Proc. Camb. Phil. Soc. (1974), **75**, 249
With 3 text-figures
PCPS 75–27
Printed in Great Britain

The static two body problem*

BY H. MÜLLER ZUM HAGEN†
King's College, London

(*Received* 29 *January* 1973)

Abstract. The problem of whether there exist static solutions of Einstein's equations for two extended bodies surrounded by vacuum is investigated. Equilibrium conditions for the general static situation are derived and their physical meaning discussed. Special cases of the problem are solved by using these equilibrium conditions. In one of these cases, the essential assumptions are that the space time is axisymmetric and that the matter of which the two bodies are composed consists of perfect fluid. Solutions are also obtained in which these restrictions are weakened.

1. *Introduction.* One of the important properties of the gravitational field is *attractiveness*. One can study attractiveness by considering the *focusing* effect on test-particles (congruence of geodesics) in a space-time which fulfils the energy inequality (positivity of [energy density + pressure], section 3 of (1)). This focusing effect manifests itself in the fact that neighbouring geodesics intersect or produce caustics (see p. 542 in (1)). One of the consequences of this focusing effect is the existence of singularities (theorem of Hawking and Penrose (1)).

Another way of studying the attractiveness of the gravitational field consists of considering the following situation: Take a space-time, which contains two bodies surrounded by a vacuum, and assume that these bodies are 'at rest' initially ($t = 0$). Then the space-time should develop in such a way that – due to their mutual attraction – these two bodies will get closer and closer and finally collide.

The phrase 'at rest' is supposed to mean: at rest with respect to an 'inertial frame of reference'. According to (2) it is physically very reasonable to define an 'inertial frame of reference' as a time-like congruence whose tangent vectorfield X^a fulfils

$$X_{(a;\,b)} = 0, \quad X_{[a,\,b}X_{c]} = 0. \tag{1·1a}$$

This means that our space time has to be *static*. And this implies automatically that the two bodies are at rest with respect to the above timelike congruence X^a, since

$$\underset{X}{\mathfrak{L}} T_{ab} = 0,\ddagger \quad T_{ab}X^b = 0. \tag{1·1b}$$

This is so because the dynamics is determined by the metric structure: (1·1*b*) follows from the field equations and (1·1*a*).

* This work is based on the author's Ph.D. thesis (Hamburg, 1970).

† Present address: I. Institut für theoretische Physik der Universität Hamburg, West Germany.

‡ $\mathfrak{L}$ denotes the Lie derivative with respect to X^a.

https://doi.org/10.1017/S0305004100048453 Published online by Cambridge University Press

Therefore we mean by the phrase 'the two bodies are initially at rest' that our space time is initially† static†.

Two special cases of the situation described above can be dealt with to some extent:

(*a*) Given a static field which is produced by one body. Do test particles, which are initially at rest‡, move towards the body? This problem has been treated by the author in (3).

(*b*) Can two separated bodies (not test particles), which are at rest initially, remain at rest? Or in other words (according to the above remark): Does there exist a *static* field which is produced by two separated bodies?

Problem (*b*) will be considered in this paper. The aim of this paper as well as (3) is to show how it comes about that a field, which is governed by Einstein's field equations, is attractive. In the attempt to show this one notices (in (3) and in this paper) an analogy between Newton's theory and Einsteins' theory (in the case of static fields). This analogy is a useful heuristic principal for finding new results. The analogy is conceptual and therefore quite surprising, since the basic set up of the two theories is so different.

This heuristic principal has been used, for instance, as a guide in formulating the 'convexity assumption' (Assumption 5·6 in section 5). This assumption is needed for excluding equilibrium configurations such as peculiarly shaped 'non-convex' bodies, which can remain in equilibrium, even if one assumes positivity of the matter density and the 'right' asymptotic assumptions. An example of such a peculiar body would be a ring with a ball placed in the centre of the ring. A further example: One body is shaped like a horse shoe with the matter distribution greater at the ends. The other body is a small ball appropriately placed between the ends of the horse shoe. (In both of the foregoing examples the equilibrium is unstable.)

