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ON STATIONARITY IN THE ARCH(∞) MODEL

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We continue investigation of the ARCH(∞) model begun in Giraitis, Kokoszka, and Leipus (2000, *Econometric Theory* 16, 3–22). Nonrestrictive conditions for the existence of a strictly stationary solution are established. The paper generalizes the results of Nelson (1990, *Econometric Theory* 6, 318–334) and Bougerol and Picard (1992, *Journal of Econometrics* 52, 115–127) to the ARCH(∞) model.

1. INTRODUCTION

We consider a system of equations

$$X_k = \left(a + \sum_{i=1}^{\infty} b_i X_{k-i}\right)\xi_k, \qquad k \in \mathbb{Z}, \tag{1.1}$$

where $a > 0$, $b_i \geq 0$ for $i \geq 1$, and ξ_k, $k \in \mathbb{Z}$, are independent and identically distributed (i.i.d.) positive random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Any strictly stationary nonnegative solution $X = (X_k, k \in \mathbb{Z})$ to (1.1) is called the ARCH(∞) process.

Such processes naturally arise in modeling the financial data. For instance, consider the well-known GARCH(p,q) process $(r_k, k \in \mathbb{Z})$ defined by equalities

$$r_k = \sigma_k \varepsilon_k, \qquad \sigma_k^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i r_{k-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{k-i}^2, \tag{1.2}$$

where $\alpha_0 > 0$, $\alpha_i \geq 0$, $\beta_i \geq 0$ for $i \geq 1$ and ε_k, $k \in \mathbb{Z}$, are i.i.d. random variables (see Bollerslev, 1986). Setting $\xi_k = \varepsilon_k^2$ and $X_k = r_k^2$ yields

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$$X_k = \sigma_k^2 \xi_k, \qquad \sigma_k^2 = \alpha_0 + \alpha(L)X_k + \beta(L)\sigma_k^2, \tag{1.3}$$

where $\alpha(L) = \alpha_1 L + \cdots + \alpha_q L^q$, $\beta(L) = \beta_1 L + \cdots + \beta_p L^p$, and L stands for the backshift operator ($LZ_k = Z_{k-1}$).

From (1.3) we formally get

$$\sigma_k^2 = (1 - \beta(1))^{-1}\alpha_0 + (1 - \beta(L))^{-1}\alpha(L)X_k,$$

which leads to (1.1) with $a = (1 - \beta(1))^{-1}\alpha_0$ and with the b_i defined by equality $\sum_{i=1}^{\infty} b_i z^i = \alpha(z)/(1 - \beta(z))$. We will show (see Theorem 2.2, which follows) that the existence of a solution to (1.3) implies $\beta(1) < 1$ and that every strictly stationary solution to (1.3) satisfies the associated ARCH(∞) equations.

The ARCH(∞) model (1.1) was introduced by Robinson (1991) in a hypothesis testing context and later studied by Kokoszka and Leipus (2000) in the change-point problem context. Giraitis, Kokoszka, and Leipus (2000) proved that under the condition

$$\mathbf{E}\xi_0 \sum_{i=1}^{\infty} b_i < 1 \tag{1.4}$$

there exists an integrable ARCH(∞) process $X^* = (X_k^*, k \in \mathbb{Z})$ having the following Volterra series representation:

$$X_k^* = a\xi_k \sum_{l=0}^{\infty} \sum_{i_1,\ldots,i_l=1}^{\infty} b_{i_1} b_{i_2} \cdots b_{i_l} \xi_{k-i_1} \xi_{k-i_1-i_2} \cdots \xi_{k-i_1-\cdots-i_l}. \tag{1.5}$$

Condition (1.4) is not necessary for existence of an ARCH(∞) process. For instance, in the GARCH(1,1) case it reduces to

$$\mathbf{E}(\alpha_1 \varepsilon_k^2 + \beta_1) < 1.$$

On the other hand, Nelson (1990) showed that the necessary and sufficient condition for existence of a strictly stationary GARCH(1,1) process is

$$\mathbf{E} \log(\alpha_1 \varepsilon_k^2 + \beta_1) < 0 \tag{1.6}$$

(provided the expectation on the left-hand side is well defined).

This result was extended to the GARCH(p,q) case by Bougerol and Picard (1992), who showed that, under condition $\mathbf{E}\varepsilon_k^2 = 1$, a stationary solution to (1.2) exists if and only if the top Lyapunov exponent γ is strictly negative. Moreover, in this case there exists only one stationary GARCH(p,q) process.

The top Lyapunov exponent is defined by

$$\gamma = \lim_{n\to\infty} n^{-1} \log\|A_1 \cdots A_n\|, \tag{1.7}$$

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where A_k is the following $(p+q-1)\times(p+q-1)$ random matrix:

$$A_k = \begin{pmatrix} \alpha_1\varepsilon_k^2+\beta_1 & \beta_2 & \dots & \beta_{p-1} & \beta_p & \alpha_2 & \dots & \alpha_{q-1} & \alpha_q \\ 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 \\ \varepsilon_k^2 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \tag{1.8}$$

and $\|\cdot\|$ stands for an arbitrary matrix norm. This exponent is difficult to compute, but it is easily estimated by Monte-Carlo simulations using (1.7).

The aim of our paper is to establish the conditions for existence of an ARCH(∞) process similar to those of Bougerol and Picard (1992). In Section 2 we formulate these conditions, Section 3 discusses the question or uniqueness of the ARCH(∞) process, and in Section 4 we give the proofs of our results.

