MULTIVARIATE DISTRIBUTIONS WITH TIME AND CROSS-DEPENDENCE: AGGREGATION AND CAPITAL ALLOCATION

BY

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ABSTRACT

This paper investigates risk aggregation and capital allocation problems for an insurance portfolio consisting of several lines of business. The class of multivariate INAR(1) processes is proposed to model different sources of dependence between the number of claims of the portfolio. The total capital required for the whole portfolio is evaluated under the TVaR risk measure, and the contribution of each line of business is derived under the TVaR-based allocation rule. We provide the risk aggregation and capital allocation formulas in the general case of continuous and strictly positive claim sizes and then in the case of mixed Erlang claim sizes. The impact of both time dependence and cross-dependence on the behavior of risk aggregation and capital allocation is numerically illustrated.

KEYWORDS

Multivariate INAR(1) processes, time dependence, cross-dependence, risk aggregation, capital allocation.

1. INTRODUCTION

The importance of managerial decisions related to risk aggregation and capital allocation has received considerable attention in actuarial science. It generally comprises two central elements: choosing risk measures to evaluate the aggregate capital needed for the safety of an insurance portfolio and allocating the part of this capital to each line of business in order to cover unexpected large losses. There is a variety of risk measure-based approaches to allocate the total capital of a portfolio to its different lines of business for meeting financial obligations. Artzner *et al*. (1999) propose an axiomatic definition of a coherent risk measure that can be used as capital requirements to regulate the risk of a portfolio as well as to allocate each line of business. Cummins (2000) provides

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an overview of common capital allocation methods suitable for insurers and points out the importance of solvency risk in the insurance industry. Myers and Read (2001) show how option-pricing methods can be used to allocate the required capital among different lines of business. For more capital allocation methods, one may refer to Dhaene *et al*. (2003, 2012), Tsanakas (2009), and Cai *et al*. (2017) and the references therein.

The tail risk measures, such as the value at risk (VaR) and the tail value at risk (TVaR), have been widely used as criteria for the purpose of appraising capital requirements. Artzner *et al*. (1999) suggest the TVaR, also called Expected Shortfall (ES), as a coherent alternative to the non-coherent risk measure VaR. The TVaR can identically be defined as the conditional tail expectation (CTE) in the case of continuous random variables. But when the marginal distributions are discrete, these two risk measures are not the same (see, e.g., Acerbi and Tasche, 2002).

Note that continuous situations are extensively discussed in contrast with discrete cases in the literature on capital allocation. Therefore, some of the references speak in terms of CTE instead of TVaR. For example, Panjer (2002) investigates the CTE-based allocation principle for the multivariate normal case. Dhaene *et al*. (2008) extend the multivariate normal risks to the elliptically distributed risks by using the CTE-based allocation rule. See, also, Cai and Li (2010) for multivariate phase-type distributed risks; Furman and Landsman (2010) for multivariate Tweedie distributed risks; Vernic (2006, 2011) for multivariate skew-normal and Pareto distributed risks; and Zhou *et al*. (2018) for multivariate generalized Gamma distributed risks. A common feature of these works is that the dependence structure between the different risks is constructed by a multivariate distribution.

The TVaR-based allocation principle has also generated tremendous interests when risk factors of an insurance portfolio are assumed by special dependence structures. Bargès *et al*. (2009) and Cossette *et al*. (2013) evaluate the TVaR-based allocations for the risks with the dependence structures described by the Farlie-Gumbel-Morgenstern (FGM) copula. Cossette *et al*. (2015) derive closed-form expressions for the contributions of each risk under the TVaR-based allocation rule for two families of bivariate distributions with exponential marginals. Ratovomirija *et al*. (2017) use the rule based on the TVaR to determine the amount of capital allocated for mixed Erlang risks joined by Sarmanov's multivariate distribution. See, also, Cossette *et al*. (2012), Ratovomirija (2016), and Vernic (2017) and the references therein.

Our primary purpose in this paper is to investigate risk aggregation and capital allocation problems based on the rule of TVaR for an insurance portfolio of multi-lines of business by introducing time dependence and cross-dependence together. More precisely, we consider the portfolio of d ($d \geq 2$) lines of business with multivariate compound distributions. The aggregate claim amount for the whole portfolio over the first n periods is defined by the random variable (r.v.) S_n, where

$$S_n = S_{1,n} + \cdots + S_{d,n}. \tag{1.1}$$

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In (1.1), $S_{i,n}$ $(i = 1, ..., d)$ is the aggregate claim amount for the i-th line of business over the first n periods, which is denoted by

$$S_{i,n} = W_{i,1} + \cdots + W_{i,n}, \tag{1.2}$$

where $W_{i,k} = \sum_{j=1}^{N_{i,k}} X_{i,k,j}$, $k = 1, ..., n$, is the aggregate claim amount for the i-th line of business in period k with the sequence of r.v.s $\{N_{i,k}\}_{k=1}^{n}$ being the number of claims and the sequence of independent and identically distributed (i.i.d.) r.v.s $\{X_{i,k,j}\}_{j=1}^{\infty}$ $(X_{i,k,j} \sim X_i)$ being the individual claim sizes for the i-th line of business in period k. For different lines of business, the continuous and strictly positive r.v.s $X_1, \ldots, X_d$ are independent between themselves and independent of $\{N_{1,k}\}_{k=1}^{n}, \ldots, \{N_{d,k}\}_{k=1}^{n}$. The sequence $\underline{N} = \{N_{1,k}, ..., N_{d,k}\}_{k\in\mathbb{N}}$ is assumed to follow a multivariate integer-valued time series. This assumption allows for time and cross-dependence between the number of claims for different lines of business from period to period. Consequently, from (1.2), $S_{i,n}$ follows a sum of dependent compound r.v.s and $(S_{1,n}, ..., S_{d,n})$ are also dependent.

Recent developments in actuarial literature have shown that integer-valued time series can serve as an effective tool in risk modeling, exhibiting different types of dependence when applied to the number of claims. Among them, Gourieroux and Jasiak (2004) reveal that the integer-valued autoregressive process of order 1 (INAR(1)) can be an acceptable alternative for modeling time dependence between the number of claims. Cossette *et al*. (2010, 2011) extend the classical discrete-time risk model by introducing a dependence relationship in time between the number of claims. They use the Poisson INAR(1) process and the Poisson integer-valued moving average (INMA) process. See, also, Zhang *et al*. (2015) and Cossette *et al.* (2020) for further details on univariate integer-valued time series with applications in actuarial science. At the same time, an insurance portfolio may contain several lines of business. Bermúdez *et al*. (2018) adopt bivariate INAR(1) (BINAR(1)) regression models to price an automobile insurance contract with two types of coverage, taking into account both cross-dependence between the number of claims from different coverage types and time dependence between the number of claims of the same policyholder from period to period.

In this paper, we study risk aggregation and capital allocation problems within the framework of multivariate INAR(1) (MINAR(1)) processes. The motivation for such processes arises from their potential for modeling different sources of dependence, including the following: first, time dependence between the number of claims for the same line of business from period to period; second, cross-dependence between the number of claims for different lines of business; and third, a source of dependence that combines these first two sources, defined as cross-time dependence.

The empirical illustration in Bermúdez *et al*. (2018) indicates that both time and cross-dependence should be considered in order to improve the fitting

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and prediction of the ratemaking model. Inspired of this, it is natural to ask whether the incorporated dependence has impact on the quantities of interest related to risk aggregation and capital allocation, and how the insurer can allocate total capital to each line of business appropriately in this situation. To this extent, we consider two cases of MINAR(1) processes with specific dependence structures for $\{N_{1,k}, ..., N_{d,k}\}_{k\in\mathbb{N}}$ in this paper. For each case, the expression for the joint Laplace-Stieltjes transform (LST) of $(S_{1,n}, ..., S_{d,n})$ is provided. We then obtain the general formula for the cumulative distribution function (c.d.f.) of S_n. Along the lines of some recent studies (Bargès *et al.*, 2009 and Cossette *et al.*, 2012), we use a top-down approach to derive the general expressions for the TVaR of S_n and the contribution of $S_{i,n}$ with the TVaR-based allocation rule. These work are performed under the assumption of continuous and strictly positive claim sizes. Thanks to the desirable properties of the class of mixed Erlang distributions, we develop the closed-form formulas for these quantities with mixed Erlang claim sizes and examine the effect of different sources of dependence on the TVaR of S_n and the contribution of $S_{i,n}$ using the TVaR-based allocation rule with numerical examples.

The remainder of this paper is structured as follows. In Section 2, we present some preliminaries on the TVaR and TVaR capital allocation, on the MINAR(1) processes, and on the class of mixed Erlang distributions. In Section 3, the quantities of interest in regard to risk aggregation and capital allocation are studied for a basic MINAR(1) process. Section 4 contains the main results on risk aggregation and capital allocation for a full BINAR(1) process. The final Section 5 concludes the paper.

2. PRELIMINARIES

Throughout this paper, let $\mathbb{N}_0$, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ denote the sets of non-negative integers, positive integers, integers, and real numbers, respectively. For the sake of convenience, here and henceforth, let $S_{-i,n} = \sum_{l=1,l\neq i}^{d} S_{l,n}$, $\sum_{j=a}^{b} x_j = 0$ and $\prod_{j=a}^{b} x_j = 1$, for any $a > b$ and $x_j > 0$. For an m-dimensional vector $\underline{d}$, let $\underline{d}^{*k}$ denote the k-th ($k \in \mathbb{N}_0$) convolution of $\underline{d}$ with itself, where $d^{*0} = (1, 0, ..., 0) \in \mathbb{R}^m$. For an m-dimensional random vector $(X_1, ..., X_m)$, the LST of $(X_1, ..., X_m)$ is defined as

$$\mathcal{L}_{X_1,...,X_m}(z_1, ..., z_m) = \mathrm{E}\left[e^{-z_1X_1} \cdots e^{-z_mX_m}\right].$$

2.1. TVaR and TVaR-based capital allocation

Let us recall the definition of the aggregate claim amount S_n given by (1.1). The VaR at level $\kappa, 0 < \kappa < 1$, of S_n is defined by $\mathrm{VaR}_\kappa(S_n) = \inf\{x \in \mathbb{R}, F_{S_n}(x) \geq \kappa\}$, where F_{S_n} denotes the c.d.f. of S_n. The TVaR at level κ, $0 < \kappa < 1$, of S_n is

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defined by

$$\begin{aligned}\mathrm{TVaR}_\kappa(S_n) &= \frac{1}{1-\kappa}\int_\kappa^1 \mathrm{VaR}_u(S_n)du \\ &= \frac{\mathrm{E}\left[S_n \times \mathbb{1}_{\{S_n>\mathrm{VaR}_\kappa(S_n)\}}\right] + \mathrm{VaR}_\kappa(S_n)(F_{S_n}(\mathrm{VaR}_\kappa(S_n)) - \kappa)}{1-\kappa},\end{aligned} \tag{2.1}$$

where $\mathbb{1}_A$ is the indicator function such that $\mathbb{1}_A(x) = 1$ if $x \in A$, and $\mathbb{1}_A(x) = 0$, if $x \notin A$.

Note that the r.v. $S_{i,n}$ given by (1.2) is continuous under the assumption of continuous and strictly positive claim sizes, implying that the r.v. S_n is also continuous. Then, $F_{S_n}(\mathrm{VaR}_\kappa(S_n)) - \kappa = 0$ and (2.1) becomes

$$\mathrm{TVaR}_\kappa(S_n) = \frac{\mathrm{E}\left[S_n \times \mathbb{1}_{\{S_n>\mathrm{VaR}_\kappa(S_n)\}}\right]}{1-\kappa}.$$

It is well known that the VaR is not a coherent risk measure. In this paper, we consider to determine the amount of capital for the whole portfolio with the TVaR-based rule. For $0 < \kappa < 1$, the allocated capital to each risk $S_{i,n}$ under the TVaR-based allocation rule is given by

$$K_i = \frac{\mathrm{TVaR}_\kappa(S_{i,n}; S_n)}{\mathrm{TVaR}_\kappa(S_n)}K,$$

where K is a given total risk capital and

$$\mathrm{TVaR}_\kappa(S_{i,n}; S_n) = \frac{\mathrm{E}\left[S_{i,n} \times \mathbb{1}_{\{S_n>\mathrm{VaR}_\kappa(S_n)\}}\right]}{1-\kappa}.$$

It implies that the capital required for the entire portfolio is equal to the sum of the allocated capital of each risk within the portfolio, that is, $\sum_{i=1}^n K_i = K$. See for example McNeil *et al*. (2015) for details on the risk measures VaR and TVaR, and the TVaR-based allocation rule.

This paper considers the particular case that $K = \mathrm{TVaR}_\kappa(S_n)$, which coincides with the TVaR-based allocation rule suggested by Bargès *et al*. (2009), Cossette *et al*. (2012, 2013, 2015), and Ratovomirija *et al*. (2017).

2.2. MINAR(1) processes

Let $\mathbf{N}_k = (N_{1,k}, ..., N_{d,k})$ and $\mathbf{e}_k = (e_{1,k}, ..., e_{d,k})$ be non-negative integer-valued random vectors. $\mathbf{N}_k$ is independent of the i.i.d. sequence $\{\mathbf{e}_k\}_{k\in\mathbb{Z}}$. Let $\mathbf{A}$ be a $d \times d$ matrix with independent elements $\{\alpha_{ij}\}_{i,j=1}^d$. The multivariate INAR(1) process $\{\mathbf{N}_k\}_{k\in\mathbb{Z}}$ based on the binomial thinning operation is defined by

$$\mathbf{N}_k = \mathbf{A} \circ \mathbf{N}_{k-1} + \mathbf{e}_k \tag{2.2}$$

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with its i-th element given by

$$N_{i,k} = \sum_{j=1}^{d} \alpha_{ij} \circ N_{j,k-1} + e_{i,k}, \quad i = 1, ..., d. \tag{2.3}$$

In (2.2), the binomial thinning operators $\alpha_{ij}\circ$ are mutually independent, for $i, j = 1, ..., d$. Each operator is defined by $\alpha_{ij} \circ N = \sum_{k=1}^{N} \delta_k$, where $\{\delta_k\}_{k=1}^{N}$ is a sequence of i.i.d. Bernoulli r.v.s such that $\mathrm{E}[\delta_k] = \alpha_{ij}$ and $\alpha_{i,j} \in [0, 1)$.

The process $\{\mathbf{N}_k\}_{k\in\mathbb{Z}}$ satisfying (2.2) is stationary if the largest eigenvalue of the matrix $\mathbf{A}$ is less than 1 and $\mathrm{E}\,\|\mathbf{e}_k\| < \infty$ (Pedeli and Karlis, 2013b). The process $\{\mathbf{e}_k\}_{k\in\mathbb{Z}}$ entered in (2.2) is known as the innovations.

The MINAR(1) process introduced above is particularly convenient for modeling dependence structures between multivariate risks. As an example, $\mathbf{N}_k$ may be interpreted as the number of claims in period k for an insurance portfolio with d lines of business. As every INAR-type process, $\mathbf{N}_k$ is composed of two different parts. The first consists of proportions of the number of claims from the previous period $k-1$, denoted by $\mathbf{N}_{k-1}$. The time dependence between the number of claims of a fixed line of business over time arises from this part. The second part consists of the innovations that are assumed to be dependent. The cross-dependence between the number of claims from different lines of business derives from the covariance structure assumed for $\mathbf{e}_k$. For further applications, we refer the readers to Pedeli and Karlis (2013a, 2013b) and Darolles *et al*. (2019).

In the literature, there are two kinds of MINAR(1) processes, depending on the form of the matrix $\mathbf{A}$ entered in (2.2).

Definition 2.1. *If* $\mathbf{A}$ *in (2.2) takes the form of a* $d \times d$ *diagonal matrix, then the process* $\{\mathbf{N}_k\}_{k\in\mathbb{Z}}$ *satisfying (2.2) is called as the basic MINAR(1) process. In particular, the basic BINAR(1) process is generated with a diagonal matrix* $\mathbf{A}$ *for* $d = 2$.

In Definition 2.1, from (2.3) each series separately follows a univariate INAR(1) process, thus allowing for time dependence. The assumption of diagonality implies that the correlation between innovations is the only source of cross-dependence between the series $\{N_{ik}, N_{jk}\}$, for $i, j = 1, ..., d$ and $i \neq j$. Therefore, two sources of dependence are arising in the basic MINAR(1) process. Detailed properties and methods of estimation about the basic MINAR(1) process have been discussed extensively in Pedeli and Karlis (2013a).

Definition 2.2. *If* $\mathbf{A}$ *in (2.2) takes the form of a* $d \times d$ *non-diagonal matrix, then the process* $\{\mathbf{N}_k\}_{k\in\mathbb{Z}}$ *satisfying (2.2) is called as the full MINAR(1) process. In particular, the full BINAR(1) process is generated with a non-diagonal matrix* $\mathbf{A}$ *for* $d = 2$.