In the case of a static axisymmetric field, which is produced by *two black holes* (instead of two bodies), the above convexity assumption is *automatically fulfilled*. This means that one can solve the problem of whether there exists an axisymmetric static space time with *two* black holes (see the remarks at the end of section 5).

2. *Some definitions and the static field equations. Theorems, Lemmas, Assumptions and equations are all numbered successively.* Our conventions are the following: signature is $+ + + -$; Greek indices, lower case Latin indices and capital Latin indices range from 1 to 4, 1 to 3, 1 to 2 respectively. Covariant differentiation with respect to the 4-dimensional metric is denoted by ∇ and the partial differentiation is denoted by a comma. The 3-metric g_{ab} will be defined in (2·2) (see below). Within the 3-*dimensional formulae the lowering* (*and raising*) *of indices* will be done by using g_{ab}. The tensor η^{abc} is the skew symmetric tensor with $\eta^{123} = (\det g_{ab})^{-\frac{1}{2}}$. Covariant differentiation with respect to g_{ab} is denoted by a semicolon. For a point set S we define $\dot{S} \equiv$ boundary of S.

We assume a *static* space time V^4, i.e. we assume that there is a complete hyper-

† I.e. static initial conditions, see (7), p. 159.

‡ At rest means – according to the remarks made in the third paragraph – that the test particle is at rest with respect to the static congruence X^a (see (1·1*a*)) of our static space time.

https://doi.org/10.1017/S0305004100048453 Published online by Cambridge University Press

surface orthogonal time-like Killing vector field and that V^4 is simply connected. Then the metric can be reduced to

$$ds^2 = \tilde{g}(x^c)_{ab}\, dx^a\, dx^b - \exp(2U(x^c))\, dt^2, \tag{2·1}$$

where $\tilde{g}_{ab}$ is the metric tensor on some *space section* M^3 orthogonal to the time-like Killing vector.

The following version is equivalent to (2·1)

$$ds^2 = \exp(-2U)\, g_{ab}\, dx^a\, dx^b - \exp(2U)\, dt^2. \tag{2·2}$$

The version (2·2) suggests itself because of the formula for weak static fields (approximation of Newton Theory)

$$ds^2 \approx (1-2U)\, \delta_{ab}\, dx^a\, dx^b - (1-2U)\, dt^2.$$

We prefer to use the version (2·2) since with this version the conceptual analogy to Newton Theory, which has been mentioned in the introduction, will become more apparent.

Using the coordinates of (2·2) (or (2·1)) we obtain for the energy momentum tensor

$$T^{\alpha}{}_{\beta} = \begin{pmatrix} T^a{}_b & 0 \\ 0 & -\mu \end{pmatrix}. \tag{2·2A}$$

The three eigenvalues of the stress tensor $T^a{}_b$ will be called the *pressures* p_1, p_2, p_3.

As in (3) we call

$$\tau \equiv \mu + \Sigma p_a \tag{2·3}$$

the (*active*) *mass density*. This name is justified in (3).

The field equations are equivalent to

$$-\hat{G}_{ab} = T_{ab} \exp(-2U) + 2U_{ab}, \tag{2·4}$$

$$\Delta U \equiv U_{,a}{}^{;a} = 2\tau . \exp(-2U), \tag{2·5}$$

where $U_{ab} \equiv U_{,a} U_{,b} - \frac{1}{2} g_{ab}\, U^{,c} U_{,c}$ and $\hat{G}_{ab}$ denotes the Einstein tensor† with respect to the 3-metric g_{ab}. The energy conservation law $\nabla_\alpha T^{\alpha\beta} = 0$ is equivalent to

$$[T^b{}_a \exp(-2U)]_{;b} = -U_{,a} . \tau \exp(-2U) \tag{2·6a}$$

$$= -2U_a{}^b{}_{;b}. \tag{2·6b}$$

3. *Equilibrium conditions.* In this section, we shall present *equilibrium* conditions, which we shall use for the treatment of the static 2-body problem.