2. STATIONARITY CONDITIONS

Throughout the paper we will use the following representation of solution (1.5):

$$X_k^* = a\xi_k \sum_{n=0}^{\infty} \eta_{kn}, \tag{2.1}$$

where $\eta_{k0} = 1$ and

$$\eta_{kn} = \sum_{l=1}^{n} \sum_{i_1+\dots+i_l=n} b_{i_1} b_{i_2} \cdots b_{i_l} \xi_{k-i_1} \xi_{k-i_1-i_2} \cdots \xi_{k-i_1-\dots-i_l}, \qquad n \geq 1.$$

Setting

$$\eta_k(t) = \sum_{n=0}^{\infty} \eta_{kn} t^n \tag{2.2}$$

we shorten (2.1) to $X_k^* = a\xi_k \eta_k(1)$.

THEOREM 2.1. *If $\eta_0(1) < \infty$ almost surely then X_k^* is the least nonnegative solution to (1.1). Otherwise, there are no finite solutions at all.*

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The next theorem allows us, in the GARCH(p,q) case, to consider the associated ARCH(∞) equations instead of original equations (1.3).

THEOREM 2.2. *If there exists a solution to (1.3) then* $\beta(1) < 1$. *Every strictly stationary solution to (1.3) satisfies the associated* ARCH(∞) *equations.*

Remark 2.1. The necessity of condition $\beta(1) < 1$ for existence of a GARCH (p,q) process was proved in Corollary 2.3 of Bougerol and Picard (1992) under assumption $\mathbf{E}\xi_k < \infty$. In our proof no moment assumptions on ξ_k are needed.

The following theorem provides some necessary conditions for existence of ARCH(∞) processes.

THEOREM 2.3. *If an* ARCH(∞) *process exists, then*

$$\sum_{i=1}^{\infty} b_i < \infty \tag{2.3}$$

and (if $\mathbf{E}\log\xi_k$ *is well defined)*

$$\mathbf{E}\log\xi_k + \log\sum_{i=1}^{\infty} b_i < 0. \tag{2.4}$$

Remark 2.2. In the GARCH(1,1) case condition (2.4) reduces to

$$\mathbf{E}\log\xi_k + \log\frac{\alpha_1}{1-\beta_1} < 0.$$

If $\beta_1 > 0$ this condition is weaker than (1.6). Hence, condition (2.4) is not sufficient, in general.

When no moment assumptions on ξ_k are made, the role of the top Lyapunov exponent is taken by the convergence radius of the random power series (2.2).

THEOREM 2.4. *The convergence radius of power series (2.2) is almost surely equal to some non-random constant R. Hence, the condition* $R > 1$ *is sufficient and the condition* $R \geq 1$ *is necessary for existence of an* ARCH(∞) *process.*

In the GARCH(p,q) case the convergence radius R and the top Lyapunov exponent γ given by (1.7) are related by the equality

$$R = e^{-\gamma}. \tag{2.5}$$

We show this by considering (1.2) as a particular case (corresponding to $u = 1$) of the model

$$X_k = \sigma_k^2\xi_k, \qquad \sigma_k^2 = \alpha_0 + \sum_{i=1}^{q}\alpha_i u^i X_{k-i} + \sum_{i=1}^{p}\beta_i u^i \sigma_{k-i}^2. \tag{2.6}$$

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The top Lyapunov exponent $\gamma(u)$ for model (2.6) is equal to $\lim_{n\to\infty} n^{-1}\log\|A_1(u)\cdots A_n(u)\|$, where, as is easily seen from (1.8), $A_k(u) = uU^{-1}A_kU$ with U being the diagonal matrix with diagonal elements $1, u, \ldots, u^{p-1}, u, \ldots, u^{q-1}$. The limit (1.7) does not depend on the choice of the matrix norm. Therefore, setting $\|C\|' = \|U^{-1}CU\|$, we get

$$\gamma(u) = \log u + \lim_{n\to\infty} n^{-1}\log\|A_1\cdots A_n\|' = \log u + \gamma.$$

On the other hand, the coefficients in the ARCH(∞) representation of (2.6) are equal to $b_i u^i$. Therefore the convergence radius of power series corresponding to this model is Ru^{-1}. Applying our Theorem 2.4 and Theorem 1.3 of Bougerol and Picard (1992), we get the following implications: for all $u > 0$,

(i) $\gamma + \log u < 0$ implies $Ru^{-1} \geq 1$;
(ii) $Ru^{-1} > 1$ implies $\gamma + \log u < 0$.

It is easily seen that (i) and (ii) yield (2.5).

For notational convenience, in what follows we denote $\gamma = -\log R$, although we do not suppose a (1.7)-type definition for γ. We call γ a top Lyapunov exponent that allows us to compare our results with those of Bougerol and Picard (1992).

By Cauchy formula for convergence radius we get

$$\gamma = \limsup \frac{1}{n}\log \eta_{0n}. \tag{2.7}$$

Theorem 2.5, which follows, shows, in particular, that lim sup in (2.7) can be replaced by lim. In this theorem we assume that model (1.1) is *irreducible*, i.e., that the greatest common divisor δ of the set $\{i \geq 1 | b_i > 0\}$ equals 1. It is easily seen that in the case $\delta > 1$ model (1.1) can be decomposed into δ irreducible ARCH(∞) models

$$X_{k\delta+j} = \left(a + \sum_{i=1}^{\infty} b_{i\delta} X_{(k-i)\delta+j}\right)\xi_{k\delta+j}, \qquad j = 0, \ldots, \delta-1,$$

which can be considered separately.

