\le \frac{|q - q_k|}{(q_k - 1)(q - 1)} \\ &< \frac{5}{4}|q - q_k| \\ &< \frac{5}{4} q_k^{-2k-6} \\ &< q_k^{-2k-4}. \end{aligned}$$

Since $q_9 \approx 1.998$ the factor of $5/4$ appears as an easy lower bound for $1/((q_k - 1)(q - 1))$. So we have found a point $\pi_q((s_j)_{j=1}^{\infty}) \in \pi_q(S_{k-1})$ such that $|\pi_{q_k}((s_j)_{j=1}^{\infty}) - \pi_q((s_j)_{j=1}^{\infty})| < q_k^{-2k-4}$. This argument also proves the converse, that is, for an arbitrary point $\pi_q((s_j)_{j=1}^{\infty}) \in \pi_q(S_{k-1})$ the point $\pi_{q_k}((s_j)_{j=1}^{\infty}) \in \pi_{q_k}(S_{k-1})$ satisfies $|\pi_{q_k}((s_j)_{j=1}^{\infty}) - \pi_q((s_j)_{j=1}^{\infty})| < q_k^{-2k-4}$. Therefore, the desired upper bound on the Hausdorff distance holds. □

LEMMA 4.8. *If* $k \ge 9$ *and* $|q - q_k| < q_k^{-2k-6}$ *then* $d_{\mathrm{H}}(g_{q,k}(\pi_q(A_q)), g_{q_k,k}(\pi_{q_k}(A_{q_k}))) < q_k^{-2k-4}$.

Proof. Let $k \ge 9$, $|q - q_k| < q_k^{-2k-6}$ and let $(c_j)_{j=1}^{\infty}$ be the fixed expansion of 1 associated to q. Recall that A_q is defined with respect to $(c_j)_{j=1}^{\infty}$. Using Lemma 4.1 and the fact that $(c_j)_{j=1}^{\infty}$ and $1^k 0^{\infty}$ agree on the first $2k + 4$ indices, we know that the free and fixed indices on this initial segment coincide. Therefore, for every $(a_j)_{j=1}^{\infty} \in A_{q_k}$, we can find

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some $(a'_j)_{j=1}^{\infty} \in A_q$ such that $(a_j)_{j=1}^{2k+4} = (a'_j)_{j=1}^{2k+4}$, and vice versa. We show that these sequences satisfy the property that

$$|g_{q_k,k}(\pi_{q_k}((a_j)_{j=1}^{\infty})) - g_{q,k}(\pi_q((a'_j)_{j=1}^{\infty}))| < q_k^{-2k-4}. \tag{32}$$

Notice that $(a_j)_{j=1}^{\infty} \in A_{q_k}$ was arbitrary, so $g_{q_k,k}(\pi_{q_k}((a_j)_{j=1}^{\infty}))$ is an arbitrary element of $g_{q_k,k}(\pi_{q_k}(A_{q_k}))$. Therefore, it suffices to prove that (32) holds since the converse direction, where we first choose an arbitrary point in A_q, proceeds in exactly the same way.

Let $(a_j)_{j=1}^{\infty} \in A_{q_k}$ be arbitrary and pick $(a'_j)_{j=1}^{\infty} \in A_q$ such that $(a_j)_{j=1}^{2k+4} = (a'_j)_{j=1}^{2k+4}$. Then $g_{q_k,k}(\pi_{q_k}((a_j)_{j=1}^{\infty}))$ is given by $\pi_{q_k}(1^{k-1}0(a_j)_{j=1}^{\infty})$ and $g_{q,k}(\pi_q((a'_j)_{j=1}^{\infty}))$ is given by $\pi_q(1^{k-1}0(a'_j)_{j=1}^{\infty})$. Observe that by the triangle inequality,

$$\begin{aligned}|\pi_{q_k}(1^{k-1}0(a_j)_{j=1}^{\infty}) - \pi_q(1^{k-1}0(a'_j)_{j=1}^{\infty})| &\le |\pi_{q_k}(1^{k-1}0(a_j)_{j=1}^{\infty}) - \pi_{q_k}(1^{k-1}0(a'_j)_{j=1}^{\infty})| \\ &+ |\pi_{q_k}(1^{k-1}0(a'_j)_{j=1}^{\infty}) - \pi_q(1^{k-1}0(a'_j)_{j=1}^{\infty})|. \end{aligned} \tag{33}$$

Since $(a_j)_{j=1}^{\infty}$ and $(a'_j)_{j=1}^{\infty}$ agree on their first $2k+4$ entries, we know the sequences $(1^{k-1}0(a_j)_{j=1}^{\infty})$ and $(1^{k-1}0(a'_j)_{j=1}^{\infty})$ agree on their first $3k+4$ entries. Therefore, we can bound the first term as follows:

$$\begin{aligned}|\pi_{q_k}(1^{k-1}0(a_j)_{j=1}^{\infty}) - \pi_{q_k}(1^{k-1}0(a'_j)_{j=1}^{\infty})| &\le q_k^{-3k-4}|\pi_{q_k}((a_j)_{j=2k+5}^{\infty}) - \pi_{q_k}((a'_j)_{j=2k+5}^{\infty})| \\ &\le q_k^{-3k-4}\left(\frac{1}{q_k-1}\right),\end{aligned}$$

where the $1/(q_k-1)$ term appears as an upper bound for the difference between the images of any two sequences under π_{q_k}, that is, the diameter of I_{q_k}. By Lemma 4.6, the second term satisfies

$$|\pi_{q_k}(1^{k-1}0(a'_j)_{j=1}^{\infty}) - \pi_q(1^{k-1}0(a'_j)_{j=1}^{\infty})| \le \frac{|q-q_k|}{(q-1)(q_k-1)}.$$

Since both $1/(q_k-1)$ and $(q_k^{-k+2} + 1/(q-1))$ are easily bounded above by q_k, we can see that the left-hand side of (32) can be bounded as follows:

$$\begin{aligned}|g_{q_k,k}(\pi_{q_k}((a_j)_{j=1}^{\infty})) - g_{q,k}(\pi_q((a'_j)_{j=1}^{\infty}))| &\le q_k^{-3k-4}\left(\frac{1}{q_k-1}\right) + \frac{q_k^{-2k-6}}{(q-1)(q_k-1)} \\ &\le q_k^{-2k-6}\left(\frac{1}{q_k-1}\right)\left(q_k^{-k+2} + \frac{1}{q-1}\right) \\ &\le q_k^{-2k-4}.\end{aligned}$$

So we have proved that (32) holds, which proves the lemma. □

4.4. *Proof of Theorem B.*

Proof of Theorem B. Let $k \ge 9$ and $|q-q_k| < q_k^{-2k-6}$. By Lemma 4.4, it suffices to prove that (25) holds in cases (a) and (b). Using the definition of ϵ-strong interleaving alongside Lemmas 4.5, 4.7 and 4.8, we know that $(\pi_q(S_{k-1}) + 1)$ and $g_{q,k}(\pi_q(A_q))$ are interleaved. If we make the further restriction that $0 < q - q_9 < q_9^{-24}$ when $k = 9$, then by

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Lemmas 2.2 and 4.3 we know that $\tau(\pi_q(S_{k-1})+1)\times\tau(g_{q,k}(\pi_q(A_q)))>q^5\times q^{-5}=1$. Hence, we can apply Theorem N to (25) in both cases (a) and (b) and the proof is complete. □

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