Compared with Definition 2.1, an extra source of dependence is incorporated into the process by relaxing the diagonality assumption of the matrix $\mathbf{A}$ in

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Definition 2.2. More specifically, the value of each series in period k is directly not only to its own observations but also to the observations of the others series at the preceding point in period $k-1$, see (2.3). This forms the third source of dependence and is referred to cross-time dependence as in Bermúdez *et al*. (2018). Some distributional properties and methods of estimation for the full MINAR(1) process can be found in Pedeli and Karlis (2013b).

In this paper, the sequence of innovations $\{\mathbf{e}_k\}_{k\in\mathbb{Z}}$ ($\mathbf{e}_k \sim \mathbf{e}$) in (2.2) is assumed to jointly follow a multivariate Poisson distribution, which is defined by the linear equation

$$\mathbf{e}^T = (e_1, ..., e_d)^T = \mathbf{B}\underline{\xi}. \tag{2.4}$$

In (2.4), $\mathbf{B} = (\mathbf{I}_d, \mathbf{B}^*)$ is a $d \times \frac{d(d+1)}{2}$ matrix where $\mathbf{I}_d$ is the identify matrix of dimension d and $\mathbf{B}^* = (b_{1,2}, ..., b_{1,d}, b_{2,3}, ..., b_{2,d}, ..., b_{d-1,d})$ is a $d \times \frac{d(d-1)}{2}$ matrix with $b_{i,j}$ being a column vector of dimension d, whose i-th and j-th elements are one and the rest are zero, for $i, j = 1, ..., d$ and $i < j$. $\underline{\xi} = (\xi_1, ..., \xi_d, \xi_{1,2}, ..., \xi_{1,d}, \xi_{2,3}, .., \xi_{2,d}, ..., \xi_{d-1,d})^T$ is a sequence of independent r.v.s, where ξ_i and $\xi_{i,j}$ are Poisson distributed with mean λ_i and $\lambda_{i,j}$, respectively, that is, $\xi_i \sim \text{Poi}(\lambda_i)$ and $\xi_{i,j} \sim \text{Poi}(\lambda_{ij})$, for $i, j = 1, ..., d$ and $i < j$.

Remark 2.1.

(1) The multivariate Poisson distribution introduced above is appealing in that it allows for a full covariance structure among the variables. For example, consider the case of the trivariate Poisson distribution with two-way covariance structure. The innovations $\mathbf{e} = (e_1, e_2, e_3)$ *take the form*

$$\begin{cases} e_1 = \xi_1 + \xi_{1,2} + \xi_{1,3}, \\ e_2 = \xi_2 + \xi_{1,2} + \xi_{2,3}, \\ e_3 = \xi_3 + \xi_{1,3} + \xi_{2,3}. \end{cases}$$

Then, each e_i *follows marginally a Poisson distribution with parameter* $\lambda_i + \sum_{j<i} \lambda_{ji} + \sum_{j>i} \lambda_{ij}$, *for* $i, j = 1, 2, 3$. *The variance-covariance matrix is given by*

$$\begin{pmatrix} \lambda_1 + \lambda_{12} + \lambda_{13} & \lambda_{12} & \lambda_{13} \\ \lambda_{12} & \lambda_2 + \lambda_{12} + \lambda_{23} & \lambda_{23} \\ \lambda_{13} & \lambda_{23} & \lambda_3 + \lambda_{13} + \lambda_{23} \end{pmatrix}.$$

(2) If $\xi_{i,j} \sim Poi(\lambda_0)$, *for all* $i, j = 1, ..., d$ *and* $i < j$, *then (2.4) is reduced to the multivariate Poisson distribution defined with a common shock, in which all the pairs of variables have the same covariance* λ_0, *see, for example, Johson et al. (1997) and Lindskog and McNeil (2003).*

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According to (2.4), it implies that the joint probability generating function (p.g.f.) of $\mathbf{e} = (e_1, ..., e_d)$ is

$$p_{\mathbf{e}}(\mathbf{z}) = p_{e_1,...,e_d}(z_1, ..., z_d) = \prod_{i=1}^{d} e^{\lambda_i(z_i-1)} e^{\sum_{j>i}\lambda_{ij}(z_i z_j-1)}.$$

Then, the stationary marginal distribution of $\{\mathbf{N}_k\}_{k\in\mathbb{Z}}$ can be determined by the equation

$$p_{\mathbf{N}}(\mathbf{z}) = p_{\mathbf{A}\circ\mathbf{N}}(\mathbf{z}) p_{\mathbf{e}}(\mathbf{z}). \tag{2.5}$$

2.3. Mixed Erlang distributions

The mixed Erlang distribution has many attractive properties when modeling the individual claim sizes of an insurance portfolio (see, e.g., Lee and Lin, 2010 and Willmot and Lin, 2011). We begin this subsection with the definition of the mixed Erlang distribution and continue with some properties about it.

Let the probability density function (p.d.f.) and c.d.f. of an Erlang distribution of order k be

$$h(x; k, \beta) = \frac{\beta^k}{(k-1)!} x^{k-1} e^{-\beta x}, \quad x > 0,$$

and

$$H(x; k, \beta) = 1 - e^{-\beta x} \sum_{j=0}^{k-1} \frac{(\beta x)^j}{j!}, \quad x > 0,$$

where $\beta > 0$ and $k \in \mathbb{N}$.

Let X be a mixed Erlang r.v. with scale parameter β. The p.d.f. and c.d.f. of X are respectively given by

$$f_X(x) = \sum_{j=1}^{\infty} q_j h(x; j, \beta),$$

and

$$F_X(x) = \sum_{j=1}^{\infty} q_j H(x; j, \beta),$$

where $\underline{q} = \{q_j\}_{j=1}^{\infty}$ is a vector of non-negative mixing probabilities such that $\sum_{j=1}^{\infty} q_j = 1$.

Based on the analytical form of $F_X(x)$, the expression for $\mathrm{TVaR}_\kappa(X)$ is given by

$$\mathrm{TVaR}_\kappa(X) = \frac{1}{1-\kappa} \sum_{j=1}^{\infty} q_j \frac{j}{\beta} \overline{H}(\mathrm{VaR}_\kappa(X); j+1, \beta), \tag{2.6}$$

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where $\overline{H}(\mathrm{VaR}_\kappa(X); j, \beta) = 1 - H(\mathrm{VaR}_\kappa(X); j, \beta)$ and $\mathrm{VaR}_\kappa(X)$ can be computed numerically from the equation $F_X(\mathrm{VaR}_\kappa(X)) - \kappa = 0$. Here and henceforth, we use the notation $X \sim \mathrm{MixErl}(\underline{q}, \beta)$ with $\underline{q} = \{q_j\}_{j=1}^{\infty}$ for convenience.

The rest of this subsection provides some distributional properties of the mixed Erlang distribution, which will be convenient for us to identify that many other r.v.s can be expressed in the form of an Erlang mixture in the sequel.

Lemma 2.1. *Let X be a r.v. with $X \sim \mathrm{MixErl}(\underline{q}, \beta_1)$ and $\underline{q} = \{q_j\}_{j=1}^{\infty}$. Then, it follows that for any positive constant β such that $\beta_1 \le \beta$, the p.d.f. of X can be expressed as the p.d.f. of a mixed Erlang distribution with parameters $(\underline{\omega}(\underline{q}, \beta_1, \beta) = \{\omega_j(\underline{q}, \beta_1, \beta)\}_{j=1}^{\infty}, \beta)$, where*

$$\omega_j(\underline{q}, \beta_1, \beta) = \sum_{k=1}^{j} q_k \binom{j-1}{j-k} \left(\frac{\beta_1}{\beta}\right)^k \left(1 - \frac{\beta_1}{\beta}\right)^{j-k}, \quad j = 1, 2, \ldots.$$

Proof. See Lemma 2.4 in Cossette *et al*. (2013). □

Assume that $X_i \sim \mathrm{MixErl}(\underline{q}_i, \beta_i)$ with $\underline{q}_i = \{q_{ij}\}_{j=1}^{\infty}$ for $i \in \mathbb{N}$. By Lemma 2.1, each X_i can be transformed into a new mixed Erlang distribution with a common scale parameter. Without loss of generality, it will be assumed that $\beta_i = \beta$ in the sequel.

Lemma 2.2. *Let X_1 and X_2 be two independent r.v.s with $X_i \sim \mathrm{MixErl}(\underline{q}_i, \beta)$ and $\underline{q}_i = \{q_{ij}\}_{j=1}^{\infty}$, for $i = 1, 2$. Then, the r.v. $Z_2 = X_1 + X_2$ has a mixed Erlang distribution with parameters $(\underline{\sigma}(\underline{q}_1, \underline{q}_2) = \{\sigma_j(\underline{q}_1, \underline{q}_2)\}_{j=1}^{\infty}, \beta)$, where*

$$\sigma_j(\underline{q}_1, \underline{q}_2) = \begin{cases} \displaystyle\sum_{k=1}^{j-1} q_{1,k} q_{2,j-k}, & j > 1, \\ 0, & j = 1. \end{cases}$$

Proof. See Lemma 2.2 in Cossette *et al*. (2013). □

2.4. Compound distributions with mixed Erlang claim sizes

Another and quite useful expression can be given to the class of mixed Erlang distributions. To do so, we first make a notation for compound distributions. Let a r.v S have a compound distribution with

$$S = \begin{cases} \displaystyle\sum_{k=1}^{N} X_k, & N > 0, \\ 0, & N = 0, \end{cases}$$

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where N is a discrete r.v. with p.m.f. $\Pr(N=j)=p_j$ for $j\in\mathbb{N}_0$ and $\underline{X}=\{X_k\}_{k=1}^{\infty}$ is a sequence of i.i.d. r.v.s ($X_k\sim X$) and independent of N. To simplify the presentation, we denote the compound r.v. S as $S:=N\vee\underline{X}$ in what follows.

A mixed Erlang distribution X can be interpreted as a compound distribution such that X follows $X=J\vee\underline{E}$, where J is a discrete r.v. with p.m.f. $\Pr(J=0)=0$ and $\Pr(J=j)=q_j$, for $j>0$, and $\underline{E}=\{E_k\}_{k=1}^{\infty}$ is a sequence of i.i.d. r.v.s, following an exponential distribution with parameter β, that is, $E_k\sim\text{Exp}(\beta)$. This interpretation of the mixed Erlang distribution enables us to derive the expression for the TVaR of some compound distributions with mixed Erlang claim sizes, which will be examined in the following lemmas.

Lemma 2.3. *Let the r.v N have a Bernoulli distribution with mean α, that is, $N\sim\text{Bern}(\alpha)$. Suppose that $\underline{X}=\{X_k\}_{k=1}^{\infty}$ is a sequence of i.i.d. r.v.s with $X_k\sim\text{MixErl}(\underline{q},\beta)$ and $\underline{q}=\{q_j\}_{j=1}^{\infty}$. Then the c.d.f. of the compound r.v. $Y=N\vee\underline{X}$ follows*

$$F_Y(x)=p_0+\sum_{j=1}^{\infty}p_jH(x;j,\beta),\tag{2.7}$$

where $p_0=1-\alpha$ and $p_j=\alpha q_j$, for $j\geq 1$.

Proof. It simply follows from the c.d.f.s of N and $\underline{X}$. □

It is clear from (2.7) that $F_Y(x)$ combines a mixed Erlang distribution on the positive real line and a point mass at zero. In what follows, we use the notation $Y\sim\text{MixErl}(\underline{p},\beta)$ with $\underline{p}=\{p_j\}_{j=0}^{\infty}$ to represent a mixed Erlang r.v. by allowing a point mass p_0 at zero.

Lemma 2.4. *Let Y_1 and Y_2 be two independent r.v.s with $Y_i=N_i\vee\underline{X}_i$, where $N_i\sim\text{Bern}(\alpha_i)$ and $\underline{X}_i=\{X_{i,k}\}_{k=1}^{\infty}$ is a sequence of i.i.d. r.v.s. with $X_{i,k}\sim\text{MixErl}(\underline{q}_i,\beta)$ and $\underline{q}_i=\{q_{i,j}\}_{j=1}^{\infty}$, for $i=1,2$. Then, the r.v. $Z_2=Y_1+Y_2\sim\text{MixErl}(\underline{\sigma}(\underline{p}_1,\underline{p}_2),\beta)$, where $\underline{\sigma}(\underline{p}_1,\underline{p}_2)=\{\sigma_j(\underline{p}_1,\underline{p}_2)\}_{j=0}^{\infty}$ with*

$$\sigma_j(\underline{p}_1,\underline{p}_2)=\sum_{k=0}^{j}p_{1,k}p_{2,j-k},\tag{2.8}$$

where $\underline{p}_i=\{p_{i,j}\}_{j=0}^{\infty}$ with $p_{i,0}=1-\alpha_i$ and $p_{i,j}=\alpha_iq_{i,j}$, for $j\geq 1$ and $i=1,2$.

Proof. The proof follows immediately from Lemmas 2.2 and 2.3. □

Remark 2.2. *From (2.8), it follows that $\sigma_0(\underline{p}_1,\underline{p}_2)=p_{1,0}p_{2,0}=(1-\alpha_1)(1-\alpha_2)$ and $\sigma_1(\underline{p}_1,\underline{p}_2)=p_{1,0}p_{2,1}+p_{1,1}p_{2,0}=(1-\alpha_1)\alpha_2q_{2,1}+(1-\alpha_2)\alpha_1q_{1,1}$. If $\alpha_1=1$ or $\alpha_2=1$, then the r.v. Z_2 given by Lemma 2.4 has exactly a mixed Erlang distribution defined on the positive real line. In particular, if $\alpha_1=\alpha_2=1$, then the result of Lemma 2.4 is reduced to that of Lemma 2.2.*

Lemma 2.5. *Let the r.v. $Y=N\vee\underline{X}$, where $N\sim Poi(\lambda)$ and $\underline{X}=\{X_k\}_{k=1}^{\infty}$ is a sequence of i.i.d. r.v.s with $X_k\sim\text{MixErl}(\underline{q},\beta)$ and $\underline{q}=\{q_j\}_{j=1}^{\infty}$. Then the c.d.f. of*

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the compound r.v. Y is given by

$$F_Y(x) = f_{N'}(0) + \sum_{k=1}^{\infty} f_{N'}(k) H(x; k, \beta),$$

where $f_{N'}$ represents the probability mass function (p.m.f.) of a non-negative integer-valued r.v. N', which is determined by the following recursive formula:

$$f_{N'}(k+1) = \frac{\lambda}{k+1}\left(q_1 f_{N'}(k) + 2q_2 f_{N'}(k-1) + \cdots + (k+1)q_{k+1} f_{N'}(0)\right),$$

for $k \in \mathbb{N}_0$ and $f_{N'}(0) = e^{-\lambda}$.

Proof. This result is a special case of Proposition 3 in Cossette *et al*. (2012). □

3. MULTIVARIATE DISTRIBUTION WITH A BASIC MINAR(1) PROCESS

In this section, the number of claims $\{N_{1,k}, \ldots, N_{d,k}\}_{k\in\mathbb{N}}$ are supposed to follow a basic MINAR(1) process according to Definition 2.1. A general expression for the joint LST of $(S_{1,n}, \ldots, S_{d,n})$ by allowing the distribution of the innovations unconstrained is first provided.

Assuming that the innovations $\{\mathbf{e}_k\}_{k\in\mathbb{N}}$ have a multivariate Poisson distribution given by (2.4), from (2.5), the joint p.g.f. of $(N_{1,k}, \ldots, N_{d,k})$, for $k \in \mathbb{N}$, follows

$$p_{N_{1,k},\ldots,N_{d,k}}(z_1, \ldots, z_d) = \prod_{i=1}^{d} e^{\lambda'_i(z_i-1)} e^{\sum_{j>i}\lambda'_{ij}(z_i z_j - 1)}, \tag{3.1}$$

where $\lambda'_{ij} = \dfrac{\lambda_{ij}}{1-\alpha_{ii}\alpha_{jj}}$ and $\lambda'_i = \dfrac{\lambda_i + \sum_{j<i}\lambda_{ji} + \sum_{j>i}\lambda_{ij}}{1-\alpha_{ii}} - \sum_{j<i}\lambda'_{ji} - \sum_{j>i}\lambda'_{ij}$.

Explicit expressions for the TVaR of S_n and the contribution of $S_{i,n}$ under the TVaR-based allocation rule are derived. Closed-form expressions are also obtained with mixed Erlang claim sizes.