THEOREM 2.5. *Suppose that model (1.1) is irreducible. If* $E|\log \xi_k| < \infty$, *then*

$$\frac{1}{n}\log \eta_{0n} \to \gamma \quad a.s. \quad and \quad \frac{1}{n}\mathbf{E}\log \eta_{0n} \to \gamma \tag{2.8}$$

as $n \to \infty$. *Moreover,*

$$\gamma = \sup_{n\geq 1} \frac{1}{n}\mathbf{E}\log \eta_{0n}. \tag{2.9}$$

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Notice that both relations (2.8) can be used for estimating γ via Monte-Carlo simulations.

By equality (2.9), $\gamma \geq \gamma_n$, where $\gamma_n := n^{-1}\mathbf{E}\log\eta_{0n}$. Therefore, by Theorem 2.4, the condition

$$\gamma_n \leq 0 \tag{2.10}$$

is necessary for the existence of an ARCH(∞) process. The following example shows that (2.10), in some cases, is stronger than (2.4).

Example 2.1.

Let $b_1 = b_2 = 1/2$, $c > 0$, and

$$\mathbf{P}\{\log\xi_0 = c\} = 1 - \mathbf{P}\{\log\xi_0 = -c\} = p.$$

Then

$$\gamma_2 = 2^{-1}\mathbf{E}\log(b_1^2\xi_{-1}\xi_{-2} + b_2\xi_{-2}) = 2^{-1}\mathbf{E}\log(b_1^2\xi_0^2 + b_2\xi_0)$$
$$= \frac{1}{2}\left[p\log\left(\frac{e^{2c}}{4} + \frac{e^c}{2}\right) + (1-p)\log\left(\frac{e^{-2c}}{4} + \frac{e^{-c}}{2}\right)\right].$$

Therefore

$$\gamma_2 \sim 2^{-1}(3p-1)c$$

as $c \to \infty$.

Suppose $p > \frac{1}{3}$. Then, for c sufficiently large, $\gamma_2 > 0$, and therefore, by (2.10), there are no solutions to (1.1). On the other hand,

$$\mathbf{E}\log\xi_0 = (2p-1)c,$$

and therefore the condition (2.4) is satisfied if $p < \frac{1}{2}$.

Theorem 2.5 provides also some sufficient conditions for existence of ARCH(∞) processes. To derive them, notice that η_{kn} satisfies the following recursive equations:

$$\eta_{kn} = \xi_{k-n}\sum_{i=1}^{n} b_i\eta_{k,n-i}. \tag{2.11}$$

Taking $k = 0$, (2.11) can be rewritten in the matrix form

$$\begin{pmatrix} \eta_{0n} \\ \eta_{0,n-1} \\ \eta_{0,n-2} \\ \vdots \end{pmatrix} = \begin{pmatrix} b_1\xi_{-n} & b_2\xi_{-n} & b_3\xi_{-n} & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} \eta_{0,n-1} \\ \eta_{0,n-2} \\ \eta_{0,n-3} \\ \vdots \end{pmatrix} =: B_{-n}\begin{pmatrix} \eta_{0,n-1} \\ \eta_{0,n-2} \\ \eta_{0,n-3} \\ \vdots \end{pmatrix},$$

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where, by definition, $\eta_{0l} = 0$ for $l < 0$. Therefore,

$$\eta_{0n} = [B_{-n} \cdots B_{-1}](1,1), \tag{2.12}$$

where $C(i,j)$ denotes the element of the matrix C in the ith row and jth column.

Fix any $s \geq 1$ and any nonnegative functional $C \mapsto \|C\|$ defined on the set of all nonnegative infinite-dimensional matrices such that $\|C\| \geq C(1,1)$ and $\|CD\| \leq \|C\| \|D\|$. Then from (2.12) we get

$$\eta_{0,ns} \leq \|B_{-ns} \cdots B_{-(n-1)s+1}\| \cdots \|B_{-s} \cdots B_{-1}\|,$$

and therefore

$$\frac{1}{ns} \mathbf{E} \log \eta_{0,ns} \leq \frac{1}{s} \mathbf{E} \log \|B_{-s} \cdots B_{-1}\|. \tag{2.13}$$

Denote the right-hand side of (2.13) by γ'. Then, by Theorem 2.5, $\gamma \leq \gamma'$. Therefore condition $\gamma' < 0$ is sufficient for existence of an ARCH(∞) process.

For instance, setting $s = 1$ and $\|C\| = \sup_{i \geq 1} \sum_{j=1}^{\infty} t^{j-i} C(i,j)$ for some $t > 1$ we get the following sufficient condition:

$$\mathbf{E} \log \max\{t^{-1}, t^{-1} b(t) \xi_0\} < 0$$

or, equivalently,

$$\mathbf{E} \log^{+}(b(t) \xi_0) < \log t,$$

where $b(t) = \sum_{i=1}^{\infty} b_i t^i$.

3. ANALYSIS OF UNIQUENESS

In this section we investigate the uniqueness of the stationary solutions to (1.1). We make the following assumption on the coefficients b_i.

Assumption A. For any strictly stationary sequence $(Z_i, i \geq 1)$ of nonnegative random variables such that almost surely $\sum_{i=1}^{\infty} b_i Z_i < \infty$, the sum

$$S_n = \sum_{i=1}^{\infty} b_{i+n} Z_i$$

tends to 0 almost surely, as $n \to \infty$.