We shall assume throughout the rest of this paper the following.

Assumption 3·1. We are given a static field being produced by two bodies B_A ($A = 1, 2$) which are surrounded by a vacuum. This means that the static field equations (2·4, 5) are fulfilled with

$$T_{\alpha\beta} = 0 \quad \text{for} \quad x \notin B_A, \tag{3·2}$$

where the *bodies* B_A are represented as the two connected components of

$$\{x \in M^3;\ \text{(active) mass density } \tau(x) \neq 0\}. \tag{3·3}$$

† $\hat{G}_{ab} \equiv \hat{R}_{ab} - \frac{1}{2} g_{ab} \hat{R}^c{}_c$ where $\hat{R}_{ab}$ the Ricci Tensor with respect to g_{ab}.

https://doi.org/10.1017/S0305004100048453 Published online by Cambridge University Press

Note that this includes the assumption that the above set has precisely two connected components. Furthermore, we assume that $\bar{B}_A$ is compact and diffeomorphic to a 3-ball.

LEMMA 3·4. *The following equilibrium condition*

$$\int_{B_A} v^a U_{,a} \Delta U \, dV = 0 \tag{3·5}$$

holds for all vector fields with

$$\int v_{a;b} T^{ab} \exp(-2U) \, dV = 0. \tag{3·6}$$

Furthermore, a second equilibrium condition

$$\int_{\dot{V}_A} w^a U_{ab} \, dS^b = 0 \tag{3·7}$$

$$U_{ab} \equiv U_{,a} U_{,b} - \tfrac{1}{2} g_{ab} U_{,c} U^{,c}$$

holds for all vector fields w^a with

$$\int_{V_A} w_{a;b} \hat{G}^{ab} \, dV = 0 \tag{3·8}$$

and for any pointset V_A $(A = 1, 2)$ with

$$V_A \supset B_A, \quad V_1 \cap B_2 = \varnothing = V_2 \cap B_1. \tag{3·9}$$

The symbols dS^a and $\dot{V}_A$ denote the surface area element and boundary of V^A respectively.

Proof of Lemma 3·4. As $T_{ab} = 0$ on the boundary $\dot{B}_A$ of each body:

$$0 = \int_{\dot{B}_A} v^a T_{ab} \exp(-2U) \, dS^b = \int_{B_A} v^a [T_a{}^b \exp(-2U)]_{;b} \, dV,$$

where the latter follows from Stokes' Theorem and (3·6). By inserting the field equations (2·6a, 5) one proves (3·5).

As $T_{ab} = 0$ on $\dot{V}_A$ (see (3·2, 9)):

$$\begin{aligned} 0 &= -\int_{\dot{V}_A} w^a T_a{}^b \exp(-2U) \, dS_b = \int_{\dot{V}_A} w^a (\hat{G}_a{}^b + 2U_a{}^b) \, dS_b \\ &= \int_{V_A} (w^a \hat{G}_a{}^b)_{;b} \, dV + 2 \int_{\dot{V}_A} w^a U_a{}^b \, dS_b, \end{aligned}$$

where the latter follows from using the field equations (2·4) and then Stokes theorem. By inserting $\hat{G}_a{}^b{}_{;b} = 0$ and (3·8) one proves (3·7).∎

Remarks on Lemma 3·4. In (4·4) and (5·29) we shall give vectorfields fufilling (3·6, 8). Then we shall apply the above equilibrium conditions to the static 2-body problem.