3.1. Joint LST of $(S_{1,n}, \ldots, S_{d,n})$

Proposition 3.1. *Assuming that the number of claims $\{N_{1,k}, \ldots, N_{d,k}\}_{k\in\mathbb{N}}$ follow a basic MINAR(1) process, the joint LST of $(S_{1,n}, \ldots, S_{d,n})$ is given by*

$$\begin{aligned}\mathcal{L}_{S_{1,n},\ldots,S_{d,n}}(z_1, \ldots, z_d) &= p_{N_{1,1},\ldots,N_{d,1}}(G_{1,n}(\mathcal{L}_{X_1}(z_1)), \ldots, G_{d,n}(\mathcal{L}_{X_d}(z_d))) \\ &\quad \times \prod_{k=2}^{n} p_{e_{1,k},\ldots,e_{d,k}}(G_{1,n-k+1}(\mathcal{L}_{X_1}(z_1)), \ldots, G_{d,n-k+1}(\mathcal{L}_{X_d}(z_d))),\end{aligned} \tag{3.2}$$

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where

$$G_{i,m}(z_i) = \begin{cases} \left(\alpha_{ii} G_{i,m-1}(z_i) + 1 - a_{ii}\right) z_i, & m > 1, \\ z_i, & m = 1, \end{cases} \tag{3.3}$$

for $i = 1, ..., d$ *and* $m = 1, ..., n$.

Proof. See Appendix. □

Remark 3.1. *From (3.3),* $G_{i,m}(\mathcal{L}_{X_i}(z_i))$ *just corresponds to the LST of a r.v. associated with* X_i*, for* $i = 1, ..., d$ *and* $m = 1, ..., n$*. Without loss of generality, suppose that* $\{X'_{i,m}\}_{m=1}^{n}$ *is a sequence of independent r.v.s such that the LST of* $X'_{i,m}$ *follows*

$$\mathcal{L}_{X'_{i,m}}(z) = G_{i,m}(\mathcal{L}_{X_i}(z)). \tag{3.4}$$

By letting $z_1 = \cdots = z_d = z$ *in (3.2), the form of* $\mathcal{L}_{S_{1,n},...,S_{d,n}}(z, ..., z)$ *indicates that the r.v.* $S_n = \sum_{i=1}^{d} S_{i,n}$ *can be viewed as a sum of n independent compound r.v.s.*

Under the assumption of multivariate Poisson innovations, combining with (3.1), it follows that S_n has a compound Poisson distribution, which can be expressed as

$$S_n = M_n \vee \underline{D}^{(n)}, \tag{3.5}$$

where $M_n \sim \text{Poi}(\lambda_{S_n})$ with

$$\lambda_{S_n} = \sum_{i=1}^{d} \left(\lambda'_i + \sum_{j>i} \lambda'_{ij} \right) + (n-1) \sum_{i=1}^{d} \left(\lambda_i + \sum_{j>i} \lambda_{ij} \right) \tag{3.6}$$

and $\underline{D}^{(n)} = \{D_j^{(n)}\}_{j=1}^{\infty}$ is a sequence of i.i.d. r.v.s ($D_j^{(n)} \sim D^{(n)}$) with common c.d.f.

$$\begin{aligned} F_{D^{(n)}}(x) = &\sum_{i=1}^{d} \left(\frac{\lambda'_i}{\lambda_{S_n}} F_{X'_{i,n}}(x) + \frac{\sum_{j>i} \lambda'_{ij}}{\lambda_{S_n}} F_{X'_{i,n}+X'_{j,n}}(x) \right) \\ &+ \sum_{i=1}^{d} \left(\frac{\lambda_i}{\lambda_{S_n}} \sum_{k=2}^{n} F_{X'_{i,n-k+1}}(x) + \frac{\sum_{j>i} \lambda_{ij}}{\lambda_{S_n}} \sum_{k=2}^{n} F_{X'_{i,n-k+1}+X'_{j,n-k+1}}(x) \right). \end{aligned} \tag{3.7}$$

Proposition 3.2. *Assuming that the number of claims* $\{N_{1,k}, ..., N_{d,k}\}_{k \in \mathbb{N}}$ *follow a basic MINAR(1) process with multivariate Poisson innovations given by (2.4),*

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the expression for $\mathrm{TVaR}_\kappa(S_n)$ *is*

$$\mathrm{TVaR}_\kappa(S_n) = \frac{\sum_{k=1}^{\infty} f_{M_n}(k)\mathrm{E}\left[\sum_{j=1}^{k} D_j^{(n)} \times \mathbb{1}_{\sum_{j=1}^{k} D_j^{(n)} > \mathrm{VaR}_\kappa(S_n)}\right]}{1-\kappa},$$

where f_{M_n} *represents the p.m.f. of* M_n.

Proof. The result can be identified from (3.5), (3.6) and (3.7). □

3.2. TVaR-based capital allocation

By letting $z_i = z$ and $z_1 = \cdots = z_{i-1} = z_{i+1} = \cdots = z_d = 0$ in (3.2), it follows that the r.v. $S_{i,n}$ can be expressed as

$$S_{i,n} = M_{i,n} \vee \underline{D}^{(i,n)}, \tag{3.8}$$

where $M_{i,n} \sim \mathrm{Poi}(\lambda_{S_{i,n}})$ with

$$\lambda_{S_{i,n}} = \lambda_i' + \sum_{j<i} \lambda_{ji}' + \sum_{j>i} \lambda_{ij}' + (n-1)\left(\lambda_i + \sum_{j<i} \lambda_{ji} + \sum_{j>i} \lambda_{ij}\right)$$

and $\underline{D}^{(i,n)} = \{D_j^{(i,n)}\}_{j=1}^{\infty}$ is a sequence of i.i.d. r.v.s ($D_j^{(i,n)} \sim D^{(i,n)}$) with common c.d.f.

$$\begin{aligned} F_{D^{(i,n)}}(x) = {} & \frac{\lambda_i' + \sum_{j<i} \lambda_{ji}' + \sum_{j>i} \lambda_{ij}'}{\lambda_{S_{i,n}}} F_{X_{i,n}'}(x) \\ & + \frac{\lambda_i + \sum_{j<i} \lambda_{ji} + \sum_{j>i} \lambda_{ij}}{\lambda_{S_{i,n}}} \sum_{k=2}^{n} F_{X_{i,n-k+1}'}(x). \end{aligned} \tag{3.9}$$

By letting $z_1 = \cdots = z_{i-1} = z_{i+1} = \cdots = z_d = z$ and $z_i = 0$ in (3.2), it follows that the r.v. $S_{-i,n} = \sum_{l=1, l\neq i}^{d} S_{l,n}$ has a compound Poisson distribution, which can be further expressed as a sum of two independent items, that is,

$$S_{-i,n} = J_{-i,n} \vee \underline{D}^{(-i,n)} + J_n^{(i)} \vee \underline{C}^{(-i,n)}. \tag{3.10}$$

where $J_{-i,n} \sim \mathrm{Poi}(\lambda_{J_{-i,n}})$ and $J_n^{(i)} \sim \mathrm{Poi}(\lambda_{J_n^{(i)}})$ with

$$\lambda_{J_{-i,n}} = \sum_{l=1, l\neq i}^{d} \lambda_l' + \sum_{l=1, l\neq i}^{d} \sum_{j>l, j\neq i} \lambda_{lj}' + (n-1)\left(\sum_{l=1, l\neq i}^{d} \lambda_l + \sum_{l=1, l\neq i}^{d} \sum_{j>l, j\neq i} \lambda_{lj}\right)$$

and

$$\lambda_{J_n^{(i)}} = \sum_{l<i} \lambda_{li}' + \sum_{l>i} \lambda_{il}, + (n-1)\left(\sum_{l<i} \lambda_{li} + \sum_{l>i} \lambda_{il}\right),$$

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$\underline{D}^{(-i,n)} = \{D_j^{(-i,n)}\}_{j=1}^{\infty}$ and $\underline{C}^{(-i,n)} = \{C_j^{(-i,n)}\}_{j=1}^{\infty}$ are two independent sequences of i.i.d. r.v.s ($D_j^{(-i,n)} \sim D^{(-i,n)}$, $C_j^{(-i,n)} \sim C^{(-i,n)}$), independent of $\underline{D}^{(i,n)}$ with common c.d.f.s

$$\begin{aligned} F_{D^{(-i,n)}}(x) = &\sum_{l\neq i} \frac{\lambda'_l}{\lambda_{J_{-i,n}}} F_{X'_{l,n}}(x) + \sum_{l\neq i} \sum_{j>l, j\neq i} \frac{\lambda'_{lj}}{\lambda_{J_{-i,n}}} F_{X'_{l,n}+X'_{j,n}}(x) \\ &+ \sum_{l\neq i} \frac{\lambda_l}{\lambda_{J_{-i,n}}} \sum_{k=2}^{n} F_{X'_{l,n-k+1}}(x) \\ &+ \sum_{l\neq i} \sum_{j>l, j\neq i} \frac{\lambda_{lj}}{\lambda_{J_{-i,n}}} \sum_{k=2}^{n} F_{X'_{l,n-k+1}+X'_{j,n-k+1}}(x) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} F_{C^{(-i,n)}}(x) = &\sum_{l<i} \frac{\lambda'_{li}}{\lambda_{J_n^{(i)}}} F_{X'_{l,n}}(x) + \sum_{l>i} \frac{\lambda'_{il}}{\lambda_{J_n^{(i)}}} F_{X'_{l,n}}(x) \\ &+ \sum_{l<i} \frac{\lambda_{li}}{\lambda_{J_n^{(i)}}} \sum_{k=2}^{n} F_{X'_{l,n-k+1}}(x) + \sum_{l>i} \frac{\lambda_{il}}{\lambda_{J_n^{(i)}}} \sum_{k=2}^{n} F_{X'_{l,n-k+1}}(x). \end{aligned} \tag{3.12}$$

We define the pair of r.v.s $(M_{i,n}, M_{-i,n})$, whose components are given by $M_{i,n} = J_{i,n} + J_n^{(i)}$ and $M_{-i,n} = J_{-i,n} + J_n^{(i)}$, where $J_{-i,n}$, $J_n^{(i)}$ and $J_{i,n}$ are independent with $J_{i,n} \sim \text{Poi}(\lambda_{S_{i,n}} - \lambda_{J_n^{(i)}})$. The joint p.m.f. of $(M_{i,n}, M_{-i,n})$ is

$$f_{M_{i,n},M_{-i,n}}(k_i, k_{-i}) = \sum_{k=0}^{\min(k_i,k_{-i})} f_{J_{i,n}}(k_i - k) f_{J_{-i,n}}(k_{-i} - k) f_{J_n^{(i)}}(k), \tag{3.13}$$

for $k_i, k_{-i} \in \mathbb{N}_0$ and $i = 1, ..., d$, where $f_{J_{i,n}}$, $f_{J_{-i,n}}$ and $f_{J_n^{(i)}}$ represent the p.m.f.s of $J_{i,n}$, $J_{-i,n}$ and $J_n^{(i)}$, respectively.

Then, we have all the elements to derive the explicit expression for $\text{TVaR}_\kappa(S_{i,n}; S_n)$.

Proposition 3.3. *Assuming that the number of claims* $\{N_{1,k}, ..., N_{d,k}\}_{k\in\mathbb{N}}$ *follow a basic MINAR(1) process with multivariate Poisson innovations given by (2.4), the expression for* $\text{TVaR}_\kappa(S_{i,n}; S_n)$ *is*

$$\text{TVaR}_\kappa(S_{i,n}; S_n) = \frac{\sum_{k_i=1}^{\infty} \sum_{k_{-i}=0}^{\infty} \sum_{k=0}^{\min(k_i,k_{-i})} f_{J_{i,n}}(k_i - k) f_{J_{-i,n}}(k_{-i} - k) f_{J_n^{(i)}}(k) \Delta_{k_i,k_{-i},k}}{1-\kappa},$$

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where, for $i = 1, ..., d$,

$$\Delta_{k_i,k_{-i},k} = E\left[\left(\sum_{j_1=1}^{k_i} D_{j_1}^{(i,n)}\right) \times \mathbb{1}_{\sum_{j_1=1}^{k_i} D_{j_1}^{(i,n)} + \sum_{j_2=1}^{k-i-k} D_{j_2}^{(-i,n)} + \sum_{j_3=1}^{k} C_{j_3}^{(-i,n)} > \mathrm{TVaR}_\kappa(S_n)}\right].$$

Proof. The desired result follows by using (3.8), (3.10) and (3.13). □

3.3. Mixed Erlang claim sizes

Assume that $X_i \sim \mathrm{MixErl}(\underline{q}_i, \beta)$ with $\underline{q}_i = \{q_{i,j}\}_{j=1}^{\infty}$, for $i = 1, ..., d$. From Lemmas 2.3 and 2.4, the form of $\mathcal{L}_{X'_{i,m}}$ given by (3.4) just corresponds to the LST of a mixed Erlang r.v., that is, $X'_{i,m} \sim \mathrm{MixErl}(\underline{p}^{(i,m)}, \beta)$ with $\underline{p}^{(i,m)} = \{p_j^{(i,m)}\}_{j=1}^{\infty}$, where

$$p_j^{(i,m)} = \begin{cases} \sum_{k=0}^{j-1} p_k^{(\alpha_{ii};\, i,m-1)} q_{i,j-k}, & m > 1, \\ q_{i,j}, & m = 1, \end{cases} \tag{3.14}$$

with $p_0^{(\alpha_{ii};\, i,m)} = 1 - \alpha_{ii}$ and $p_j^{(\alpha_{ii};i,m)} = \alpha_{ii} p_j^{(i,m)}$, for $j \in \mathbb{N}$, $i = 1, ..., d$ and $m = 1, ..., n$.

Then, from Lemma 2.2, the independent sum $X'_{i,m} + X'_{j,m}$ has a mixed Erlang distribution, that is, $X'_{i,m} + X'_{j,m} \sim \mathrm{MixErl}(\underline{\sigma}(\underline{p}^{(i,m)}, \underline{p}^{(j,m)}), \beta)$ with $\underline{\sigma}(\underline{p}^{(i,m)}, \underline{p}^{(j,m)}) = \{\sigma_l(\underline{p}^{(i,m)}, \underline{p}^{(j,m)})\}_{l=1}^{\infty}$, where

$$\sigma_l(\underline{p}^{(i,m)}, \underline{p}^{(j,m)}) = \begin{cases} \sum_{k=1}^{l-1} p_k^{(i,m)} p_{l-k}^{(j,m)}, & l > 1, \\ 0, & l = 1, \end{cases} \tag{3.15}$$

for $m = 1, ..., n$, $i, j = 1, ..., d$ and $i \neq j$.

Therefore, we can identify that the r.v. $D^{(n)}$ with c.d.f. given by (3.7) has a mixed Erlang distribution, that is, $D^{(n)} \sim \mathrm{MixErl}(\underline{d}^{(n)}, \beta)$ with $\underline{d}^{(n)} = \{d_s^{(n)}\}_{s=1}^{\infty}$, where, from (3.14) and (3.15),

$$d_s^{(n)} = \sum_{i=1}^{d} \left(\frac{\lambda'_i}{\lambda_{S_n}} p_s^{(i,n)} + \frac{\sum_{j>i} \lambda'_{ij}}{\lambda_{S_n}} \sigma_s(\underline{p}^{(i,n)}, \underline{p}^{(j,n)}) \right)$$
$$+ \sum_{i=1}^{d} \left(\frac{\lambda_i}{\lambda_{S_n}} \sum_{k=2}^{n} p_s^{(i,n-k+1)} + \frac{\sum_{j>i} \lambda_{ij}}{\lambda_{S_n}} \sum_{k=2}^{n} \sigma_s(\underline{p}^{(i,n-k+1)}, \underline{p}^{(j,n-k+1)}) \right).$$

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According to Lemma 2.5 and Proposition 3.5, it follows that

$$\mathrm{TVaR}_\kappa(S_n) = \frac{\sum_{k=1}^{\infty} f_{M_n'}(k)\frac{k}{\beta}\overline{H}(\mathrm{VaR}_\kappa(S_n); k+1, \beta)}{1-\kappa},$$

where $f_{M_n'}$ represents the p.m.f. of a non-negative integer-valued r.v. M_n', which is determined by the recursive formula

$$f_{M_n'}(k+1) = \frac{\lambda_{S_n}}{k+1}\left(d_1^{(n)} f_{M_n'}(k) + 2d_2^{(n)} f_{M_n'}(k-1) + \cdots + (k+1)d_{k+1}^{(n)} f_{M_n'}(0)\right),$$

for $k \in \mathbb{N}_0$ with $f_{M_n'}(0) = e^{-\lambda_{S_n}}$ and the parameter λ_{S_n} is given by (3.6).

