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The integration of generalized hypergeometric functions

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1. In the usual notation for generalized hypergeometric functions we let

$$
\begin{aligned}
{}_AF_B[(a);(b);z] &= \sum_{m=0}^{\infty} \frac{((a))_m}{((b))_m} \frac{z^m}{m!} \\
&= \sum_{m=0}^{\infty} \frac{(a_1)_m (a_2)_m \ldots (a_A)_m}{(b_1)_m (b_2)_m \ldots (b_B)_m} \frac{z^m}{m!},
\end{aligned}
$$

where $$(a_r)_m = a_r(a_r+1)(a_r+2)\ldots(a_r+m-1), \quad (a_r)_0 = 1,$$

and (a) denotes the sequence of parameters

$$a_1, a_2, \ldots, a_r, \ldots, a_A.$$

Throughout the present paper we shall suppose that there are A of the a parameters, B of the b parameters, and so on. Thus $((a))_m$ is to be interpreted as

$$\prod_{i=1}^{A} (a_i)_m,$$

and similar interpretations hold for $((b))_m$, etc.

The power series ${}_AF_B[(a);(b);z]$ converges under the following conditions:

(i) $A < B+1$ for all $z, a_1, \ldots, a_A, b_1, \ldots, b_B$, real or complex; or

(ii) $A = B+1$, and $|z| < 1$, for all $a_1, \ldots, a_A, b_1, \ldots, b_B$,

$$\text{or} \quad z = 1, \quad \text{and} \quad R(\Sigma b - \Sigma a) > 0,$$

$$\text{or} \quad z = -1, \quad \text{and} \quad R(\Sigma b - \Sigma a + 1) > 0.$$

The integration of the generalized hypergeometric function ${}_AF_B[z]$ with respect to the argument is quite straightforward. For instance, we have Dr L. J. Slater's generalized integrals ((7), p. 288)

$$\int_0^z {}_AF_B[(a);(b);z]\,dz = \frac{((b)-1)}{((a)-1)} \{{}_AF_B[(a)-1;(b)-1;z]-1\}$$

and

$$
\begin{aligned}
&\int_0^z \int_0^z \cdots \cdots \int_0^z {}_AF_B[(a);(b);z]\,d^nz \\
&\quad = (-)^{(A+B)n} \frac{(1-(b))_n}{(1-(a))_n} {}_AF_B[(a)-n;(b)-n;z] - \sum_{r=0}^{n} (-)^{(A+B)r} \frac{(1-(b))_r}{(1-(a))_r} \frac{z^{n-r}}{(n-r)!},
\end{aligned}
$$

which are immediate consequences of the definition of the function involved.

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In the case of the product of two generalized hypergeometric functions Dr Slater has recently proved that if

$$I \equiv \int_0^\infty e^{-zt} t^{\lambda-1} {}_AF_B[(a); (b); xt] \, {}_{A'}F_{B'}[(a'); (b'); yt] \, dt,$$

where $R(\lambda)$ and $R(z)$ are both positive, then ((8), p. 54)

$$I = \frac{\Gamma(\lambda)}{z^\lambda} \sum_{m=0}^{\infty} \frac{((a))_m (\lambda)_m}{m! \, ((b))_m} \left(\frac{x}{z}\right)^m {}_{A'+1}F_{B'}\left[(a'), \lambda+m; (b'); \frac{y}{z}\right]. \tag{1·1}$$

For a generalization of (1·1) we put

$$I' \equiv \int_0^\infty t^{\mu-1} {}_AF_B[(a); (b); xt] \, {}_{A'}F_{B'}[(a'); (b'); yt] \, K_\nu(zt) \, dt,$$

where, for convergence, $A \leqslant B$ and $A' \leqslant B'$.

On substituting power series expansions of the hypergeometric functions, if we interchange the order of the double summation and the integration, we find, by means of (6) (p. 51 (27)), that

$$I' = \frac{2^{\mu-2} \, \Gamma\{\frac{1}{2}(\mu+\nu)\} \, \Gamma\{\frac{1}{2}(\mu-\nu)\}}{z^\mu} \times F\left[\begin{matrix} \frac{1}{2}(\mu+\nu), \frac{1}{2}(\mu-\nu): & \frac{1}{2}(a), \frac{1}{2}(a)+\frac{1}{2}; & \frac{1}{2}(a'), \frac{1}{2}(a')+\frac{1}{2}; & \frac{x^2}{X^2}, \frac{y^2}{Y^2} \\ — \quad : & \frac{1}{2}, \frac{1}{2}(b), \frac{1}{2}(b)+\frac{1}{2}; & \frac{1}{2}, \frac{1}{2}(b'), \frac{1}{2}(b')+\frac{1}{2}; & \end{matrix}\right], \tag{1·2}$$

where $X = 2^{1-A+B} z$, $Y = 2^{1-A'+B'} z$, $A \leqslant B, A' \leqslant B', R(\mu) > |R(\nu)|; R(z) > 0$ for $A < B$, $A' < B'$, or $R(z) > R(x) + R(y)$ when $A = B$, $A' = B'$.