By stationarity, S_n has the same distribution as the random variable $\sum_{i=1}^{\infty} b_{i+n} Z_{i+n} = \sum_{i=n+1}^{\infty} b_i Z_i$, which tends to 0 almost surely. This implies that $S_n \to 0$ in probability and, consequently, some subsequence of S_n tends to 0 almost surely. Therefore Assumption A holds if the sequence $(b_i, i \geq 1)$ is decreasing starting from some i_0.

In the case of the GARCH(p,q) model the coefficients in its ARCH(∞) representation are decreasing starting from $i_0 = \max(p,q)$ (see Bollerslev, 1986), and hence Assumption A is satisfied.

The main result in this section is as follows.

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THEOREM 3.1. *Suppose that the top Lyapunov exponent γ for model (1.1) is strictly negative and the coefficients b_i satisfy Assumption A. Then X^*, given in (1.5), is the unique strictly stationary solution to (1.1).*

Remark 3.1. Even in the GARCH(p,q) case, our theorem is stronger than the uniqueness part of Theorem 1.3 in Bougerol and Picard (1992) (because we do not need any moment assumption on ξ_k). On the other hand, it does not cover the GARCH(p,q) case completely because we do not know whether condition $\gamma < 0$ is necessary for existence of a GARCH(p,q) process when ξ_k is nonintegrable.

Remark 3.2. Equations (1.1) can have infinitely many solutions, if stationarity is not required. Take, e.g., the ARCH(1) case

$$X_k = (a + b_1 X_{k-1})\xi_k. \tag{3.1}$$

The general solution to (3.1) has the form

$$X_k = X_k^* + Cb_1^k \xi_1 \cdots \xi_k \quad \text{if } k \geq 0;$$

$$X_{-k} = X_{-k}^* + C/(b_1^k \xi_0 \xi_{-1} \cdots \xi_{-k+1}) \quad \text{if } k \geq 1,$$

where C is an arbitrary nonnegative random variable. Clearly, the only choice $C = 0$ almost surely leads to the stationarity of X.

If Assumption A is not fulfilled, we do not know whether X^* is the only stationary ARCH(∞) process. Nevertheless, our last theorem shows that it is the only ARCH(∞) process that can arise "in nature," i.e., can be represented as the weak limit of the ARCH(∞) process, starting from some $s > -\infty$.

In what follows we denote $\mathcal{F}_n = \sigma(\xi_k, k \leq n)$.

THEOREM 3.2. *Let $X_{-s}, \ldots, X_0$ be $\mathcal{F}_0$-measurable and $X_1, X_2, \ldots$ be calculated recursively from (1.1) assuming $X_k = 0$ for $k < -s$. If $\eta_0(1) < \infty$ almost surely, then, for all p and $k_1, \ldots, k_p \in \mathbb{Z}$, $(X_{n+k_1}, \ldots, X_{n+k_p})$ converges in distribution to $(X_{k_1}^*, \ldots, X_{k_p}^*)$, as $n \to \infty$.*

We complete this section with the example of the summable sequence $(b_i, i \geq 1)$, which does not satisfy Assumption A.

Example 3.1.

Let

$$b_i = \begin{cases} 1/i & \text{if } i = k^2 \quad \text{for some } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Z_1, Z_2, \ldots$ be independent copies of a random variable Z with the density $p(x) = 1/x^2$, $x \geq 1$.

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It is easy to check that conditions of the three series theorem are satisfied and therefore $\sum_{i=1}^{\infty} b_i Z_i < \infty$ almost surely. Let us show that almost surely

$$\limsup_{n\to\infty} \sum_{i=1}^{\infty} b_{i+n} Z_i \geq 1. \tag{3.2}$$

Set $A = \{\forall k\ \exists i \geq k\ Z_i \geq i\}$. Because, for all $k \geq 1$,

$$\begin{aligned}\mathbf{P}\{\forall i \geq k\ Z_i \leq i\} &= \lim_{n\to\infty} \mathbf{P}\{Z \leq k\}\cdots\mathbf{P}\{Z \leq k+n\} \\ &= \lim_{n\to\infty}\left(1-\frac{1}{k}\right)\cdots\left(1-\frac{1}{k+n}\right) = 0,\end{aligned}$$

we get $\mathbf{P}(A) = 1$. Let us show that (3.2) holds for all $\omega \in A$.

Fix $\omega \in A$ and find a sequence (i_k) such that, for all k, $i_k \geq k$ and $Z_{i_k}(\omega) \geq i_k$. Define the sequences (m_k) and (n_k) by relations $m_k^2 \leq i_k < (m_k+1)^2$ and $n_k = (m_k+2)^2 - i_k$. Clearly, $m_k \to \infty$, $i_k/m_k^2 \to 1$, and $n_k/i_k \to 0$ as $k \to \infty$. Moreover,

$$n_k \geq (m_k+2)^2 - (m_k+1)^2 = 2m_k + 3 \to \infty.$$

Hence,

$$\sum_{i=1}^{\infty} b_{i+n_k} Z_i(\omega) \geq b_{i_k+n_k} Z_{i_k}(\omega) = \frac{Z_{i_k}(\omega)}{i_k+n_k} \geq \frac{i_k}{i_k+n_k} \to 1.$$