Also, we get that the r.v.s $D^{(i,n)}$, $D^{(-i,n)}$, and $C^{(-i,n)}$ with c.d.f.s given by (3.9), (3.11), and (3.12) have a mixed Erlang distribution, for $i = 1, ..., d$, that is, $D^{(i,n)} \sim \mathrm{MixErl}(\underline{d}^{(i,n)}, \beta)$, $D^{(-i,n)} \sim \mathrm{MixErl}(\underline{d}^{(-i,n)}, \beta)$, and $C^{(-i,n)} \sim \mathrm{MixErl}(\underline{c}^{(-i,n)}, \beta)$ with $\underline{d}^{(i,n)} = \{d_s^{(i,n)}\}_{s=1}^{\infty}$, $\underline{d}^{(-i,n)} = \{d_s^{(-i,n)}\}_{s=1}^{\infty}$ and $\underline{c}^{(-i,n)} = \{c_s^{(-i,n)}\}_{s=1}^{\infty}$, where from (3.14) and (3.15),

$$d_s^{(i,n)} = \frac{\lambda_i' + \sum_{j<i}\lambda_{ji}' + \sum_{j>i}\lambda_{ij}'}{\lambda_{S_{i,n}}} p_s^{(i,n)} + \frac{\lambda_i + \sum_{j<i}\lambda_{ji} + \sum_{j>i}\lambda_{ij}}{\lambda_{S_{i,n}}} \sum_{k=2}^{n} p_s^{(i,n-k+1)},$$

$$\begin{aligned} d_s^{(-i,n)} = &\sum_{l\neq i} \frac{\lambda_l'}{\lambda_{J_{-i,n}}} p_s^{(l,n)} + \sum_{l\neq i}\sum_{j>l, j\neq i} \frac{\lambda_{lj}'}{\lambda_{J_{-i,n}}} \sigma_s(\underline{p}^{(l,n)}, \underline{p}^{(j,n)}) \\ &+ \sum_{l\neq i} \frac{\lambda_l}{\lambda_{J_{-i,n}}} \sum_{k=2}^{n} p_s^{(l,n-k+1)} + \sum_{l\neq i}\sum_{j>l, j\neq i} \frac{\lambda_{lj}}{\lambda_{J_{-i,n}}} \sum_{k=2}^{n} \sigma_s\left(\underline{p}^{(l,n-k+1)}, \underline{p}^{(j,n-k+1)}\right) \end{aligned}$$

and

$$\begin{aligned} c_s^{(-i,n)} = &\sum_{l<i} \frac{\lambda_{li}'}{\lambda_{J_n^{(i)}}} p_s^{(l,n)} + \sum_{l>i} \frac{\lambda_{il}'}{\lambda_{J_n^{(i)}}} p_s^{(l,n)} \\ &+ \sum_{l<i} \frac{\lambda_{li}}{\lambda_{J_n^{(i)}}} \sum_{k=2}^{n} p_s^{(l,n-k+1)} + \sum_{l>i} \frac{\lambda_{il}}{\lambda_{J_n^{(i)}}} \sum_{k=2}^{n} p_s^{(l,n-k+1)}. \end{aligned}$$

According to Proposition 3.6, it follows that

$$\mathrm{TVaR}_\kappa(S_{i,n}; S_n) = \frac{\sum_{k_i=1}^{\infty}\sum_{k_{-i}=0}^{\infty}\sum_{k=0}^{\min(k_i,k_{-i})} f_{J_{i,n}}(k_i-k) f_{J_{-i,n}}(k_{-i}-k) f_{J_n^{(i)}}(k)\Omega_{k_i,k_{-i},k}}{1-\kappa},$$

where, for $i = 1, ..., d$,

$$\Omega_{k_i,k_{-i},k} = \sum_{j=1}^{\infty}\sum_{l=1}^{j-1} d_l^{(i,n)*k_i} \sigma_{j-l}(\underline{d}^{(-i,n)*(k_{-i}-k)}, \underline{c}^{(-i,n)*k}) \frac{l}{\beta}\overline{H}(\mathrm{VaR}_\kappa(S_n); j+1, \beta)$$

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TABLE 1.

VALUES OF $\mathrm{VaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_5)$, AND $\mathrm{TVaR}_\kappa(S_{i,5};S_5)$ $(i=1,2,3)$ WITH $\underline{\alpha}=(0.25,0.2,0.15)$ AND $\underline{\lambda}=(0.2,0.4,0.3)$.

κ	$\mathrm{VaR}_\kappa(S_5)$	$\mathrm{TVaR}_\kappa(S_5)$	$\mathrm{TVaR}_\kappa(S_{1,5};S_5)$	$\mathrm{TVaR}_\kappa(S_{2,5};S_5)$	$\mathrm{TVaR}_\kappa(S_{3,5};S_5)$
0.95	93.238	102.631	39.194	28.138	35.299
0.96	95.539	104.700	39.873	28.794	36.033
0.97	98.403	107.296	40.715	29.624	36.957
0.98	102.270	110.834	41.847	30.768	38.219
0.99	108.505	116.602	43.652	32.664	40.286

TABLE 2.

VALUES OF $\mathrm{VaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_5)$, AND $\mathrm{TVaR}_\kappa(S_{i,5};S_5)$ $(i=1,2,3)$ WITH $\underline{\alpha}=(0.5,0.4,0.3)$ AND $\underline{\lambda}=(0.2,0.4,0.3)$.

κ	$\mathrm{VaR}_\kappa(S_5)$	$\mathrm{TVaR}_\kappa(S_5)$	$\mathrm{TVaR}_\kappa(S_{1,5};S_5)$	$\mathrm{TVaR}_\kappa(S_{2,5};S_5)$	$\mathrm{TVaR}_\kappa(S_{3,5};S_5)$
0.95	125.189	137.457	58.763	36.931	41.764
0.96	128.196	140.160	59.784	37.773	42.603
0.97	131.939	143.551	61.051	38.839	43.660
0.98	136.991	148.169	62.754	40.308	45.107
0.99	145.132	155.695	65.469	42.743	47.483

with

$$\sigma_j(\underline{d}^{(-i,n)*(k_{-i}-k)},\underline{c}^{(-i,n)*k})=\begin{cases}\displaystyle\sum_{s=1}^{j-1} d_s^{(-i,n)*(k_{-i}-k)}c_{j-s}^{(-i,n)*k}, & j>1,\\ 0, & j=1.\end{cases}$$

3.4. Numerical illustration

In this subsection, we consider an insurance portfolio consisting of three lines of business with $n=5$ periods. We provide two examples to numerically illustrate the impact of time and cross-dependence on the closed-form expressions for $\mathrm{VaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_5)$, and $\mathrm{TVaR}_\kappa(S_{i,5};S_5)$, for $i=1,2,3$.

The individual claim sizes are assumed to follow a mixed Erlang distribution, where $X_i\sim\mathrm{MixErl}(\underline{q}_i,\beta)$ with $\underline{q}_i=\{q_{ij}\}_{j=1}^{\infty}$, for $i=1,2,3$. We set $\underline{q}_1=(0.2,0.3,0.5)$, $\underline{q}_2=(0.3,0.4,0.3)$, $\underline{q}_3=(0.5,0.1,0.4)$, and $\beta=1$. The number of claims $\{N_{1,k},\ldots,N_{d,k}\}_{k\in\mathbb{N}}$ are assumed to follow a basic MINAR(1) process with multivariate Poisson innovations given by (2.4).

Example 3.4. Let the parameters $(\lambda_1,\lambda_2,\lambda_3)=(1,0.8,1.2)$. Tables 1–3 display the values of $\mathrm{VaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_5)$, and $\mathrm{TVaR}_\kappa(S_{i,5};S_5)$, for $\kappa=0.95,0.96,0.97,0.98,0.99$ and $i=1,2,3$, with different time dependence parameters $\underline{\alpha}=(\alpha_{11},\alpha_{22},\alpha_{33})$ and cross-dependence parameters

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TABLE 3.

VALUES OF $\mathrm{VaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_5)$, AND $\mathrm{TVaR}_\kappa(S_{i,5}; S_5)$ $(i = 1, 2, 3)$ WITH $\underline{\alpha} = (0.5, 0.4, 0.3)$ AND $\underline{\lambda} = (0.4, 0.8, 0.6)$.

κ	$\mathrm{VaR}_\kappa(S_5)$	$\mathrm{TVaR}_\kappa(S_5)$	$\mathrm{TVaR}_\kappa(S_{1,5}; S_5)$	$\mathrm{TVaR}_\kappa(S_{2,5}; S_5)$	$\mathrm{TVaR}_\kappa(S_{3,5}; S_5)$
0.95	165.202	179.692	67.622	52.857	59.213
0.96	168.768	182.883	68.160	54.167	60.556
0.97	173.798	186.881	68.782	55.857	62.241
0.98	179.169	192.317	69.561	58.197	64.560
0.99	188.766	201.160	70.662	62.101	68.396

$\underline{\lambda} = (\lambda_{12}, \lambda_{13}, \lambda_{23})$. For the purpose of comparison, we set $\underline{\alpha} = (0.25, 0.2, 0.15)$ and $\underline{\lambda} = (0.2, 0.4, 0.3)$ in Table 1, $\underline{\alpha} = (0.5, 0.4, 0.3)$ and $\underline{\lambda} = (0.2, 0.4, 0.3)$ in Table 2, and $\underline{\alpha} = (0.5, 0.4, 0.3)$ and $\underline{\lambda} = (0.4, 0.8, 0.6)$ in Table 3.

From Tables 1–2, we can observe that the values of $\mathrm{VaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_5)$, and $\mathrm{TVaR}_\kappa(S_{i,5}; S_5)$ $(i = 1, 2, 3)$ increase with the time dependence level. Similarly, these values increase when the cross-dependence level increases with a comparison between Tables 2 and 3.

Example 3.5. Let the parameters $(\lambda_1, \lambda_2, \lambda_3) = (1, 0.8, 1.2)$. In Figure 1, the cross-dependence parameters $\underline{\lambda} = (\lambda_{12}, \lambda_{13}, \lambda_{23})$ are fixed to be $(0.2, 0.4, 0.3)$. We calculate the contribution $\mathrm{TVaR}_{0.99}(S_{i,5}; S_5)$ for the i-th line of business $(i = 1, 2, 3)$ and the total of them with different time dependence parameters $\underline{\alpha} = (\alpha_{11}, \alpha_{22}, \alpha_{33})$. We consider the following seven cases: $\underline{\alpha} = (0.75, 0.5, 0.25)$ for case 1, $\underline{\alpha} = (0.75, 0.25, 0.5)$ for case 2, $\underline{\alpha} = (0.5, 0.75, 0.25)$ for case 3, $\underline{\alpha} = (0.5, 0.5, 0.5)$ for case 4, $\underline{\alpha} = (0.5, 0.25, 0.75)$ for case 5, $\underline{\alpha} = (0.25, 0.75, 0.5)$ for case 6, and $\underline{\alpha} = (0.25, 0.5, 0.75)$ for case 7.

Similarly, the time dependence parameters $\underline{\alpha} = (\alpha_{11}, \alpha_{22}, \alpha_{33})$ are fixed to be $(0.5, 0.4, 0.3)$ in Figure 2. The values of $\mathrm{TVaR}_{0.99}(S_{i,5}; S_5)$ $(i = 1, 2, 3)$ and the total of them are presented with different cross-dependence parameters $\underline{\lambda} = (\lambda_{12}, \lambda_{13}, \lambda_{23})$. We also consider seven cases with $\underline{\lambda} = (0.75, 0.5, 0.25)$ for case 1, $\underline{\lambda} = (0.75, 0.25, 0.5)$ for case 2, $\underline{\lambda} = (0.5, 0.75, 0.25)$ for case 3, $\underline{\lambda} = (0.5, 0.5, 0.5)$ for case 4, $\underline{\lambda} = (0.5, 0.25, 0.75)$ for case 5, $\underline{\lambda} = (0.25, 0.75, 0.5)$ for case 6, and $\underline{\lambda} = (0.25, 0.5, 0.75)$ for case 7.

In Figure 1, although the mean of the time dependence parameters is equal to 0.5 for each case, the allocation for i-th line of business $(i = 1, 2, 3)$, and the total of them show a different behavior. This reveals that both the parameters $(\alpha_{11}, \alpha_{22}, \alpha_{33})$ have effect on the contribution of a certain line of business. For example, letting $\alpha_{11} = 0.75$ in the first two cases, the values of $\mathrm{TVaR}_{0.99}(S_{1,5}; S_5)$ are equal to 136.933 with $(\alpha_{22}, \alpha_{33}) = (0.5, 0.25)$ and to 134.329 with $(\alpha_{22}, \alpha_{33}) = (0.25, 0.5)$, respectively. A similar pattern can be observed from Figure 2.

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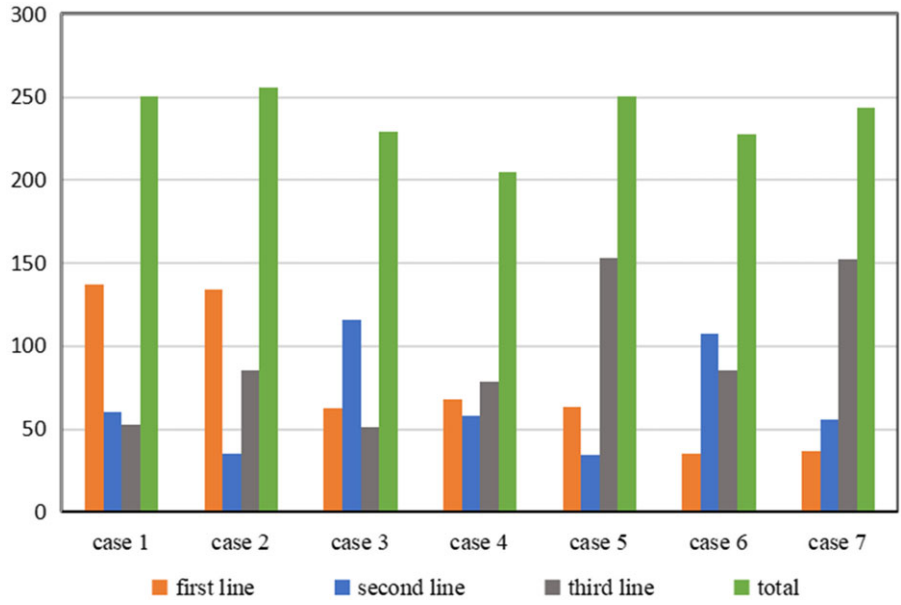


FIGURE 1. Values of $\mathrm{TVaR}_{0.99}(S_{i,5}; S_5)$ $(i = 1, 2, 3)$ and $\mathrm{TVaR}_{0.99}(S_5)$ with different $\underline{\alpha} = (\alpha_{11}, \alpha_{22}, \alpha_{33})$.

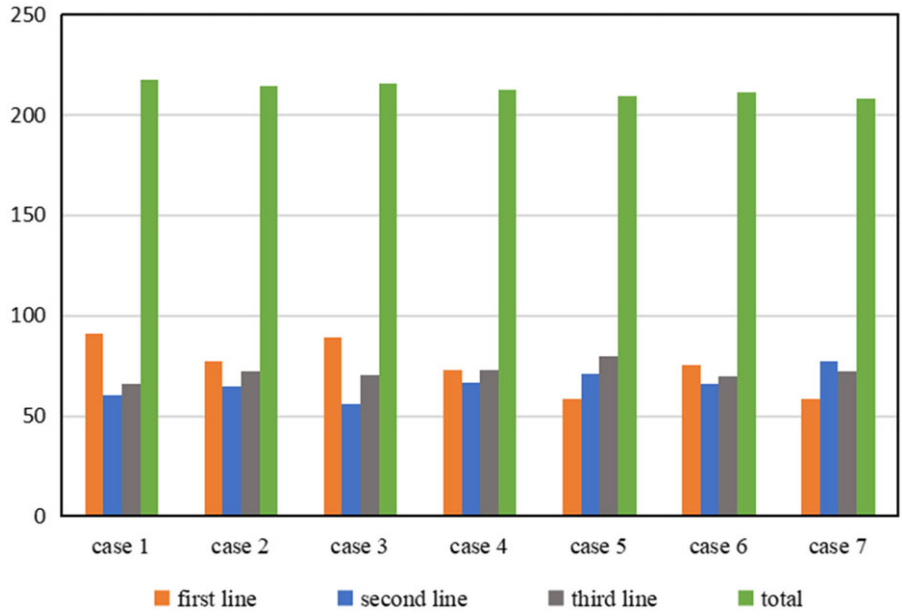


FIGURE 2. Values of $\mathrm{TVaR}_{0.99}(S_{i,5}; S_5)$ $(i = 1, 2, 3)$ and $\mathrm{TVaR}_{0.99}(S_5)$ with different $\underline{\lambda} = (\lambda_{12}, \lambda_{13}, \lambda_{23})$.

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4. BIVARIATE DISTRIBUTION WITH A FULL BINAR(1) PROCESS

In this section, a full BINAR(1) process according to Definition 2.2 is proposed to describe the dependence structure of the number of claims for different lines of business by relaxing the assumption of diagonality of the matrix **A**. We first present a general expression for the joint LST of $(S_{1,n}, S_{2,n})$ in Proposition 4.1 without any distributional assumption on the innovations.