When the aforesaid conditions are satisfied, by making use of the known formula ((6), p. 50 (26)) we can also show that

$$\int_0^\infty e^{-\frac{1}{2}zt} t^{\mu-1} {}_AF_B[(a); (b); xt] \, {}_{A'}F_{B'}[(a'); (b'); yt] \, K_\nu(\tfrac{1}{2}zt) \, dt = \frac{\sqrt{\pi} \, \Gamma(\mu+\nu) \, \Gamma(\mu-\nu)}{\Gamma(\mu+\frac{1}{2}) \, z^\mu} F\left[\begin{matrix} \mu+\nu, \mu-\nu: & (a); & (a'); & \frac{x}{z}, \frac{y}{z} \\ \mu+\frac{1}{2}: & (b); & (b'); & \end{matrix}\right]. \tag{1·3}$$

Here as well as in what follows the notation adopted for double hypergeometric functions is due to Burchnall and Chaundy ((4), p. 112; see also (9), p. 245) in preference, for the sake of brevity, to an earlier one introduced by Kampé de Fériet ((1), p. 150).

The double series in (1·2) and (1·3) simplify in several special cases and we thus obtain a number of known results. Moreover, if we set $\mu = \lambda + \frac{1}{2}$ and $\nu = \pm \frac{1}{2}$, and (in the case of (1·2)) make a free use of Legendre's duplication formula, we shall arrive at the integral (1·1), since

$$K_{\pm\frac{1}{2}}(x) = \left[\frac{\pi}{2x}\right]^{\frac{1}{2}} e^{-x}.$$

2. *Further integrals.* In this section we establish another generalized result by means of the integral

$$\int_0^\infty t^{\mu-1} {}_AF_B[(a); (b); -x^2t^2] \, {}_{A'}F_{B'}[(a'); (b'); -y^2t^2] \, K_\nu(2zt) \, dt = \frac{\Gamma\{\frac{1}{2}(\mu-\nu)\} \, \Gamma\{\frac{1}{2}(\mu+\nu)\}}{4z^\mu} \times F\left[\begin{matrix} \frac{1}{2}(\mu-\nu), \frac{1}{2}(\mu+\nu): & (a); & (a'); & -\frac{x^2}{z^2}, -\frac{y^2}{z^2} \\ — \quad : & (b); & (b'); & \end{matrix}\right], \tag{2·1}$$

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which holds whenever $A < B$, $A' < B'$, $R(\mu) > |R(\nu)|$; $R(z) > 0$ for $A < B-1$, $A' < B'-1$, or each of the four numbers $R(z \pm ix \pm iy)$ is positive if $A = B-1$, $A' = B'-1$.

The proof of this formula runs parallel to that of (1·2); and the complete result follows by an appeal to the theory of analytic continuation.

The integrals (1·2) and (2·1) can be exhibited fairly easily to transform into each other, since

$$ {}_AF_B[(a);\,(b);\,z] = {}_{2A}F_{2B+1}[\tfrac{1}{2}(a),\,\tfrac{1}{2}(a)+\tfrac{1}{2};\,\tfrac{1}{2},\,\tfrac{1}{2}(b),\,\tfrac{1}{2}(b)+\tfrac{1}{2};\,z^2/4^{1-A+B}]. $$

Now consider the integral

$$ J \equiv \int_0^\infty t^{\mu-1}\,{}_AF_B[(a);\,(b);\,-x^2t^2]\,{}_{A'}F_{B'}[(a');\,(b');\,-y^2t^2]\,J_\nu(2zt)\,dt. $$

We begin with the formula (2·1) which is true when

$$ R(\mu) > |R(\nu)|, \quad R(z) > 0, $$

it being assumed that the numbers x and y are real and positive.

If we replace z by $\delta + iz$, where both δ and z are real, and use the asymptotic expansion of

$$ K_\nu[2(\delta+iz)t], $$

it is easily seen that the integral converges uniformly for $\delta \geqslant 0$, provided that it is convergent when $\delta = 0$ and $z \neq 0$. It follows that the integral is continuous when $\delta = 0$ and therefore (2·1) remains valid when z is replaced by $\pm iz$, where the new z is real and positive.