4. THE PROOFS

Proof of Theorem 2.1. Let $X = (X_k,\ k \in \mathbb{Z})$ be any nonnegative solution to (1.1). We show, by induction, that, for all $n \geq 1$, the following identity holds:

$$X_k = a\xi_k \sum_{i=0}^{n-1} \eta_{ki} + \xi_k \sum_{i=0}^{n-1} \eta_{ki} \sum_{j=n}^{\infty} b_{j-i} X_{k-j}. \tag{4.1}$$

If $n = 1$, (4.1) coincides with (1.1). Assuming that (4.1) is true we show that it holds with n replaced by $n+1$:

$$\begin{aligned}X_k &= a\xi_k \sum_{i=0}^{n-1} \eta_{ki} + \xi_k \sum_{i=0}^{n-1} \eta_{ki}\left(b_{n-i} X_{k-n} + \sum_{j=n+1}^{\infty} b_{j-i} X_{k-j}\right) \\ &= a\xi_k \sum_{i=0}^{n-1} \eta_{ki} + \xi_k \sum_{i=0}^{n-1} \eta_{ki}\left(b_{n-i} a\xi_{k-n} + b_{n-i}\xi_{k-n} \sum_{j=n+1}^{\infty} b_{j-n} X_{k-j}\right. \\ &\qquad\qquad \left. + \sum_{j=n+1}^{\infty} b_{j-i} X_{k-j}\right) \\ &= a\xi_k \sum_{i=0}^{n} \eta_{ki} + \xi_k \sum_{i=0}^{n} \eta_{ki} \sum_{j=n+1}^{\infty} b_{j-i} X_{k-j}\end{aligned}$$

(the last equality follows from (2.11)). Hence, (4.1) holds for all $n \geq 1$.

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By nonnegativity of X_k, for all n, $X_k \geq a\xi_k \sum_{i=1}^{n-1} \eta_{ki}$. Therefore

$$X_k \geq a\xi_k \sum_{i=1}^{\infty} \eta_{ki} = a\xi_k \eta_k(1) = X_k^*. \qquad \blacksquare$$

Proof of Theorem 2.2. Let $X = (X_k,\ k \in \mathbb{Z})$ be any nonnegative solution to (1.3). Then, for all $n \geq 1$, from (1.3) we get

$$\begin{aligned}\sigma_k^2 &= \alpha_0(1 + \beta(1) + \cdots + [\beta(1)]^{n-1}) \\ &\quad + \alpha(L)(1 + \beta(L) + \cdots + [\beta(L)]^{n-1})X_k + [\beta(L)]^n \sigma_k^2 \\ &\geq \alpha_0(1 + \beta(1) + \cdots + [\beta(1)]^{n-1}). \end{aligned} \tag{4.2}$$

Because $\alpha_0 > 0$, the sequence $1 + \beta(1) + \cdots + [\beta(1)]^{n-1}$ is bounded, and therefore $\beta(1) < 1$.

To prove the second part of the theorem, suppose $\beta(1) < 1$ and define

$$B = \begin{pmatrix} \beta_1 & \dots & \beta_{p-1} & \beta_p \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}, \qquad \theta_k = \begin{pmatrix} \sigma_k^2 \\ \sigma_{k-1}^2 \\ \vdots \\ \sigma_{k-p+1}^2 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then, by (1.3),

$$\theta_k = B\theta_{k-1} + (\alpha_0 + \alpha(L)X_k)e_1,$$

which implies, for all $n \geq 1$,

$$\sigma_k^2 = e_1^\top B^n \theta_{k-n} + \sum_{m=0}^{n-1} (\alpha_0 + \alpha(L)L^m X_k) e_1^\top B^m e_1. \tag{4.3}$$

Because $\beta(1) < 1$, the spectral radius of the matrix B is less than 1, and therefore B^n tends to zero matrix as $n \to \infty$. By stationarity,

$$\|B^n \theta_{k-n}\| \leq \|B^n\| \, \|\theta_{k-n}\| \to 0$$

in probability ($\|\cdot\|$ stands for arbitrary matrix norm). It remains to note that the second term in (4.3) converges almost surely to the right-hand side of the associated ARCH(∞) equations. $\blacksquare$

Proof of Theorem 2.3. By assumption of the theorem, $\eta_0(1) < \infty$ almost surely. Because $\eta_0(1) \geq \sum_{i=1}^{\infty} b_i \xi_{-i}$, by the three series theorem we obtain

$$\sum_{i=1}^{\infty} \mathbf{P}\{b_i \xi_0 > c\} < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} b_i \mathbf{E}[\xi_0 \mathbf{1}_{\{b_i \xi_0 \leq c\}}] < \infty$$

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for all $c > 0$. The first of these inequalities implies that $b_n \xi_0 \xrightarrow{P} 0$ and hence $b_n \to 0$ as $n \to \infty$. Fix $\epsilon > 0$ and find n_0 such that $b_n \le \epsilon$ for $n \ge n_0$. Then

$$\mathbf{E}[\xi_0 \mathbf{1}_{\{\xi_0 \le 1/\epsilon\}}] \sum_{i=n_0}^{\infty} b_i \le \sum_{i=n_0}^{\infty} b_i \mathbf{E}[\xi_0 \mathbf{1}_{\{b_i \xi_0 \le 1\}}] < \infty,$$

which implies (2.3) if ϵ is sufficiently small.