It is interesting that $(-U_{,a} \Delta U \, dV)$ in (3·5) can be interpreted as the 'gravitational force'† acting on the active mass element $[\tau \, dV]$, since $(-U_{,a} \Delta U \, dV) . [\tau \, dV]^{-1}$ is the

† This is the concept described in (3).

https://doi.org/10.1017/S0305004100048453 Published online by Cambridge University Press

initial acceleration of a test particle which is initially at rest.† This is so because of (2·5) and because the geodesic (motion of particles) can be written as

$$D_r \dot{x}^a = -U^{,a}.[2\dot{x}^b\dot{x}_b + \tfrac{1}{2}\exp(-2U)], \quad \dot{x}^a \equiv \frac{dx^a}{dr}$$

for some suitable parameter r (see also (3)). $D_r \equiv {}_{;a}\dot{x}^a$ denotes the covariant differentiation along the curve $x^a(r)$.

Within *Newton's Theory* the integral in (3·5) is some component of the total force (the component in the direction of v^a) and one chooses as v^a any constant vectorfield. Such a vectorfield fulfils (3·6) (as well as (3·8)).

Remark on (3·7). The equilibrium condition (3·7) is in some sense the surface-integral version of (3·5). This is easily seen if one bears in mind that $U_{,a}\Delta U = U_a{}^b{}_{;b}$, which implies

$$\int_{B_A} v^a U_{,a}\Delta U\, dV = \int_{\dot{B}_A} v^a U_{ab}\, dS^b \tag{3·10}$$

for *constant* v^a (i.e. $v_{a;b} = 0$).

U_{ab} plays the role of the stress-tensor of the gravitational field (see (2·4, 6)), since $U_{,a}$ is in some sense the 'gravitational force' as pointed out on page 252 (at the bottom).

As pointed out above one chooses within Newton Theory a constant vector w^a (i.e. $w_{a;b} = 0$). This of course fulfils (3·8).

4. *Equilibrium conditions in the axisymmetric case.* We shall give explicitly a vector for which the precondition (3·8) is fulfilled. We shall relate our equilibrium condition to the one previously used (see (4), p. 312).

We shall call our static spacetime *axisymmetric*‡ if the following holds: There exists a hypersurface orthogonal Killing vector Y^α which is orthogonal to our static Killing vector X^α; the orbits of Y^α are closed; there exists a reasonable *axis* $\mathfrak{A}$; 'reasonable' meaning that

$$\mathfrak{A} \equiv \{x \in M^3;\, Y^a(x) = 0\} = \{x \in M^3;\, x \in \dot{M}^2\} \tag{4·1}$$

is a connected 1-dim submanifold, where M^2 is one of the 2-spaces (in M^3) which are orthogonal to Y^a. By Y^a is meant the projection of Y^a into M^2. (By rotating M^2 around the axis we obtain M^3.)

We shall say that the vacuum region of a spacetime can be covered by a *Weyl coordinate* system if

$$\left.\begin{aligned} ds^2 &= e^{-2U}[e^{2V}((dx^1)^2 + (dx^2)^2) + (x^1)^2(dx^3)^2] - e^{2U}dt^2, \\ x^3 &= \mathrm{mod}\, 2\pi, \quad x^1 \geqslant 0, \quad U = U(x^A), \quad V = V(x^A), \end{aligned}\right\} \tag{4·2}$$

for all points not belonging to the axis $\mathfrak{A}$.

THEOREM 4·3. *We assume that our static spacetime is axisymmetric. Then the vectorfield*

$$w^a \equiv -\tfrac{1}{4}\eta^{abc}Y_{b;c} \tag{4·4}$$

† See footnote ‡ in section 1 (page 250).

‡ We do not assume $\mathfrak{L}\, Y^\alpha = 0$ (Lie-derivative) since this is implied by the fact that our space is static and axisymmetric.