Making use of the formula

$$ \pi i\,J_\nu(z) = e^{-\frac{1}{2}\nu\pi i}K_\nu(-iz) - e^{\frac{1}{2}\nu\pi i}K_\nu(iz) $$

we thus find that

$$ J = \frac{\Gamma\{\frac{1}{2}(\mu+\nu)\}}{2z^\mu\,\Gamma\{1-\frac{1}{2}(\mu-\nu)\}}\,F\left[\begin{matrix}\frac{1}{2}(\mu-\nu),\,\frac{1}{2}(\mu+\nu): & (a); & (a'); & \dfrac{x^2}{z^2},\,\dfrac{y^2}{z^2}\\ \text{—} \quad : & (b); & (b'); & \end{matrix}\right], \tag{2·2} $$

provided $A < B$, $A' < B'$, $R(\mu+\nu) > 0$, and $x, y, z > 0$.

In the 'confluent' case $A = A' = 0$, $B = B' = 1$, the double hypergeometric series in (2·1) and (2·2) reduce to Appell's F_4, and we obtain the integrals evaluated earlier by Bailey (see (2) and (3)).

Next we remark that when $A = A' = 2$, $B = B' = 3$, and the parameters are suitably chosen, an application of the formula ((10), section 5·41 (1)) to (2·1) and (2·2) will express the integrals

$$ \int_0^\infty t^{k-1}J_\lambda(xt)\,J_\mu(xt)\,J_\nu(yt)\,J_\rho(yt)\begin{bmatrix}K_\sigma(zt)\\ J_\sigma(zt)\end{bmatrix}dt $$

in terms of hypergeometric series, under appropriate conditions of convergence.

In particular, when $x = y$ and the parameters are properly specialized, the last integral gives us the elegant formula

$$ \int_0^\infty t^{\mu-1}\sin^2 at\,J_\nu(bt)\,dt = \frac{2^{\mu+1}\,\Gamma\{\frac{1}{2}(\mu+\nu+2)\}\,a^2}{b^{\mu+2}\,\Gamma\{\frac{1}{2}(\nu-\mu)\}} \times {}_3F_2\left[\tfrac{1}{2}(\mu-\nu+2),\,\tfrac{1}{2}(\mu+\nu+2),\,1;\,\tfrac{3}{2},\,2;\,\frac{4a^2}{b^2}\right], \tag{2·3} $$

provided $R(\mu+\nu+2) > 0$, $R(\mu) < \frac{3}{2}$ and $0 < a < \frac{1}{2}b$.

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3. Finally, we make use of the formula ((5), p. 177 (16)) and obtain a double-integral analogue of (1·3) in the form

$$\int_0^\infty \int_0^\infty e^{-\frac{1}{2}z(s+t)} s^{\alpha-1} t^{\beta-1} \, {}_AF_B[(a); (b); x(s+t)] \, {}_{A'}F_{B'}[(a'); (b'); y(s+t)] \, K_\nu[\tfrac{1}{2}z(s+t)] \, ds \, dt$$

$$= \frac{2^{2\mu-1} \Gamma(\alpha) \Gamma(\beta) \Gamma(\mu+\nu) \Gamma(\mu-\nu)}{\Gamma(2\mu) z^\mu} F\left[\begin{matrix} \mu+\nu, \mu-\nu: (a); (a'); \\ \mu+\frac{1}{2}: \quad (b); (b'); \end{matrix} \frac{x}{z}, \frac{y}{z}\right], \qquad (3\cdot1)$$

where $\alpha+\beta = \mu$, and $R(\alpha) > 0$, $R(\beta) > 0$ in addition to the conditions stated earlier.

It is not difficult to give similar analogues of the integral formulae (1·2), (2·1) and (2·2), nor does it seem worthwhile to record the various steps through which (3·1) is reduced to the result

$$\int_0^\infty \int_0^\infty e^{-z(s+t)} s^{\alpha-1} t^{\beta-1} \, {}_AF_B[(a); (b); x(s+t)] \, {}_{A'}F_{B'}[(a'); (b'); y(s+t)] \, ds \, dt$$

$$= \frac{\Gamma(\alpha) \Gamma(\beta)}{z^{\alpha+\beta}} \sum_{m=0}^{\infty} \frac{((a))_m (\alpha+\beta)_m}{m! ((b))_m} \left(\frac{x}{z}\right)^m {}_{A'+1}F_{B'}\left[(a'), \alpha+\beta+m; (b'); \frac{y}{z}\right], \qquad (3\cdot2)$$

which furnishes an interesting double-integral analogue of Dr Slater's formula already referred to.

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