Furthermore, the innovations $\{e_{1,k}, e_{2,k}\}_{k\in\mathbb{N}}$ are assumed to follow a bivariate Poisson distribution according to (2.4), whose joint p.g.f. is

$$p_{e_{1,k},e_{2,k}}(z_1, z_2) = \exp\big(\lambda_1(z_1-1) + \lambda_2(z_2-1) + \lambda_{12}(z_1z_2-1)\big). \tag{4.1}$$

Theoretically, the stationary distribution of $\{N_{1,k}, N_{2,k}\}_{k\in\mathbb{N}}$ can be determined by (2.5) with (4.1). Pedeli and Karlis (2013b) state that $\{N_{1,k}, N_{2,k}\}_{k\in\mathbb{N}}$ is an infinite sum of bivariate Hermite vectors. However, a convenient expression for the joint distribution of $\{N_{1,k}, N_{2,k}\}_{k\in\mathbb{N}}$ is not easy to derive since there are some mathematical difficulties when dealing with this kind of infinite sum. In Lemma 4.1, we provide a method to approximate the stationary distribution of $\{N_{1,k}, N_{2,k}\}_{k\in\mathbb{N}}$.

Lemma 4.1. *If $\{N_{1,k}, N_{2,k}\}_{k\in\mathbb{N}}$ follow a full BINAR(1) process according to Definition 2.2, in which the innovations $\{\mathbf{e}_k\}_{k\in\mathbb{N}}$ have a bivariate Poisson distribution given by (4.1). Then, the joint distribution of $\{N_{1,k}, N_{2,k}\}_{k\in\mathbb{N}}$ can be approximated as a bivariate Hermite distribution with joint p.g.f.*

$$\begin{aligned} p_{N_{1,k},N_{2,k}}(z_1, z_2) = \exp\big(&\omega_1(z_1-1) + \omega_2(z_2-1) + \omega_3(z_1^2-1) \\ &+ \omega_4(z_2^2-1) + \omega_5(z_1z_2-1)\big), \end{aligned}$$

where the parameters $\underline{\omega} = (\omega_1, ..., \omega_5)$ are determined by the following linear equations

$$\begin{cases} \omega_1 + 2\omega_3 + \omega_5 = \mu_1, \\ \omega_2 + 2\omega_4 + \omega_5 = \mu_2, \\ \omega_1 + 4\omega_3 + \omega_5 = \sigma_1^2, \\ \omega_2 + 4\omega_4 + \omega_5 = \sigma_2^2, \\ \omega_5 = \sigma_{12}, \end{cases}$$

with

$$\mu_1 = \mathrm{E}[N_{1,k}] = \frac{(1-\alpha_{22})(\lambda_1+\lambda_{12}) + \alpha_{12}(\lambda_2+\lambda_{12})}{(1-\alpha_{11})(1-\alpha_{22}) - \alpha_{12}\alpha_{21}},$$

$$\mu_2 = \mathrm{E}[N_{2,k}] = \frac{(1-\alpha_{11})(\lambda_2+\lambda_{12}) + \alpha_{21}(\lambda_1+\lambda_{12})}{(1-\alpha_{11})(1-\alpha_{22}) - \alpha_{12}\alpha_{21}},$$

$$\begin{aligned} \sigma_1^2 &= \mathrm{Var}(N_{1,k}) \\ &= \frac{\alpha_{12}^2\sigma_2^2 + 2\alpha_{11}\alpha_{12}\sigma_{12} + \alpha_{11}(1-\alpha_{11})\mu_1 + \alpha_{12}(1-\alpha_{12})\mu_2 + (\lambda_1+\lambda_{12})}{1-\alpha_{11}^2}, \end{aligned}$$

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$$\sigma_2^2 = \mathrm{Var}(N_{2,k}) = \frac{\alpha_{21}^2\sigma_1^2 + 2\alpha_{21}\alpha_{22}\sigma_{12} + \alpha_{22}(1-\alpha_{22})\mu_2 + \alpha_{21}(1-\alpha_{21})\mu_1 + (\lambda_2+\lambda_{12})}{1-\alpha_{22}^2}$$

and

$$\sigma_{12} = \mathrm{Cov}(N_{1,k}, N_{2,k}) = \frac{\alpha_{11}\alpha_{21}\sigma_1^2 + \alpha_{12}\alpha_{22}\sigma_2^2 + \lambda_{12}}{1-\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}}.$$

Proof. Note that the expressions for $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$, and σ_{12} have been given by Pedeli and Karlis (2013b). The proof then follows by matching the first two moments of $(N_{1,k}, N_{2,k})$. □

Remark 4.1. *Numerical illustrations gathered in Online Appendix suggest that the method proposed by Lemma 4.1 works well. It admits that the marginals of the full BINAR(1) process follow a univariate Hermite distribution. This leads to the BINAR(1) process allowing overdispersion, which coincides with the full BINAR(1) processes considered in Pedeli and Karlis (2013b) and Bermudez et al. (2018).*

Within this framework, we provide explicit expressions for the TVaR of S_n and the contribution of $S_{i,n}$ under the TVaR-based allocation rule. Closed-form expressions are also derived with mixed Erlang claim sizes.

For reasons of simplicity, this section only focuses on the bivariate case. The derivation for the multivariate case with $d > 2$ is not straightforward since the complexity of the process increases sharply.

4.1. Joint LST of $(S_{1,n}, S_{2,n})$

Proposition 4.1. *Assuming that the number of claims $\{N_{1,k}, N_{2,k}\}_{k\in\mathbb{N}}$ follow a full BINAR(1) process, the joint LST of $(S_{1,n}, S_{2,n})$ is given by*

$$\begin{aligned}\mathcal{L}_{S_{1,n},S_{2,n}}(z_1,z_2) = {} & p_{N_{1,1},N_{2,1}}\left(G_{1,n}(\mathcal{L}_{X_1}(z_1),\mathcal{L}_{X_2}(z_2)), G_{2,n}(\mathcal{L}_{X_1}(z_1),\mathcal{L}_{X_2}(z_2))\right)\\ & \times \prod_{k=2}^{n} p_{e_{1,k},e_{2,k}}(G_{1,n-k+1}(\mathcal{L}_{X_1}(z_1),\mathcal{L}_{X_2}(z_2)),\\ & \times G_{2,n-k+1}(\mathcal{L}_{X_1}(z_1),\mathcal{L}_{X_2}(z_2))),\end{aligned} \tag{4.2}$$

where

$$G_{1,m}(z_1,z_2) = \begin{cases} \prod_{i=1}^{2}\left(\alpha_{i1}G_{i,m-1}(z_1,z_2) + 1 - a_{i1}\right) z_1, & m > 1,\\ z_1, & m = 1,\end{cases} \tag{4.3}$$

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and

$$G_{2,m}(z_1,z_2)=\begin{cases}\prod_{i=1}^{2}\left(\alpha_{i2}G_{i,m-1}(z_1,z_2)+1-a_{i2}\right)z_2, & m>1,\\ z_2, & m=1,\end{cases} \tag{4.4}$$

for $m=1,\ldots,n$.

Proof. See Appendix. □

Remark 4.2. *By letting* $z_1=z_2=z$ *in (4.2),* $G_{i,m}(\mathcal{L}_{X_1}(z),\mathcal{L}_{X_2}(z))$ *from (4.3) and (4.4) just corresponds to the LST of a r.v. associated with* X_1 *and* X_2*, for* $i=1,2$ *and* $m=1,\ldots,n$*. Without loss of generality, suppose that* $\{X'_{i,m}\}_{m=1}^{n}$ *is a sequence of independent r.v.s such that the LST of* $X'_{i,m}$ *is*

$$\mathcal{L}_{X'_{i,m}}(z)=G_{i,m}(\mathcal{L}_{X_1}(z),\mathcal{L}_{X_2}(z)). \tag{4.5}$$

This shows that $\mathcal{L}_{S_{1,n},S_{2,n}}(z,z)$ *corresponds to the LST of a sum of n independent compound r.v.s.*

Under the assumption of bivariate Poisson innovations, from Lemma 4.1, it follows that the r.v. S_n is compound Poisson distributed, which can be expressed as

$$S_n=M_n\vee\underline{D}^{(n)}, \tag{4.6}$$

where $M_n\sim\text{Poi}(\lambda_{S_n})$ with

$$\lambda_{S_n}=\sum_{k=1}^{5}\omega_k+(n-1)(\lambda_1+\lambda_2+\lambda_{12}) \tag{4.7}$$

and $\underline{D}^{(n)}=\{D_j^{(n)}\}_{j=1}^{\infty}$ is a sequence of i.i.d. r.v.s ($D_j^{(n)}\sim D^{(n)}$) with common c.d.f.

$$\begin{aligned}F_{D^{(n)}}(x)=&\frac{\omega_1}{\lambda_{S_n}}F_{X'_{1,n}}(x)+\frac{\omega_3}{\lambda_{S_n}}F_{X'_{1,n}+\tilde{X}'_{1,n}}(x)+\frac{\lambda_1}{\lambda_{S_n}}\sum_{k=2}^{n}F_{X'_{1,n-k+1}}(x)\\&+\frac{\omega_2}{\lambda_{S_n}}F_{X'_{2,n}}(x)+\frac{\omega_4}{\lambda_{S_n}}F_{X'_{2,n}+\tilde{X}'_{2,n}}(x)+\frac{\lambda_2}{\lambda_{S_n}}\sum_{k=2}^{n}F_{X'_{2,n-k+1}}(x)\\&+\frac{\omega_5}{\lambda_{S_n}}F_{X'_{1,n}+X'_{2,n}}(x)+\frac{\lambda_{12}}{\lambda_{S_n}}\sum_{k=2}^{n}F_{X'_{1,n-k+1}+X'_{2,n-k+1}}(x),\end{aligned} \tag{4.8}$$

and $\tilde{X}'_{i,m}$ being the independent copy of $X'_{i,m}$, for $i=1,2$ and $m=1,\ldots,n$.

Proposition 4.2. *Assuming that the number of claims* $\{N_{1,k},N_{2,k}\}_{k\in\mathbb{N}}$ *follow a full BINAR(1) process with bivariate Poisson innovations, the expression for*

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$\mathrm{TVaR}_\kappa(S_n)$ *is given by*

$$\mathrm{TVaR}_\kappa(S_n) = \frac{\sum_{k=1}^{\infty} f_{M_n}(k) E\Big[\sum_{j=1}^{k} D_j^{(n)} \times \mathbb{1}_{\sum_{j=1}^{k} D_j^{(n)} > \mathrm{VaR}_\kappa(S_n)}\Big]}{1-\kappa},$$

where f_{M_n} *represents the p.m.f. of* M_n.

Proof. Combining with (4.6) (4.7) and (4.8), the proof follows. □

4.2. TVaR-based capital allocation

We now turn our focus to the expressions for the c.d.f.s of $S_{i,n}$ and $S_{-i,n}$, for $i = 1, 2$. Combining with (4.3) and (4.4), suppose that $\{X_{j,m}^{(1)}\}_{m=1}^{n}$ and $\{X_{j,m}^{(2)}\}_{m=1}^{n}$ form two sequences of independent r.v.s such that

$$\mathcal{L}_{X_{j,m}^{(1)}}(z) = G_{j,m}(\mathcal{L}_{X_1}(z), 1) \tag{4.9}$$

and

$$\mathcal{L}_{X_{j,m}^{(2)}}(z) = G_{j,m}(1, \mathcal{L}_{X_2}(z)), \tag{4.10}$$

for $j = 1, 2$ and $m = 1, ..., n$.

By letting $(z_1, z_2) = (z, 0)$ or $(z_1, z_2) = (0, z)$ in (4.2), it implies that $\mathcal{L}_{S_{1,n},S_{2,n}}(z, 0)$ or $\mathcal{L}_{S_{1,n},S_{2,n}}(0, z)$ corresponds to the LST of a finite sum of independent compound Poisson r.v.s, that is,

$$S_{i,n} = M_{i,n} \vee \underline{D}^{(i,n)}, \tag{4.11}$$

where $M_{i,n} \sim \mathrm{Poi}(\lambda_{S_{i,n}})$ with $\lambda_{S_{i,n}} = \lambda_{S_n}$ given by (4.7) and $\underline{D}^{(i,n)} = \{D_j^{(i,n)}\}_{j=1}^{\infty}$ is a sequence of i.i.d. r.v.s ($D_j^{(i,n)} \sim D^{(i,n)}$) with common c.d.f.

$$\begin{aligned} F_{D^{(i,n)}}(x) = {} & \frac{\omega_1}{\lambda_{S_{i,n}}} F_{X_{1,n}^{(i)}}(x) + \frac{\omega_3}{\lambda_{S_{i,n}}} F_{X_{1,n}^{(i)} + \tilde{X}_{1,n}^{(i)}}(x) + \frac{\lambda_1}{\lambda_{S_{i,n}}} \sum_{k=2}^{n} F_{X_{1,n-k+1}^{(i)}}(x) \\ & + \frac{\omega_2}{\lambda_{S_{i,n}}} F_{X_{2,n}^{(i)}}(x) + \frac{\omega_4}{\lambda_{S_{i,n}}} F_{X_{2,n}^{(i)} + \tilde{X}_{2,n}^{(i)}}(x) + \frac{\lambda_2}{\lambda_{S_{i,n}}} \sum_{k=2}^{n} F_{X_{2,n-k+1}^{(i)}}(x) \\ & + \frac{\omega_5}{\lambda_{S_{i,n}}} F_{X_{1,n}^{(i)} + X_{2,n}^{(i)}}(x) + \frac{\lambda_{12}}{\lambda_{S_{i,n}}} \sum_{k=2}^{n} F_{X_{1,n-k+1}^{(i)} + X_{2,n-k+1}^{(i)}}(x), \end{aligned} \tag{4.12}$$

and $\tilde{X}_{j,m}^{(i)}$ being the independent copy of $X_{j,m}^{(i)}$, for $i = 1, 2$, $j = 1, 2$ and $m = 1, ..., n$.

Similarly, it follows that $S_{-i,n}$ has a compound Poisson distribution. As proceed in Section 3.2, $S_{-i,n}$ can be expressed as a sum of two independent items, that is,

$$S_{-i,n} = J_{-i,n} \vee \underline{D}^{(-i,n)} + J_n \vee \underline{C}^{(-i,n)}. \tag{4.13}$$

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In (4.13), $J_n \sim \text{Poi}(\lambda_{J_n})$ and $J_{-i,n} \sim \text{Poi}(\lambda_{J_{-i,n}})$ with $\lambda_{J_n} = \omega_5 + (n-1)\lambda_{12}$ and $\lambda_{J_{-i,n}} = \sum_{k=1}^{4} \omega_k + (n-1)(\lambda_1 + \lambda_2)$, $\underline{D}^{(-i,n)} = \{D_j^{(-i,n)}\}_{j=1}^{\infty}$ and $\underline{C}^{(-i,n)} = \{C_j^{(-i,n)}\}_{j=1}^{\infty}$ are two independent sequences of i.i.d. r.v.s ($D_j^{(-i,n)} \sim D^{(-i,n)}$, $C_j^{(-i,n)} \sim C^{(-i,n)}$), independent of $\underline{D}^{(i,n)}$, with common c.d.f.s

$$F_{D^{(-i,n)}}(x) = \frac{\omega_1}{\lambda_{J_{-i,n}}} F_{X_{1,n}^{(-i)}}(x) + \frac{\omega_3}{\lambda_{J_{-i,n}}} F_{X_{1,n}^{(-i)} + \tilde{X}_{1,n}^{(-i)}}(x) + \frac{\lambda_1}{\lambda_{J_{-i,n}}} \sum_{k=2}^{n} F_{X_{1,n-k+1}^{(-i)}}(x)$$
$$+ \frac{\omega_2}{\lambda_{J_{-i,n}}} F_{X_{2,n}^{(-i)}}(x) + \frac{\omega_4}{\lambda_{J_{-i,n}}} F_{X_{2,n}^{(-i)} + \tilde{X}_{2,n}^{(-i)}}(x) + \frac{\lambda_2}{\lambda_{J_{-i,n}}} \sum_{k=2}^{n} F_{X_{2,n-k+1}^{(-i)}}(x) \tag{4.14}$$

and

$$F_{C^{(-i,n)}} = \frac{\omega_5}{\lambda_{J_n}} F_{X_{1,n}^{(-i)} + X_{2,n}^{(-i)}}(x) + \frac{\lambda_{12}}{\lambda_{J_n}} \sum_{k=2}^{n} F_{X_{1,n-k+1}^{(-i)} + X_{2,n-k+1}^{(-i)}}(x). \tag{4.15}$$

In (4.14) and (4.15), for $j = 1, 2$ and $k = 1, ..., n$, $X_{j,k}^{(-i)} = X_{j,k}^{(2)}$ if $i = 1$, $X_{j,k}^{(-i)} = X_{j,k}^{(1)}$ if $i = 2$, and $\tilde{X}_{j,n}^{(-i)}$ is the independent copy of $X_{j,n}^{(-i)}$.