https://doi.org/10.1017/S0305004100048453 Published online by Cambridge University Press

fulfils $w_{a;b}\hat{G}^{ab} = 0$. *This means that* (3·8) *is fulfilled and the equilibrium condition* (3·7) *holds for the above* w^a. *In addition let us assume that* M^2 (*being one of the* 2-*spaces orthogonal to the rotational Killing vector* Y^a) *is diffeomorphic to the half plane* $\{(x, y) \in R^2;\ x > 0\}$. *Let* C_A $(A = 1, 2)$ *be a curve, which lies in* M^2 *and fulfils* $C_A = M^2 \cap \dot{V}_A$, *where* V_A *fulfils* (3·9). (C_A *begins and ends on our axis* $\mathfrak{A}$.) *Then, as a special case of* (3·7), *the following equilibrium conditions hold*

$$\int_{\tilde{C}_A} w^a U_{ab}\, dS^b = 0 \quad (A = 1, 2) \tag{4·5}$$

for the above w^a, *where the* (*closed*) 2-*surface*† $\tilde{C}_A$ *is generated*† *by rotating the curve* C_A *around the axis* $\mathfrak{A}$.

Remark. If we assume that the vacuum region can be covered by a Weyl coordinate system, then the above equilibrium condition (4·5) is equivalent to

$$\int_{C_A} x^1 . [(U_{,1}{}^2 - U_{,2}{}^2)\, dx^1 + 2U_{,1}U_{,2}\, dx^2] = 0 \tag{4·6}$$

(see (5·12, 13, 15)) where C_A is the curve described in the above Theorem. The 'equilibrium condition' (4·6) has been stated in (4), p. 313. But there was no physical interpretation for it in the sense of the above remarks on Lemma 3·4. In (4), p. 313, the 'equilibrium condition' (4·6) has been derived from

$$V(x^A) \to 0 \quad \text{for} \quad x' \to 0 \tag{4·7}$$

(see (4), p. 313), which follows from the regularity of ds^2 on the axis $\mathfrak{A}$

$$((x^1, x^2) \in \mathfrak{A} \Leftrightarrow x^1 = 0).$$

Proof of Theorem 4·3. Y^a is a hypersurface orthogonal Killing vector. From this it follows:

$$w^a = \eta^{abc} W_{,b} Y_c\, W^{-1}, \quad W^2 \equiv Y_a Y^a, \tag{4·8}$$

$$w^a{}_{;b}\hat{G}_a{}^b = w^A{}_{:B}\hat{G}_A{}^B, \tag{4·9}$$

where in the latter a coordinate system with $Y^a = \delta^a_3$, $g_{A3} = 0$ has been used. The *colon* means differentiation with respect to the 2-metric which is induced in the surface orthogonal to Y^a.

On the other hand we have

$$\hat{G}^A{}_B = W^{-1}(W^{,A}{}_{:B} - \delta^A_B\, W^{,C}{}_{:C}), \tag{4·10}$$

where we have used the fact that the *Einstein tensor of any* 2-*metric is zero*. By inserting (4·8, 10) into (4·9) we prove $w^a{}_{;b}\hat{G}_a{}^b = 0$.

(3·7) is fulfilled for $w^a = -\frac{1}{4}\eta^{abc}Y_{b;c}$. Furthermore, there exist two sets V_A $(A = 1, 2)$ which satisfy $\dot{V}_A = \tilde{C}_A$ and (3·9). This proves (4·5). ∎

5. *The static two-body problem.* We shall now state the Theorems on the static two-body problem. We begin by stating and discussing the assumptions of Theorem 1.

† $\tilde{C}_A \equiv \bigcup_{x \in C_A} T_x$, where T_x is the closed trajectory of our rotational Killing vector Y^a.

https://doi.org/10.1017/S0305004100048453 Published online by Cambridge University Press

Apart from Assumption 3·1 (in section 3) we shall assume

Assumption 5·1. The (active) mass density τ is not negative.

Assumption 5·2. The metric g_{ab} is asymptotically Euclidean and U tends sufficiently fast towards zero at infinity. In more precise terms: For some compact region K of M^3 the set $M^3 \backslash K$ is diffeomorphic to R^3 minus some compact set. In this region coordinates can be introduced such that

$$\left.\begin{array}{ll} (a)\ g_{ab} = \delta_{ab} + O(|x|^{-1}), & (b)\ g_{ab,c} = O(|x|^{-2}), \\ (c)\ U = O(|x|^{-1}), & (d)\ U_{,a} = O(|x|^{-2}), \end{array}\right\} \tag{5·3}$$

where $|x|^2 \equiv \sum_1^3 (x^a)^2$.