Let us prove the second part of theorem. Define $b(1) = \sum_{i=1}^{\infty} b_i$ and, for $i \ge 1$, $p_i = b_i / b(1)$ and $\tilde{\xi}_i = b(1)\xi_{-i}$. Notice that

$$\eta_0(1) = \sum_{l=0}^{\infty} \sum_{i_1, \dots, i_l = 1}^{\infty} p_{i_1} \cdots p_{i_l} \tilde{\xi}_{i_1} \cdots \tilde{\xi}_{i_1 + \dots + i_l} < \infty \quad \text{a.s.} \tag{4.4}$$

We will prove that this yields

$$\tilde{\xi}_1 \cdots \tilde{\xi}_n \to 0 \quad \text{a.s.} \tag{4.5}$$

as $n \to \infty$.

Let Ω' be a set of all sequences $(i_l, l \ge 1)$ endowed with the cylindrical σ-field $\mathcal{F}'$. Let $\mathbf{P}'$ be a probability on $(\Omega', \mathcal{F}')$ defined by equalities

$$\mathbf{P}'\{(i_l, l \ge 1) | i_1 = i_1^*, \dots, i_s = i_s^*\} = p_{i_1^*} \cdots p_{i_s^*}.$$

For $r \ge 1$ and $\omega' = (i_l, l \ge 1) \in \Omega'$ define $\nu_r(\omega') = i_r$. Then $\nu_1, \nu_2, \dots$ are i.i.d. random variables on $(\Omega', \mathcal{F}', \mathbf{P}')$ such that $\mathbf{P}'\{\nu_r = i\} = p_i$ for all $i \ge 1$.

Set

$$A = \{(\omega, \omega') | \tilde{\xi}_{\nu_1(\omega')}(\omega) \cdots \tilde{\xi}_{\nu_1(\omega') + \dots + \nu_l(\omega')}(\omega) \to 0 \quad \text{as } l \to \infty\}. \tag{4.6}$$

By (4.4), for almost all $\omega \in \Omega$ and for all $\epsilon > 0$

$$\epsilon \sum_{l=0}^{\infty} \mathbf{P}'\{\tilde{\xi}_{\nu_1}(\omega) \cdots \tilde{\xi}_{\nu_1 + \dots + \nu_l}(\omega) > \epsilon\}$$

$$\le \sum_{l=0}^{\infty} \mathbf{E}'[\tilde{\xi}_{\nu_1}(\omega) \cdots \tilde{\xi}_{\nu_1 + \dots + \nu_l}(\omega)] = \eta_0(1) < \infty$$

($\mathbf{E}'$ stands for the expectation with respect to $\mathbf{P}'$). By the Borel–Cantelli lemma, this implies that

$$\mathbf{P}'\{\omega' | (\omega, \omega') \in A\} = 1$$

for almost all $\omega \in \Omega$ and therefore

$$(\mathbf{P} \times \mathbf{P}')(A) = 1. \tag{4.7}$$

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On the other hand, for all Borel sets $A_1,\dots,A_l$ we have

$$
\begin{aligned}
&(\mathbf{P}\times\mathbf{P}')\{\tilde{\xi}_{\nu_1}\in A_1,\dots,\tilde{\xi}_{\nu_1+\dots+\nu_l}\in A_l\}\\
&\quad=\sum_{i_1,\dots,i_l=1}^{\infty} p_{i_1}\cdots p_{i_l}\mathbf{P}\{\tilde{\xi}_{i_1}\in A_1,\dots,\tilde{\xi}_{i_1+\dots+i_l}\in A_l\}\\
&\quad=\sum_{i_1,\dots,i_l=1}^{\infty} p_{i_1}\cdots p_{i_l}\mathbf{P}\{\tilde{\xi}_{i_1}\in A_1\}\cdots\mathbf{P}\{\tilde{\xi}_{i_1+\dots+i_l}\in A_l\}\\
&\quad=\mathbf{P}\{\tilde{\xi}_0\in A_1\}\cdots\mathbf{P}\{\tilde{\xi}_0\in A_l\}.
\end{aligned}
$$

This means that $\tilde{\xi}_{\nu_1},\dots,\tilde{\xi}_{\nu_1+\dots+\nu_l}$ are i.i.d. random variables with the same distribution as $\tilde{\xi}_0$. Hence, by (4.6) and (4.7), relation (4.5) holds.

Relation (4.5) implies (2.4). Indeed, if $\mathbf{E}\log\tilde{\xi}_k>0$, then, by the strong law of large numbers, the sum $S_n:=\sum_{i=1}^n\log\tilde{\xi}_i\to+\infty$ as $n\to\infty$. If $\mathbf{E}\log\tilde{\xi}_k=0$, then, by Theorem 6.1.4 and Corollary 6.1.1 in Stout (1974), $\limsup S_n=+\infty$. (Because $X_k^*<\infty$ almost surely, (1.5) implies that $\mathbf{P}\{\tilde{\xi}_k\neq 1\}>0$ so that Stout's Corollary 6.1.1 is applicable.) ■

Proof of Theorem 2.4. By (4.1) with $X_k=X_k^*=a\xi_k\eta_k(1)$, $\eta_k(1)$ can be written in the form

$$\eta_k(1)=\sum_{i=0}^{n-1}\eta_{ki}\zeta_{kni},$$

where

$$\zeta_{kni}=1+\sum_{j=n}^{\infty}b_{j-i}\xi_{k-j}\eta_{k-j}(1).$$

Therefore

$$\{\eta_k(1)<\infty\}=\bigcap_{i\in I}\{\zeta_{kni}<\infty\},$$

where

$$I=\{i\,|\,i=i_1+\cdots+i_l \text{ for some } l\geq 1 \text{ and } i_1,\dots,i_l\geq 1 \text{ such that } b_{i_1}\cdots b_{i_l}>0\}.$$