We then define the pair of r.v.s $(M_{i,n}, M_{-i,n})$, whose components are given by $M_{i,n} = J_{i,n} + J_n$ and $M_{-1,n} = J_{-i,n} + J_n$, where J_n, $J_{-i,n}$ and $J_{i,n}$ are independent and $J_{i,n} \sim \text{Poi}\big(\sum_{k=1}^{4} \omega_k + (n-1)(\lambda_1 + \lambda_2)\big)$, $i = 1, 2$. The joint p.m.f. of $(M_{i,n}, M_{-i,n})$ is given by

$$f_{M_{i,n}, M_{-i,n}}(k_1, k_2) = \sum_{k=0}^{\min(k_1, k_2)} f_{J_{i,n}}(k_1 - k) f_{J_{-i,n}}(k_2 - k) f_{J_n}(k), \tag{4.16}$$

for $k_1, k_2 \in \mathbb{N}_0$ and $i = 1, 2$, where $f_{J_n}, f_{J_{i,n,}}$ and $f_{J_{-i,n}}$ represent the p.m.f.s of J_n, $J_{i,n,}$ and $J_{-i,n}$, respectively.

In summary, we have all the elements to find the expression for $\text{TVaR}_\kappa(S_{i,n}; S_n)$.

Proposition 4.3. *Assuming that the number of claims $\{N_{1,k}, N_{2,k}\}_{k \in \mathbb{N}}$ follow a full BINAR(1) process with bivariate Poisson innovations, the expression for $\text{TVaR}_\kappa(S_{i,n}; S_n)$ is given by*

$$\text{TVaR}_\kappa(S_{i,n}; S_n) = \frac{\sum_{k_i=1}^{\infty} \sum_{k_{-i}=0}^{\infty} \sum_{k=0}^{\min(k_i, k_{-i})} f_{J_{i,n}}(k_i - k) f_{J_{-i,n}}(k_{-i} - k) f_{J_n}(k) \Delta_{k_i, k_{-i}, k}}{1 - \kappa},$$

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where, for $i = 1, 2$,

$$\Delta_{k_i,k_{-i},k} = \mathrm{E}\left[\left(\sum_{j_1=1}^{k_i} D_{j_1}^{(i,n)}\right) \times \mathbb{1}_{\sum_{j_1=1}^{k_i} D_{j_1}^{(i,n)} + \sum_{j_2=1}^{k_{-i}-k} D_{j_2}^{(-i,n)} + \sum_{j_3=1}^{k} C_{j_3}^{(-i,n)} > \mathrm{TVaR}_k(S_n)}\right].$$

Proof. Using (4.11), (4.13) and (4.16), the desired result follows. □

4.3. Mixed Erlang claim sizes

Assume that $X_i \sim \mathrm{MixErl}(\underline{q}_i, \beta)$ with $\underline{q}_i = \{q_{i,j}\}_{j=1}^{\infty}$, for $i = 1, 2$. According to Lemmas 2.3 and 2.4, the form of $\mathcal{L}_{X'_{i,m}}$ given by (4.5) implies that the r.v. $X'_{i,m}$ has a mixed Erlang distribution, that is, $X'_{i,m} \sim \mathrm{MixErl}(\underline{p}^{(i,m)}, \beta)$ with $\underline{p}^{(i,m)} = \{p_j^{(i,m)}\}_{j=1}^{\infty}$, where

$$p_j^{(i,m)} = \begin{cases} \displaystyle\sum_{k=0}^{j-1} \sigma_k(\underline{p}^{(\alpha_{i1};\,1,m-1)}, \underline{p}^{(\alpha_{i2};\,2,m-1)}) q_{i,j-k}, & m > 1, \\ q_{i,j}, & m = 1. \end{cases} \tag{4.17}$$

In (4.17), $\underline{p}^{(\alpha_{il};\,l,m)} = \{p_j^{(\alpha_{il};\,l,m)}\}_{j=0}^{\infty}$ and $\underline{\sigma}(\underline{p}^{(\alpha_{i1};\,1,m)}, \underline{p}^{(\alpha_{i2};\,2,m)}) = \{\sigma_j(\underline{p}^{(\alpha_{i1};\,1,m)}, \underline{p}^{(\alpha_{i2};\,2,m)})\}_{j=0}^{\infty}$ where

$$p_j^{(\alpha_{il};\,l,m)} = \begin{cases} \alpha_{il} p_j^{(l,m)}, & j \geq 1, \\ 1 - \alpha_{il}, & j = 0, \end{cases}$$

and $\sigma_j(\underline{p}^{(\alpha_{i1};\,1,m)}, \underline{p}^{(\alpha_{i2};\,2,m)}) = \displaystyle\sum_{k=0}^{j} p_k^{(\alpha_{i1};1,m)} p_{j-k}^{(\alpha_{i2};2,m)}$, for $i, l = 1, 2$, and $m = 1, ..., n$.

From Lemma 2.4, the r.v.s $X'_{i,m} + \tilde{X}'_{i,m}$ and $X'_{1,m} + X'_{2,m}$ in (4.8) have a mixed Erlang distribution, that is, $X'_{i,m} + \tilde{X}'_{i,m} \sim \mathrm{MixErl}(\underline{\sigma}(\underline{p}^{(i,m)}, \underline{p}^{(i,m)}), \beta)$ and $X'_{1,m} + X'_{2,m} \sim \mathrm{MixErl}(\underline{\sigma}(\underline{p}^{(1,m)}, \underline{p}^{(2,m)}), \beta)$, where $\underline{\sigma}(\underline{p}^{(i,m)}, \underline{p}^{(i,m)}) = \{\sigma_j(\underline{p}^{(i,m)}, \underline{p}^{(i,m)})\}_{j=1}^{\infty}$ and $\underline{\sigma}(\underline{p}^{(1,m)}, \underline{p}^{(2,m)}) = \{\sigma_j(\underline{p}^{(1,m)}, \underline{p}^{(2,m)})\}_{j=1}^{\infty}$, with

$$\sigma_j(\underline{p}^{(i,m)}, \underline{p}^{(i,m)}) = \begin{cases} \displaystyle\sum_{k=1}^{j-1} p_k^{(i,m)} p_{j-k}^{(i,m)}, & j > 1, \\ 0, & j = 1, \end{cases} \tag{4.18}$$

and

$$\sigma_j(\underline{p}^{(1,m)}, \underline{p}^{(2,m)}) = \begin{cases} \displaystyle\sum_{k=1}^{j-1} p_k^{(1,m)} p_{j-k}^{(2,m)}, & j > 1, \\ 0, & j = 1, \end{cases} \tag{4.19}$$

for $i = 1, 2$ and $m = 1, ..., n$.

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Based on the above analysis, we have that the r.v. $D^{(n)}$ with c.d.f. given by (4.8) has a mixed Erlang distribution, that is,

$$D^{(n)} \sim \text{MixErl}(\underline{d}^{(n)}, \beta) \tag{4.20}$$

with $\underline{d}^{(n)} = \{d_j^{(n)}\}_{j=1}^{\infty}$, where, from (4.17), (4.18) and (4.19),

$$\begin{aligned} d_j^{(n)} =& \frac{\omega_1}{\lambda_{S_n}} p_j^{(1,n)} + \frac{\omega_3}{\lambda_{S_n}} \sigma_j(\underline{p}^{(1,n)}, \underline{p}^{(1,n)}) + \frac{\lambda_1}{\lambda_{S_n}} \sum_{k=2}^{n} p_j^{(1,n-k+1)} \\ &+ \frac{\omega_2}{\lambda_{S_n}} p_j^{(2,n)} + \frac{\omega_4}{\lambda_{S_n}} \sigma_j(\underline{p}^{(2,n)}, \underline{p}^{(2,n)}) + \frac{\lambda_2}{\lambda_{S_n}} \sum_{k=2}^{n} p_j^{(2,n-k+1)} \\ &+ \frac{\omega_5}{\lambda_{S_n}} \sigma_j(\underline{p}^{(1,n)}, \underline{p}^{(2,n)}) + \frac{\lambda_{12}}{\lambda_{S_n}} \sum_{k=2}^{n} \sigma_j(\underline{p}^{(1,n-k+1)}, \underline{p}^{(2,n-k+1)}). \end{aligned} \tag{4.21}$$

According to Lemma 2.5 and Proposition 4.2, from (4.20) and (4.21), the expression for $\text{TVaR}_\kappa(S_n)$ follows

$$\text{TVaR}_\kappa(S_n) = \frac{\sum_{k=1}^{\infty} f_{M_n'}(k) \frac{k}{\beta} \overline{H}(\text{VaR}_\kappa(S_n); k+1, \beta)}{1-\kappa}, \tag{4.22}$$

where $f_{M_n'}$ represents the p.m.f. of a non-negative integer-valued r.v. M_n', which is determined by the recursive formula

$$f_{M_n'}(k+1) = \frac{\lambda_{S_n}}{k+1} \left(d_1^{(n)} f_{M_n'}(k) + 2 d_2^{(n)} f_{M_n'}(k-1) + \cdots + (k+1) d_{k+1}^{(n)} f_{M_n'}(0) \right),$$

for $k \in \mathbb{N}_0$ with $f_{M_n'}(0) = e^{-\lambda_{S_n}}$ and the parameter λ_{S_n} is given by (4.7).

Applying the same methodology, we can get that the r.v. $X_{k,m}^{(i)}$ in (4.12) has a mixed Erlang distribution with a (possible) point mass at zero, for $i, k = 1, 2$ and $m = 1, ..., n$, that is, $X_{k,m}^{(i)} \sim \text{MixErl}(\underline{p}^{(i;k,m)}, \beta)$.

For $i = 1$, $\underline{p}^{(1;1,m)} = \{p_j^{(1;1,m)}\}_{j=1}^{\infty}$ and $\underline{p}^{(1;2,m)} = \{p_j^{(1;2,m)}\}_{j=0}^{\infty}$ where

$$p_j^{(1;1,m)} = \begin{cases} \sum_{k=0}^{j-1} \sigma_k(\underline{p}^{(\alpha_{11};\,1;\,1,m-1)}, \underline{p}^{(\alpha_{12};\,1;2,m-1)}) q_{1,j-k}, & m > 1, \\ q_{1,j}, & m = 1, \end{cases} \tag{4.23}$$

and

$$p_j^{(1;2,m)} = \begin{cases} \sigma_j(\underline{p}^{(\alpha_{21};\,1;1,m-1)}, \underline{p}^{(\alpha_{22};\,1;2,m-1)}), & m > 1, \\ \mathbb{1}_{\{j=0\}}, & m = 1. \end{cases} \tag{4.24}$$

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For $i=2$, $\underline{p}^{(2;1,m)}=\{p_j^{(2;1,m)}\}_{j=0}^{\infty}$ and $\underline{p}^{(2;2,m)}=\{p_j^{(2;2,m)}\}_{j=1}^{\infty}$ where

$$p_j^{(2;1,m)}=\begin{cases}\sigma_j(\underline{p}^{(\alpha_{11};\,2;\,1,m-1)},\underline{p}^{(\alpha_{12};\,2;\,2,m-1)}), & m>1,\\ \mathbb{1}_{\{j=0\}}, & m=1,\end{cases} \tag{4.25}$$

and

$$p_j^{(2;2,m)}=\begin{cases}\displaystyle\sum_{k=0}^{j-1}\sigma_k(\underline{p}^{(\alpha_{21};\,2;\,1,m-1)},\underline{p}^{(\alpha_{22};\,2;\,2,m-1)})q_{2,j-k}, & m>1,\\ q_{2,j}, & m=1.\end{cases} \tag{4.26}$$

In (4.23)–(4.26), $\underline{p}^{(\alpha_{ik};\,l;\,k,m)}=\{p_j^{(\alpha_{ik};\,l;\,k,m)}\}_{j=0}^{\infty}$, where

$$p_j^{(\alpha_{ik};\,l;\,k,m)}=\begin{cases}\alpha_{ik}p_j^{(l;k,m)}, & j\geq 1,\\ 1-\alpha_{ik}, & j=0,\end{cases}$$

and $\underline{\sigma}(\underline{p}^{(\alpha_{i1};l;\,1,m)},\underline{p}^{(\alpha_{i2};\,l;\,2,m)})=\{\sigma_j(\underline{p}^{(\alpha_{i1};l;\,1,m)},\underline{p}^{(\alpha_{i2};\,l;\,2,m)})\}_{j=0}^{\infty}$, where

$$\sigma_j(\underline{p}^{(\alpha_{i1};\,l;\,1,m)},\underline{p}^{(\alpha_{i2};\,l;\,2,m)})=\sum_{s=0}^{j}p_s^{(\alpha_{i1};l;\,1,m)}p_{j-s}^{(\alpha_{i2};l;\,2,m)},$$

for $i,k,l=1,2$ and $m=1,\ldots,n$.

According to (4.9) and (4.10), we have that the r.v.s $X_{k,n}^{(i)}+\tilde{X}_{k,n}^{(i)}$ and $X_{1,m}^{(i)}+X_{2,m}^{(i)}$, for $k=1,2$ and $m=1,\ldots,n$, follow a mixed Erlang distribution on the positive real line and a (possible) point mass at zero. Let $X_{k,n}^{(i)}+\tilde{X}_{k,n}^{(i)}\sim$ MixErl$(\underline{\sigma}(\underline{p}^{(i;k,n)},\underline{p}^{(i;k,n)}),\beta)$ and $X_{1,m}^{(i)}+X_{2,m}^{(i)}\sim$ MixErl$(\underline{\sigma}(\underline{p}^{(i;1,m)},\underline{p}^{(i;2,m)},\beta)$.

For $i=1$, $\underline{\sigma}(\underline{p}^{(1;1,n)},\underline{p}^{(1;1,n)})=\{\sigma_j(\underline{p}^{(1;1,n)}),\underline{p}^{(1;1,n)})\}_{j=1}^{\infty}$ with

$$\sigma_j(\underline{p}^{(1;1,n)},\underline{p}^{(1;1,n)})=\begin{cases}\displaystyle\sum_{k=1}^{j-1}p_k^{(1;\,1,n)}p_{j-k}^{(1;\,1,n)}, & j>1,\\ 0, & j=1,\end{cases}$$

and $\underline{\sigma}(\underline{p}^{(1;2,n)}),\underline{p}^{(1;2,n)})=\{\sigma_j(\underline{p}^{(1;2,n)},\underline{p}^{(1;2,n)})\}_{j=0}^{\infty}$ with

$$\sigma_j(\underline{p}^{(1;2,n)},\underline{p}^{(1;2,n)})=\sum_{k=0}^{j}p_k^{(1;\,2,n)}p_{j-k}^{(1;\,2,n)}.$$

For $i=2$, $\underline{\sigma}(\underline{p}^{(2;1,n)},\underline{p}^{(2;1,n)})=\{\sigma_j(\underline{p}^{(2;1,n)}),\underline{p}^{(2;1,n)})\}_{j=0}^{\infty}$ with

$$\sigma_j(\underline{p}^{(2;1,n)},\underline{p}^{(2;\,1,n)})=\sum_{k=0}^{j}p_k^{(2;\,1,n)}p_{j-k}^{(2;\,1,n)},$$

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and $\underline{\sigma}(\underline{p}^{(2;\,2,n)}), \underline{p}^{(2;\,2,n)}) = \{\sigma_j(\underline{p}^{(2;\,2,n)}, \underline{p}^{(2;2,n)})\}_{j=1}^{\infty}$ with

$$\sigma_j(\underline{p}^{(2;\,2,n)}, \underline{p}^{(2;\,2,n)}) = \begin{cases} \sum_{k=1}^{j-1} p_k^{(2;\,2,n)} p_{j-k}^{(2;\,2,n)}, & j > 1, \\ 0, & j = 1. \end{cases}$$

For $i = 1, 2$ and $m = 1, ..., n$, $\underline{\sigma}(\underline{p}^{(i;1,m)}, \underline{p}^{(i;2,m)}) = \{\sigma_j(\underline{p}^{(i;1,m)}, \underline{p}^{(i;2,m)})\}_{j=1}^{\infty}$ with

$$\sigma_j(\underline{p}^{(i;1,m)}, \underline{p}^{(i;2,m)}) = \sum_{k=1}^{j} p_k^{(i;1,m)} p_{j-k}^{(i;2,m)}.$$

Based on the above analysis, from (4.12), (4.14), and (4.15), we have that, for $i = 1, 2$, $D^{(i,n)} \sim \text{MixErl}(\underline{d}^{(i,n)}, \beta)$, $D^{(-i,n)} \sim \text{MixErl}(\underline{d}^{(-i,n)}, \beta)$, and $C^{(-i,n)} \sim \text{MixErl}(\underline{c}^{(-i,n)}, \beta)$ with $\underline{d}^{(i,n)} = \{d_j^{(i,n)}\}_{j=0}^{\infty}$, $\underline{d}^{(-i,n)} = \{d_j^{(-i,n)}\}_{j=0}^{\infty}$ and $\underline{c}^{(-i,n)} = \{c_j^{(-i,n)}\}_{j=1}^{\infty}$.

For $j > 0$, we have

$$\begin{aligned} d_j^{(i,n)} =& \frac{\omega_1}{\lambda_{S_{i,n}}} p_j^{(i;1,n)} + \frac{\omega_3}{\lambda_{S_{i,n}}} \sigma_j(\underline{p}^{(i;1,n)}, \underline{p}^{(i;1,n)}) + \frac{\lambda_1}{\lambda_{S_{i,n}}} \sum_{k=2}^{n} p_j^{(i;1,n-k+1)} \\ &+ \frac{\omega_2}{\lambda_{S_{i,n}}} p_j^{(i;2,n)} + \frac{\omega_4}{\lambda_{S_{i,n}}} \sigma_j(\underline{p}^{(i;2,n)}, \underline{p}^{(i;2,n)}) + \frac{\lambda_2}{\lambda_{S_{i,n}}} \sum_{k=2}^{n} p_j^{(i;2,n-k+1)} \\ &+ \frac{\omega_5}{\lambda_{S_{i,n}}} \sigma_j(\underline{p}^{(i;1,n)}, \underline{p}^{(i;2,n)}) + \frac{\lambda_{12}}{\lambda_{S_{i,n}}} \sum_{k=2}^{n} \sigma_j(\underline{p}^{(i;1,n-k+1)}, \underline{p}^{(i;2,n-k+1)}), \end{aligned}$$

$$\begin{aligned} d_j^{(-i,n)}(x) =& \frac{\omega_1}{\lambda_{J_{-i,n}}} p_j^{(-i;1,n)} + \frac{\omega_3}{\lambda_{J_{-i,n}}} \sigma_j(\underline{p}^{(-i;1,n)}, \underline{p}^{(-i;1,n)}) + \frac{\lambda_1}{\lambda_{J_{-i,n}}} \sum_{k=2}^{n} p_j^{(-i;1,n-k+1)} \\ &+ \frac{\omega_2}{\lambda_{J_{-i,n}}} p_j^{(-i;2,n)} + \frac{\omega_4}{\lambda_{J_{-i,n}}} \sigma_j(\underline{p}^{(-i;2,n)}, \underline{p}^{(-i;2,n)}) + \frac{\lambda_2}{\lambda_{J_{-i,n}}} \sum_{k=2}^{n} p_j^{(-i;2,n-k+1)} \end{aligned}$$

and

$$c_j^{(-i,n)} = \frac{\omega_5}{\lambda_{J_n}} \sigma_j(\underline{p}^{(-i;1,n)}, \underline{p}^{(-i;2,n)}) + \frac{\lambda_{12}}{\lambda_{J_n}} \sum_{k=2}^{n} \sigma_j(\underline{p}^{(-i;1,n-k_1)}, \underline{p}^{(-i;2,n-k+1)}),$$

where, for $k = 1, 2$ and $m = 1, ..., n$, the superscript $(-i; k, m)$ of $\underline{p}^{(-i;k,m)}$ denotes $(1; k, m)$ for $i = 2$ and $(2; k, m)$ for $i = 1$, respectively.

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For $j=0$,

$$d_0^{(i,n)}=\begin{cases}\dfrac{\omega_2}{\lambda_{S_{1,n}}}p_0^{(1;2,n)}+\dfrac{\omega_4}{\lambda_{S_{1,n}}}\sigma_0(\underline{p}^{(1;2,n)},\underline{p}^{(1;2,n)})+\dfrac{\lambda_2}{\lambda_{S_{1,n}}}\displaystyle\sum_{k=2}^{n}p_0^{(1;2,n-k+1)}, & i=1,\\ \dfrac{\omega_1}{\lambda_{S_{2,n}}}p_0^{(2;1,n)}+\dfrac{\omega_3}{\lambda_{S_{2,n}}}\sigma_0(\underline{p}^{(2;1,n)},\underline{p}^{(2;1,n)})+\dfrac{\lambda_1}{\lambda_{S_{2,n}}}\displaystyle\sum_{k=2}^{n}p_0^{(2;1,n-k+1)}, & i=2,\end{cases}$$

and

$$d_0^{(-i,n)}=\begin{cases}\dfrac{\omega_1}{\lambda_{J_{-i,n}}}p_0^{(2;1,n)}+\dfrac{\omega_3}{\lambda_{J_{-i,n}}}\sigma_0(\underline{p}^{(2;1,n)},\underline{p}^{(2;1,n)})+\dfrac{\lambda_1}{\lambda_{J_{-i,n}}}\displaystyle\sum_{k=2}^{n}p_0^{(2;1,n-k+1)}, & i=1,\\ \dfrac{\omega_2}{\lambda_{J_{-i,n}}}p_0^{(1;2,n)}+\dfrac{\omega_4}{\lambda_{J_{-i,n}}}\sigma_0(\underline{p}^{(1;2,n)},\underline{p}^{(1;2,n)})+\dfrac{\lambda_2}{\lambda_{J_{-i,n}}}\displaystyle\sum_{k=2}^{n}p_0^{(1;2,n-k+1)}, & i=2.\end{cases}$$

Finally, according to Proposition 4.3, it follows that

$$\mathrm{TVaR}_\kappa(S_{i,n};S_n)=\frac{\displaystyle\sum_{k_i=1}^{\infty}\sum_{k_{-i}=0}^{\infty}\sum_{k=0}^{\min(k_i,k_{-i})}f_{J_{i,n}}(k_i-k)f_{J_{-i,n}}(k_{-i}-k)f_{J_n}(k)\Omega_{k_i,k_{-i},k}}{1-\kappa},$$

where, for $i=1,2$,

$$\Omega_{k_i,k_{-i},k}=\sum_{j=1}^{\infty}\sum_{l=1}^{j-1}d_l^{(i,n)*k_i}\sigma_{j-l}(\underline{d}^{(-i,n)*(k_{-i}-k)},\underline{c}^{(-i,n)*k})\frac{l}{\beta}\overline{H}(\mathrm{VaR}_\kappa(S_n);j+1,\beta)$$

with $\sigma_j(\underline{d}^{(-i,n)*(k_{-i}-k)},\underline{c}^{(-i,n)*k})=\displaystyle\sum_{s=0}^{j-1}d_s^{(-i,n)*(k_{-i}-k)}c_{j-s}^{(-i,n)*k}$.

4.4. Numerical illustration

In this subsection, we consider an insurance portfolio consisting of two lines of business with $n=5$ periods. We provide two examples to numerically illustrate the impact of cross-time dependence on the closed-form expressions for $\mathrm{VaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_{1,5};S_5)$ and $\mathrm{TVaR}_\kappa(S_{2,5};S_5)$.

The individual claim sizes are assumed to follow a mixed Erlang distribution, where $X_i\sim\mathrm{MixErl}(\underline{q}_i,\beta)$ with $\underline{q}_i=\{q_{ij}\}_{j=1}^{\infty}$, for $i=1,2$. We set $\underline{q}_1=(0.2,0.3,0.5)$, $\underline{q}_2=(0.3,0.4,0.3)$ and $\beta=1$. The number of claims $\{N_{1,k},N_{2,k}\}_{k\in\mathbb{N}}$ are assumed to follow a full BINAR(1) process with bivariate Poisson innovations.

Example 4.4. Let the parameters $(\lambda_1,\lambda_2,\lambda_{12})=(1.2,0.8,0.2)$ and $(\alpha_{11},\alpha_{22})=(0.5,0.4)$. Tables 4–6 contain the values of $\mathrm{VaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_{1,5};S_5)$, and $\mathrm{TVaR}_\kappa(S_{2,5};S_5)$, for $\kappa=0.95,0.96,0.97,0.98,0.99$, with different cross-time dependence parameters $(\alpha_{12},\alpha_{21})$. For the purpose of

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TABLE 4.

VALUES OF $\mathrm{VaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_{1,5}; S_5)$, AND $\mathrm{TVaR}_\kappa(S_{2,5}; S_5)$ WITH $(\alpha_{12}, \alpha_{21}) = (0.2, 0.1)$.

κ	$\mathrm{VaR}_\kappa(S_5)$	$\mathrm{TVaR}_\kappa(S_5)$	$\mathrm{TVaR}_\kappa(S_{1,5}; S_5)$	$\mathrm{TVaR}_\kappa(S_{2,5}; S_5)$
0.95	109.801	123.511	77.476	46.035
0.96	113.129	126.536	79.840	46.696
0.97	117.286	130.341	82.552	47.789
0.98	122.925	135.544	85.457	50.087
0.99	132.072	144.067	90.153	53.914

TABLE 5.

VALUES OF $\mathrm{VaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_{1,5}; S_5)$, AND $\mathrm{TVaR}_\kappa(S_{2,5}; S_5)$ WITH $(\alpha_{12}, \alpha_{21}) = (0.3, 0.2)$.

κ	$\mathrm{VaR}_\kappa(S_5)$	$\mathrm{TVaR}_\kappa(S_5)$	$\mathrm{TVaR}_\kappa(S_{1,5}; S_5)$	$\mathrm{TVaR}_\kappa(S_{2,5}; S_5)$
0.95	145.047	163.100	98.147	64.953
0.96	149.428	167.079	100.762	66.317
0.97	154.901	172.089	103.771	68.318
0.98	162.324	178.939	107.900	71.039
0.99	174.363	190.162	114.020	76.142

TABLE 6.

VALUES OF $\mathrm{VaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_{1,5}; S_5)$, AND $\mathrm{TVaR}_\kappa(S_{2,5}; S_5)$ WITH $(\alpha_{12}, \alpha_{21}) = (0.4, 0.3)$.

κ	$\mathrm{VaR}_\kappa(S_5)$	$\mathrm{TVaR}_\kappa(S_5)$	$\mathrm{TVaR}_\kappa(S_{1,5}; S_5)$	$\mathrm{TVaR}_\kappa(S_{2,5}; S_5)$
0.95	214.447	239.890	137.358	102.529
0.96	220.764	245.483	140.537	104.946
0.97	228.646	252.465	144.240	108.225
0.98	239.320	261.866	150.061	111.805
0.99	256.595	276.673	158.152	118.521

comparison, we set $(\alpha_{12}, \alpha_{21}) = (0.2, 0.1)$ in Table 4, $(\alpha_{12}, \alpha_{21}) = (0.3, 0.2)$ in Table 5, and $(\alpha_{12}, \alpha_{21}) = (0.4, 0.3)$ in Table 6.

From Tables 4–6, we can observe that the values of $\mathrm{VaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_{1,5}; S_5)$, and $\mathrm{TVaR}_\kappa(S_{2,5}; S_5)$ significantly increase with the cross-time dependence level.

Example 4.5. In Table 7, we compare the values of $\mathrm{VaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_{1,5}; S_5)$, and $\mathrm{TVaR}_\kappa(S_{2,5}; S_5)$ using a basic BINAR(1) process and a full BINAR(1) process for $\kappa = 0.95, 0.96, 0.97, 0.98, 0.99$, respectively. For the basic BINAR(1) process, we set the parameters $(\alpha_{11}, \alpha_{22}) = (0.0349, 0.0627)$ and $(\lambda_1, \lambda_2, \lambda_{12}) = (0.0601, 0.0768, 0.0138)$. For the full BINAR(1) process, we set the parameters

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TABLE 7.

COMPARISON OF $\mathrm{VaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_{1,5};S_5)$, AND $\mathrm{TVaR}_\kappa(S_{2,5};S_5)$ WITH DIFFERENT BINAR(1) PROCESSES.

Model	κ	$\mathrm{VaR}_\kappa(S_5)$	$\mathrm{TVaR}_\kappa(S_5)$	$\mathrm{TVaR}_\kappa(S_{1,5};S_5)$	$\mathrm{TVaR}_\kappa(S_{2,5};S_5)$
Basic BINAR(1)	0.95	7.456	9.977	4.914	5.063
	0.96	8.041	10.541	5.195	5.346
	0.97	8.785	11.259	5.552	5.707
	0.98	9.818	12.258	6.049	6.209
	0.99	11.544	13.935	6.882	7.053
Full BINAR(1)	0.95	8.150	10.678	5.257	5.421
	0.96	8.773	11.282	5.558	5.724
	0.97	9.562	12.046	5.936	6.110
	0.98	10.634	13.087	6.456	6.631
	0.99	12.406	14.811	7.294	7.487

$$\mathbf{A} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} 0.033 & 0.058 \\ 0.075 & 0.063 \end{pmatrix}$$

and $(\lambda_1, \lambda_2, \lambda_{12}) = (0.0583, 0.0756, 0.0137)$. Note that all of these parameters are estimated from an identical automobile insurance database, which contains information for 14,386 policyholders with two lines of business. Please see Table 3 of Bermúdez et al. (2018) for more details.

Table 7 shows that the values of $\mathrm{VaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_5)$, $\mathrm{TVaR}_\kappa(S_{1,5};S_5)$, and $\mathrm{TVaR}_\kappa(S_{2,5};S_5)$ using the basic BINAR(1) process are less than the corresponding ones using the full BINAR(1) process. This reminds us to be aware of additional capital needed for the portfolio if a more complicated structure, allowing for cross-time dependence, is drew into the model.

5. CONCLUSIONS

In this paper, we have utilized the class of MINAR(1) processes to model dependence structures between the number of claims of an insurance portfolio. Following Bermúdez *et al*. (2018), we have focused our attention on three sources of dependence: cross-dependence, time dependence, and cross-time dependence. By developing the joint LST of the aggregate claim amount for different lines of business, we have derived the closed-form expressions for the TVaR of the aggregate claim amount and the contribution of each line of business under the TVaR-based allocation rule. The findings of this work reveal that different dependence structures incorporated into the models give significantly different risk aggregation and capital allocation patterns. An interesting avenue for future research is to study how this class of processes can be used to analyze the predictive distribution of aggregate claims for an insurance portfolio.

The present work only considers the multivariate Poisson distributions as the innovation of the MINAR(1) processes. The assumption of multivariate

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Poisson innovation might be extended with other innovations. It should be noted that Propositions 3.1 and 4.1 are still valid with any other innovations. However, such extension needs more thought for deriving the risk aggregation and capital allocation formulas since the stationary distribution of the MINAR(1) process is not easy to obtain, even in the bivariate case. Another extension of this work might introduce a higher order MINAR process to construct a more general dependence structure, while the mathematical representation of the produced model may be quite complicated.

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Supplementary Material

To view supplementary material for this article, please visit https://doi.org/10.1017/asb.2022.8.

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APPENIDX A. PROOFS

Proof of Proposition 3.1. From (1.2), we can rewrite $S_{i,n}$ $(i = 1, ..., d)$ as

$$S_{i,n} = W_{i,1} + \cdots + W_{i,n} = \sum_{j=1}^{M_{i,n}} \tilde{X}_{i,j},$$

where $M_{i,n} = N_{i,1} + \cdots + N_{i,n}$ and $\{\tilde{X}_{i,j}\}_{j=1}^{\infty}$ form a sequence of i.i.d. r.v.s distributed as X_i, for $i = 1, ..., d$.

The joint LST of $(S_{1,n}, ..., S_{d,n})$ can be expressed as

$$\begin{aligned}\mathcal{L}_{S_{1,n},...,S_{d,n}}(z_1, ..., z_d) &= \mathrm{E}\Big[\mathcal{L}_{X_1}(z_1)^{N_{1,1}} \cdots \mathcal{L}_{X_1}(z_1)^{N_{1,n}} \cdots \\ &\qquad \times \mathcal{L}_{X_d}(z_d)^{N_{d,1}} \cdots \mathcal{L}_{X_d}(z_d)^{N_{d,n}}\Big] \\ &= \mathrm{E}\Big[\mathcal{L}_{X_1}(z_1)^{M_{1,n}} \cdots \mathcal{L}_{X_d}(z_d)^{M_{d,n}}\Big] \\ &= p_{M_{1,n},...,M_{d,n}}(\mathcal{L}_{X_1}(z_1), ..., \mathcal{L}_{X_d}(z_d)). \end{aligned} \tag{A1}$$

In order to find $\mathcal{L}_{S_{1,n},...,S_{d,n}}(z_1, ..., z_d)$, we need to derive the expression for $p_{M_{1,n},...,M_{d,n}}(z_1, ..., z_d)$, that is, the joint p.g.f. of $(M_{1,n}, ..., M_{d,n})$. We first consider the cases for $n = 1, 2$ and then for the general case $n > 2$.