Because the ζ_{kni} are $\mathcal{F}_{k-n}$-measurable, the event $\{\eta_k(1)<\infty\}$ lies in $\mathcal{F}_{k-n}$ for all $n\geq 0$. Replacing b_i by $b_i t^i$ we get that

$$\{\eta_k(t)<\infty\}\in\bigcap_{n\geq 1}\mathcal{F}_{k-n}.$$

By the zero-one law, one of the following two alternatives holds:

$$\text{either}\quad \eta_k(t)<\infty \quad\text{a.s.}\quad \text{or}\quad \eta_k(t)=\infty \quad\text{a.s.} \tag{4.8}$$

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Define

$$R = \sup\{t \geq 0 |\ \eta_0(t) < \infty \quad \text{a.s.}\}. \tag{4.9}$$

Let (t_n) and (u_n) be two sequences of nonnegative numbers with the same limit R and such that $t_n < R < u_n$ for all n (in the case $R = 0$ or $R = \infty$ we consider only one of these sequences). Set

$$D = \bigcap_n \{\eta_0(t_n) < \infty, \eta_0(u_n) = \infty\}.$$

Then, by (4.9) and (4.8), $\mathbf{P}(D) = 1$ and R is the convergence radius of $\eta_0(t, \omega)$ for all $\omega \in D$. ■

Proof of Theorem 2.5. We begin with the proof of some modification of Kingman's subadditive ergodic theorem (see Kingman, 1973).

LEMMA 4.1. *Let $y = (y_{st} |\ s, t \in \mathbb{N}, s < t)$ be a family of random variables satisfying the following conditions:*

(i) for all $s < t$, $\mathbf{E}y_{st}^- < \infty$;
(ii) the process y is subadditive, i.e., for all $s < t < u$, $y_{su} \leq y_{st} + y_{tu}$;
(iii) the process y is stationary, i.e., its finite-dimensional distributions are the same as those of the "shifted" process $y' = (y_{s+1,t+1} | s < t)$ (this condition implies that $\mathbf{E}y_{st}$ depends only on the difference $t - s$, i.e., $\mathbf{E}y_{st} = g_{t-s}$ for some sequence $(g_t) \subset (-\infty;\infty]$);
(iv) y is the process with independent increments, i.e., for all $t_0 < t_1 < \cdots < t_n$, the random variables $y_{t_0 t_1}, \ldots, y_{t_{n-1} t_n}$ are independent;
(v) the g.c.d. of the set $T = \{t |\ g_t < \infty\}$ equals 1.

Then $y_{0n}/n \to \theta$ almost surely and $\mathbf{E}y_{0n}/n \to \theta$, as $n \to \infty$, where the "time constant" θ is defined as $\inf_{n \geq 1} g_n/n$.

Proof. From (ii) and (iii) we get

$$g_{s+t} \leq g_s + g_t. \tag{4.10}$$

Therefore $g_s < \infty$ and $g_t < \infty$ imply $g_{s+t} < \infty$; i.e., the set T is additive: $s + t \in T$ for all $s, t \in T$. It is well known that then T contains each sufficiently large multiple of its g.c.d. (see Feller, 1996, Ch. XIII, Sect. 11, Lemma 1). Hence, there exists some t_0 such that $g_t < \infty$ for all $t \geq t_0$.

Fix $k \geq 1$ and consider two sequences of integers, (m_n) and (r_n), uniquely defined by the following conditions:

$$n = m_n k + r_n, \qquad t_0 \leq r_n < t_0 + k.$$

From (4.10) we get

$$\frac{g_n}{n} \leq \frac{m_n g_k + g_{r_n}}{n} \to \frac{g_k}{k},$$

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as $n \to \infty$, and therefore

$$\limsup_{n\to\infty} g_n/n \le \inf_{k\ge 1} g_k/k \le \liminf_{n\to\infty} g_n/n.$$

Hence, $g_n/n \to \theta$, as $n \to \infty$.

Now consider the family of random variables $z = (z_{st} | s < t)$, where $z_{st} = y_{st_0, tt_0}$. It can be easily verified that z satisfies all conditions of Kingman's subadditive ergodic theorem. Besides that, the time constant for z equals

$$\lim_{n\to\infty} \mathbf{E} z_{0n}/n = \lim_{n\to\infty} g_{nt_0}/n = t_0\theta.$$

Assumption (iv) yields that the σ-field of events invariant under the shift $z \mapsto z'$ is trivial. Therefore $z_{0n}/n \to t_0\theta$ and

$$\frac{y_{0,nt_0}}{nt_0} \to \theta \quad \text{a.s.},$$

as $n \to \infty$.