For $n = 1$,

$$\begin{aligned} p_{M_{1,1},...,M_{d,1}}(z_1, ..., z_d) &= \mathrm{E}\Big[z_1^{N_{1,1}} \cdots z_d^{N_{d,1}}\Big] \\ &= p_{N_{1,1},...,N_{d,1}}(z_1, ..., z_d) \\ &= p_{N_{1,1},...,N_{d,1}}(G_{1,1}(z_1), ..., G_{d,1}(z_d)). \end{aligned} \tag{A2}$$

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For $n = 2$

$$\begin{aligned} p_{M_{1,2},\ldots,M_{d,2}}(z_1, \ldots, z_d) &= \mathrm{E}\left[z_1^{N_{1,1}+N_{1,2}} \cdots z_d^{N_{d,1}+N_{d,2}}\right] \\ &= \mathrm{E}\left[z_1^{N_{1,1}} z_1^{\alpha_{11}\circ N_{1,1}+e_{1,1}} \cdots z_d^{N_{d,1}} z_d^{\alpha_{dd}\circ N_{d,1}+e_{d,1}}\right] \\ &= \mathrm{E}\left[z_1^{N_{1,1}+\alpha_{11}\circ N_{1,1}} \cdots z_d^{N_{d,1}+\alpha_{dd}\circ N_{d,1}}\right]\mathrm{E}\left[z_1^{e_{1,2}} \cdots z_d^{e_{d,2}}\right]. \end{aligned} \tag{A3}$$

Note that

$$\begin{aligned} \mathrm{E}\left[z_1^{N_{1,1}+\alpha_{11}\circ N_{1,1}} \cdots z_d^{N_{d,1}+\alpha_{dd}\circ N_{d,1}}\right] &= \mathrm{E}\left[z_1^{N_{1,1}}(\alpha_{11}z_1 + 1 - \alpha_{11})^{N_{1,1}} \cdots z_d^{N_{d,1}}\right. \\ &\quad \left. \times (\alpha_{dd}z_d + 1 - \alpha_{dd})^{N_{d,1}}\right] \\ &= p_{N_{1,1},\ldots,N_{d,1}}(G_{1,2}(z_1), \ldots, G_{d,2}(z_d)) \end{aligned} \tag{A4}$$

and

$$\mathrm{E}\left[z_1^{e_{1,2}} \cdots z_d^{e_{d,2}}\right] = p_{e_{1,2},\ldots,e_{d,2}}(G_{1,1}(z_1), \ldots, G_{d,1}(z_d)). \tag{A5}$$

From (A4) and (A5), (A3) becomes

$$\begin{aligned} p_{M_{1,2},\ldots,M_{d,2}}(z_1, \ldots, z_d) &= p_{N_{1,1},\ldots,N_{d,1}}(G_{1,2}(z_1), \ldots, \\ &\quad G_{d,2}(z_d))p_{e_{1,2},\ldots,e_{d,2}}(G_{1,1}(z_1), \ldots, G_{d,1}(z_d)). \end{aligned} \tag{A6}$$

For $n > 2$,

$$\begin{aligned} p_{M_{1,n},\ldots,M_{d,n}}(z_1, \ldots, z_d) &= \mathrm{E}\left[z_1^{N_{1,1}+N_{1,2}+\cdots+N_{1,n}} \cdots z_d^{N_{d,1}+N_{d,2}+\cdots+N_{d,n}}\right] \\ &= \mathrm{E}\left[z_1^{N_{1,1}+\alpha_{11}\circ N_{1,1}+\cdots+\alpha_{11}^{n-1}\circ N_{1,1}} \cdots \right. \\ &\quad \left. z_d^{N_{d,1}+\alpha_{dd}\circ N_{d,1}+\cdots+\alpha_{dd}^{n-1}\circ N_{d,1}}\right] \\ &\quad \times \mathrm{E}\left[z_1^{e_{1,2}+\alpha_{11}\circ e_{1,2}+\cdots+\alpha_{11}^{n-2}\circ e_{1,2}} \cdots \right. \\ &\quad \left. z_d^{e_{d,2}+\alpha_{dd}\circ e_{d,2}+\cdots+\alpha_{dd}^{n-2}\circ e_{d,2}}\right] \\ &\quad \times \cdots \times \mathrm{E}\left[z_1^{e_{1,n-1}+\alpha_{11}\circ e_{1,n-1}} \cdots z_d^{e_{d,n-1}+\alpha_{dd}\circ e_{d,n-1}}\right] \\ &\quad \times \mathrm{E}\left[z_1^{e_{1,n}} \cdots z_d^{e_{d,n}}\right]. \end{aligned} \tag{A7}$$

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Note that

$$\begin{aligned}
&\mathrm{E}\Big[z_1^{N_{1,1}+\alpha_{11}\circ N_{1,1}+\cdots+\alpha_{11}^{n-1}\circ N_{1,1}}\cdots z_d^{N_{d,1}+\alpha_{dd}\circ N_{d,1}+\cdots+\alpha_{dd}^{n-1}\circ N_{d,1}}\Big]\\
&=\mathrm{E}\Big[z_1^{N_{11}+\cdots+\alpha_{11}^{n-2}\circ N_{11}}(\alpha_{11}z_1+1-\alpha_{11})^{\alpha_{11}^{n-2}\circ N_{11}}\times\cdots\\
&\quad\times z_d^{N_{d1}+\cdots+\alpha_{dd}^{n-2}\circ N_{d1}}(\alpha_{dd}z_d+1-\alpha_{dd})^{\alpha_{dd}^{n-2}\circ N_{d1}}\Big]\\
&=\mathrm{E}\Big[z_1^{N_{11}+\cdots+\alpha_{11}^{n-3}\circ N_{11}}(G_{1,2}(z_1))^{\alpha_{11}^{n-2}\circ N_{11}}\times\cdots\\
&\quad\times z_d^{N_{d1}+\cdots+\alpha_{dd}^{n-3}\circ N_{d1}}(G_{d,2}(z_d))^{\alpha_{dd}^{n-2}\circ N_{d1}}\Big]\\
&=\mathrm{E}\Big[z_1^{N_{11}+\cdots+\alpha_{11}^{n-3}\circ N_{11}}(\alpha_{11}G_{1,2}(z_1)+1-\alpha_{11})^{\alpha_{11}^{n-3}\circ N_{11}}\times\cdots\\
&\quad\times z_d^{N_{d1}+\cdots+\alpha_{dd}^{n-3}\circ N_{d1}}(\alpha_{dd}G_{d,2}(z_d)+1-\alpha_{dd})^{\alpha_{dd}^{n-3}\circ N_{d1}}\Big].
\end{aligned}$$

Recursively, the above expression follows

$$\begin{aligned}
&\mathrm{E}\Big[z_1^{N_{11}}(G_{1,n-1}(z_1))^{\alpha_{11}\circ N_{11}}\cdots z_d^{N_{d1}}(G_{d,n-1}(z_d))^{\alpha_{dd}\circ N_{d1}}\Big]\\
&=\mathrm{E}\Big[z_1^{N_{11}}(\alpha_{11}G_{1,n-1}(z_1)+1-\alpha_{11})^{N_{11}}\cdots z_d^{N_{d1}}(\alpha_{dd}G_{d,n-1}(z_d)+1-\alpha_{dd})^{N_{d1}}\Big]\\
&=\mathrm{E}\Big[G_{1,n}(z_1)^{N_{11}}\cdots G_{d,n}(z_d)^{N_{d,1}}\Big]\\
&=p_{N_{1,1},\ldots,N_{d,1}}(G_{1,n}(z_1),\ldots,G_{d,n}(z_d)).
\end{aligned}\tag{A8}$$

Similarly, for any $k=0,1,\ldots,n-2$, we have

$$\begin{aligned}
&\mathrm{E}\Big[z_1^{e_{1,n-k}+\cdots+\alpha_{11}^{k}\circ e_{1,n-k}}\cdots z_d^{e_{d,n-k}+\cdots+\alpha_{dd}^{k}\circ e_{d,n-k}}\Big]\\
&\quad=p_{e_{1,n-k},\ldots,e_{d,n-k}}(G_{1,k+1}(z_1),\ldots,G_{d,k+1}(z_d)).
\end{aligned}\tag{A9}$$

From (A8) and (A9), (A7) becomes

$$\begin{aligned}
p_{M_{1,n},\ldots,M_{d,n}}(z_1,\ldots,z_d)&=p_{N_{1,1},\ldots,N_{d,1}}(G_{1,n}(z_1),\ldots,G_{d,n}(z_d))\\
&\quad\times\prod_{k=2}^{n}p_{e_{1,k},\ldots,e_{d,k}}(G_{1,n-k+1}(z_1),\ldots,G_{d,n-k+1}(z_d)).
\end{aligned}\tag{A10}$$

From (A1), the desired result follows by substituting $\mathcal{L}_{X_1}(z_1),\ldots,\mathcal{L}_{X_d}(z_d)$ for $z_1,\ldots,z_d$ in (A2), (A6) and (A10).

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This completes the proof.

Proof of Proposition 4.1. As proceed in the proof of Proposition 3.1, we can rewrite $S_{i,n}$ $(i=1,2)$ as

$$S_{i,n} = W_{i,1} + \cdots + W_{i,n} = \sum_{j=1}^{M_{i,n}} \tilde{X}_{i,j},$$

where $M_{i,n} = N_{i,1} + \cdots + N_{i,n}$ and $\{\tilde{X}_{i,j}\}_{j=1}^{\infty}$ form a sequence of i.i.d. r.v.s distributed as X_i, for $i=1,2$.

The joint LST of $(S_{1,n}, S_{2,n})$ is

$$\begin{aligned}\mathcal{L}_{S_{1,n},S_{2,n}}(z_1,z_2) &= \mathrm{E}\Big[\mathcal{L}_{X_1}(z_1)^{N_{1,1}} \cdots \mathcal{L}_{X_1}(z_1)^{N_{1,n}} \mathcal{L}_{X_2}(z_2)^{N_{2,1}} \cdots \mathcal{L}_{X_2}(z_2)^{N_{2,n}}\Big] \\ &= \mathrm{E}\Big[\mathcal{L}_{X_1}(z_1)^{M_{1,n}} \mathcal{L}_{X_2}(z_2)^{M_{2,n}}\Big] \\ &= p_{M_{1,n},M_{2,n}}(\mathcal{L}_{X_1}(z_1), \mathcal{L}_{X_2}(z_2)). \end{aligned} \tag{A11}$$

For $n=1$,

$$\begin{aligned} p_{M_{1,1},M_{2,1}}(z_1,z_2) &= \mathrm{E}\Big[z_1^{N_{1,1}} z_2^{N_{2,1}}\Big] \\ &= p_{N_{1,1},N_{2,1}}(z_1,z_2) \\ &= p_{N_{1,1},N_{2,1}}(G_{1,1}(z_1,z_2), G_{2,1}(z_1,z_2)). \end{aligned} \tag{A12}$$

For $n=2$,

$$\begin{aligned} p_{M_{1,2},M_{2,2}}(z_1,z_2) &= \mathrm{E}\Big[z_1^{N_{1,1}+N_{1,2}} z_2^{N_{2,1}+N_{2,2}}\Big] \\ &= \mathrm{E}\Big[z_1^{N_{1,1}} z_1^{\alpha_{11}\circ N_{1,1}+\alpha_{12}\circ N_{2,1}+e_{1,1}} z_2^{N_{2,1}} z_2^{\alpha_{21}\circ N_{1,1}+\alpha_{22}\circ N_{2,1}+e_{2,1}}\Big] \\ &= \mathrm{E}\Big[z_1^{N_{1,1}+\alpha_{11}\circ N_{1,1}} z_2^{\alpha_{21}\circ N_{1,1}} z_1^{\alpha_{12}\circ N_{2,1}} z_2^{N_{2,1}+\alpha_{22}\circ N_{2,1}}\Big] \mathrm{E}\Big[z_1^{e_{1,1}} z_2^{e_{2,}}\Big]. \end{aligned} \tag{A13}$$

Note that

$$\begin{aligned} &\mathrm{E}\Big[z_1^{N_{1,1}+\alpha_{11}\circ N_{1,1}} z_2^{\alpha_{21}\circ N_{1,1}} z_1^{\alpha_{12}\circ N_{2,1}} z_2^{N_{2,1}+\alpha_{22}\circ N_{2,1}}\Big] \\ &= \mathrm{E}\Big[z_1^{N_{1,1}} (\alpha_{11} z_1 + 1 - \alpha_{11})^{N_{1,1}} (\alpha_{21} z_2 + 1 - \alpha_{21})^{N_{1,1}} \\ &\quad \times z_2^{N_{2,1}} (\alpha_{12} z_1 + 1 - \alpha_{12})^{N_{2,1}} (\alpha_{22} z_2 + 1 - \alpha_{22})^{N_{2,1}}\Big] \\ &= p_{N_{1,1},N_{2,1}}(G_{1,2}(z_1,z_2), G_{2,2}(z_1,z_2)). \end{aligned} \tag{A14}$$

and

$$\mathrm{E}\Big[z_1^{e_{1,2}} z_2^{e_{2,2}}\Big] = p_{e_{1,2},e_{2,2}}(G_{1,1}(z_1), G_{2,1}(z_2)). \tag{A15}$$

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From (A14) and (A15), (A13) becomes

$$p_{M_{1,2},M_{2,2}}(z_1,z_2)=p_{N_{1,1},N_{2,1}}(G_{1,2}(z_1,z_2),G_{2,2}(z_1,z_2))p_{e_{1,2},e_{2,2}}(G_{1,1}(z_1,z_2),G_{2,1}(z_1,z_2)). \tag{A16}$$

For $n>2$,

$$\begin{aligned}
&p_{M_{1,n},M_{2,n}}(z_1,z_2) \qquad\qquad (A17)\\
&=\mathrm{E}\left[z_1^{N_{1,1}+\cdots+N_{1,n-1}+N_{1,n}}z_2^{N_{2,1}+\cdots+N_{2,n-1}+N_{2,n}}\right]\\
&=\mathrm{E}\left[z_1^{N_{1,1}+\cdots+N_{1,n-1}+\alpha_{11}\circ N_{1,n-1}+\alpha_{12}\circ N_{2,n-1}+e_{1,n}}\right.\\
&\quad\left.\times z_2^{N_{2,1}+\cdots+N_{2,n-1}+\alpha_{21}\circ N_{1,n-1}+\alpha_{22}\circ N_{2,n-1}+e_{2,n}}\right]\\
&=\mathrm{E}\left[z_1^{N_{1,1}+\cdots+N_{1,n-1}+\alpha_{11}\circ N_{1,n-1}}z_2^{\alpha_{21}\circ N_{1,n-1}}\right.\\
&\quad\left.\times z_1^{\alpha_{12}\circ N_{2,n-1}}z_2^{N_{2,1}+\cdots+N_{2,n-1}+\alpha_{22}\circ N_{2,n-1}}\right]\\
&\quad\times\mathrm{E}\left[z_1^{e_{1,n}}z_2^{e_{2,n}}\right]\\
&=\mathrm{E}\left[z_1^{N_{1,1}+\cdots+N_{1,n-2}}(G_{1,2}(z_1,z_2))^{N_{1,n-1}}z_2^{N_{2,1}+\cdots+N_{2,n-2}}(G_{2,2}(z_1,z_2))^{N_{2,n-1}}\right]\\
&\quad\times p_{e_{1,n},e_{2,n}}(G_{1,1}(z_1,z_2),G_{2,1}(z_1,z_2))\\
&=\mathrm{E}\left[z_1^{N_{1,1}+\cdots+N_{1,n-2}}(G_{1,2}(z_1,z_2))^{\alpha_{11}\circ N_{1,n-2}+\alpha_{12}\circ N_{2,n-2}+e_{1,n-1}}\right.\\
&\quad\left.\times z_2^{N_{2,1}+\cdots+N_{2,n-2}}(G_{2,2}(z_1,z_2))^{\alpha_{21}\circ N_{1,n-2}+\alpha_{22}\circ N_{2,n-2}+e_{2,n-1}}\right]\\
&\quad\times p_{e_{1,n},e_{2,n}}(G_{1,1}(z_1,z_2),G_{2,1}(z_1,z_2))\\
&=\mathrm{E}\left[z_1^{N_{1,1}+\cdots+N_{1,n-2}}(G_{1,2}(z_1,z_2))^{\alpha_{11}\circ N_{1,n-2}}(G_{2,2}(z_1,z_2))^{\alpha_{21}\circ N_{1,n-2}}\right.\\
&\quad\left.\times z_2^{N_{2,1}+\cdots+N_{2,n-2}}(G_{1,2}(z_1,z_2))^{\alpha_{12}\circ N_{2,n-2}}(G_{2,2}(z_1,z_2))^{\alpha_{22}\circ N_{2,n-2}}\right]\\
&\quad\times p_{e_{1,n-1},e_{2,n-1}}(G_{1,2}(z_1,z_2),G_{2,2}(z_1,z_2))\\
&\quad\times p_{e_{1,n},e_{2,n}}(G_{1,1}(z_1,z_2),G_{2,1}(z_1,z_2)).
\end{aligned}$$

Recursively, (A17) becomes

$$p_{M_{1,n},M_{2,n}}(z_1,z_2)=p_{N_{1,1},N_{2,1}}(G_{1,n}(z_1,z_2),G_{2,n}(z_1,z_2))\times\prod_{k=2}^{n}p_{e_{1,k},e_{2,k}}(G_{1,n-k+1}(z_1,z_2),G_{2,n-k+1}(z_1,z_2)). \tag{A18}$$

From (A11), the proof follows by substituting $\mathcal{L}_{X_1}(z_1)$ and $\mathcal{L}_{X_2}(z_2)$ for z_1 and z_2 in (A12), (A16) and (A18).

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