Fix an arbitrary $1 \le k < t_0$. To complete the proof, it suffices to show that

$$\frac{y_{0,nt_0+k}}{nt_0+k} \to \theta \quad \text{a.s.},$$

as $n \to \infty$. From the inequality $g_{t_0+k} < \infty$ we get $y_{0,t_0+k} < \infty$ almost surely and therefore

$$\frac{y_{0,t_0+k}}{nt_0+k} \to 0 \quad \text{a.s.},$$

as $n \to \infty$. By stationarity condition (iii), the distribution of $(y_{t_0+k,nt_0+k} | n \ge 2)$ coincides with that of $(y_{0,(n-1)t_0} | n \ge 2)$. Hence

$$\frac{y_{t_0+k,nt_0+k}}{nt_0+k} \to \theta \quad \text{a.s.},$$

as $n \to \infty$, and

$$\limsup_{n\to\infty} \frac{y_{0,nt_0+k}}{nt_0+k} \le \lim_{n\to\infty} \left[\frac{y_{0,t_0+k}}{nt_0+k} + \frac{y_{t_0+k,nt_0+k}}{nt_0+k} \right] = \theta \quad \text{a.s.}$$

Similar arguments imply the inequality

$$\liminf_{n\to\infty} \frac{y_{2t_0-k,(n-2)t_0}}{nt_0+k} \ge \lim_{n\to\infty} \left[\frac{y_{0,(n+2)t_0}}{nt_0+k} - \frac{y_{0,2t_0-k}}{nt_0+k} \right] = \theta \quad \text{a.s.}$$

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Applying once again stationarity condition (iii) we get

$$\liminf_{n\to\infty} \frac{y_{0,nt_0+k}}{nt_0+k} \geq \theta \quad \text{a.s.} \qquad \blacksquare$$

We proceed with the proof of Theorem 2.5. Define

$$y_{st} = -\log \eta_{-s,t-s}.$$

For all $u,v > 0$ the inequalities

$$\begin{aligned}\log^+(uv) &\leq \log[\max(u,1)\max(v,1)] \\ &= \log\max(u,1) + \log\max(v,1) = \log^+ u + \log^+ v\end{aligned}$$

hold. Therefore

$$\begin{aligned}y_{st}^- = (-y_{st})^+ &\leq \log^+\left[c \max_{i_1+\cdots+i_l=t-s} \xi_{-s-i_1}\cdots\xi_{-s-i_1-\cdots-i_l}\right] \\ &\leq \log^+ c + \max_{i_1+\cdots+i_l=t-s} [\log^+\xi_{-s-i_1} + \cdots + \log^+\xi_{-s-i_1-\cdots-i_l}] \\ &\leq \log^+ c + \sum_{i_1+\cdots+i_l=t-s} [\log^+\xi_{-s-i_1} + \cdots + \log^+\xi_{-s-i_1-\cdots-i_l}]\end{aligned}$$

with some $c \in \mathbb{R}$. This implies $\mathbf{E}y_{st}^- < \infty$, and therefore condition (i) of Lemma 4.1 is satisfied.

The process y is obviously stationary and has independent increments. It is easy to verify that y is also subadditive. Finally, from the inequality

$$y_{0n} \leq -\log(b_n \xi_n)$$

we get $g_n < \infty$, provided $b_n > 0$. By the irreducibility assumption, condition (v) of Lemma 4.1 is satisfied too.

From Lemma 4.1 we get

$$y_{0n}/n \to -\gamma \quad \text{a.s.},$$

as $n \to \infty$, where γ is the top Lyapunov exponent defined by (2.9). $\blacksquare$

Proof of Theorem 3.1. It suffices to prove that the second term in (4.1) tends to 0 in probability as $n \to \infty$.

Fix any q such that $e^\gamma < q < 1$. By (2.8), $\eta_{kn} \leq q^n$ for all sufficiently large n. Therefore, for all n,

$$\eta_{kn} \leq Cq^n,$$

where C is almost surely a finite random variable. This implies that the second term in (4.1) does not exceed $C\xi_k S_n$, where

$$S_n = \sum_{i=0}^{n-1} q^i \sum_{j=n}^{\infty} b_{j-i} X_{k-j}.$$

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By stationarity, S_n has the same distribution as the sum $\sum_{i=0}^{n-1} q^i \tilde{\zeta}_{n-i}$, where

$$\tilde{\zeta}_j = \sum_{r=0}^{\infty} b_{j+r} X_{-r}.$$

By Assumption A, $\tilde{\zeta}_n \to 0$ almost surely as $n \to \infty$. Therefore the convergence of S_n to 0 follows from the well known fact that if $\sum_{i=0}^{\infty} |a_i| < \infty$ and $b_n \to 0$ then $\sum_{i=0}^{n} a_i b_{n-i} \to 0$. ■

Proof of Theorem 3.2. For any X satisfying the conditions of the theorem, (4.1) holds with $n = k$. Let $(\widetilde{X}_{-s}, \ldots, \widetilde{X}_0)$ be a random vector independent of the ξ_n and distributed as $(X_{-s}, \ldots, X_0)$. Then, by stationarity, the random vector $(X_{n+k_1}, \ldots, X_{n+k_p})$ has the same distribution as $(\widetilde{X}_{k_1}^{(n)}, \ldots, \widetilde{X}_{k_p}^{(n)})$, where

$$\widetilde{X}_k^{(n)} = a\xi_k \sum_{i=0}^{n+k-1} \eta_{ki} + \xi_k \sum_{i=0}^{n+k-1} \eta_{ki} \sum_{r=0}^{s} b_{n-i+r} \widetilde{X}_{-r}. \tag{4.11}$$

It suffices to prove that

$$\widetilde{X}_k^{(n)} \to X_k^* \quad \text{a.s.} \tag{4.12}$$

as $n \to \infty$.

The first term in (4.11) converges to $a\xi_k \eta_k(1) = X_k^*$. On the other hand, by Theorem 2.3, $b_n \to 0$. Therefore (4.12) is implied by the same argument as in the proof of Theorem 3.1. ■

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