Remark. Assumptions 5·2, 5·3 and 5·6 (see below) are absolutely essential. There exist equilibrium configurations where just one of these assumptions is not fulfilled and the other two are fulfilled. This occurs also in Newton's Theory.

Assumption 5·4. (*a*) The space time is axisymmetric. Our space section M^3 is diffeomorphic to R^3. (*b*) The vacuum can be covered by a Weyl coordinate system.

Remark. The assumption of axial symmetry is a genuine restriction. The physical situation is then such that the 2 axisymmetric bodies are lined up on the axis $\mathfrak{A}$. However this situation is still characteristic enough to get some clues about the static 2-body problem.

The Assumption 5·4*a* implies 5·4*b* *locally* for points with $(Y^a Y_a)_{,b} \neq 0$.

From Assumption 4 it follows that we can construct in the *vacuum* region M_0^3 the *flat* metric

$$\bar{g}_{ab} \equiv (g_{ab} - Y_a Y_b W^{-2})(W_{,c} W^{,c}) + Y_a Y_b W^{-2}, \tag{5·5}$$

where $W^2 \equiv Y^a Y^b g_{ab}$ (and Y^a is the projection of our Killing vector Y^α into M^3). This means that we can talk about 2-*planes* of the space $(M_0^3, \bar{g}_{ab})$, which is an *Euclidean* 3-space with two compact sets removed (see below: the remarks made on (5·16)).

Assumption 5·6. The 2 bodies can be separated by a plane P (as defined in the last paragraph) in such a way that B_1 lies 'above P' and B_2 'below P'. More precisely: Let M_A^3 $(A = 1, 2)$ be the two connected components of $M^3 \backslash P$ (the plane P lies in the vacuum region). Then

$$\bar{B}_A \supset M_A^3. \tag{5·7}$$

Remark. In the last but one paragraph of the introduction the Assumption 5·6 has been called the convexity assumption. Two interesting counter-examples have been given in that paragraph.

Assumption 5·8. The matter consists of perfect fluid, i.e.

$$T_{\alpha\beta} = u_\alpha u_\beta (\mu + p) + p g_{\alpha\beta}, \tag{5·9}$$

and the energy density μ is positive (see (2·2A)).

Remark. This assumption is a real restriction and it will be *weakened* in Theorem 5·32.

https://doi.org/10.1017/S0305004100048453 Published online by Cambridge University Press

THEOREM 5·10. *There exists no static space time which fulfils Assumptions* 3·1, 5·1, 5·2, 5·4, 5·6 *and* 5·8. *This means no two bodies as described above can remain in static equilibrium.*

Remark. In Theorem 5·32 we shall state a generalized version of the above Theorem.

Proof of Theorem 5·10. By using our equilibrium condition (4·5) we shall find a further equilibrium condition (see (5·15)), which looks much the same as the original one but which we are able to evaluate. The aim of this evaluation is to show that the integrand in our equilibrium condition is positive (see (5·15), or compare (4·5)). This means that the left-hand side of our equilibrium condition cannot be zero. This is a contradiction, thus proving our Theorem. (See the paragraph with the heading 'Central part of the proof'.)

We shall now derive a result which we shall need later on (in (5·18)). By evaluating (2·6a) for perfect fluid we obtain

$$p_{,a} = -(p+\mu)\, U_{,a}.$$

This means that the surfaces $U = \text{const.}$ coincide with the surface $p = \text{const.}$ As $p = 0$ on the surface $\dot{B}_A$ of the body we obtain

$$U = \text{const}\dagger \quad \text{on} \quad \dot{B}_A. \tag{5·11}$$

As C_A lies in the vacuum region we can evaluate the left-hand side of the *equilibrium condition* (4·5) in Weyl coordinates (see (4·2)). Therefore by using the field equations we obtain:

$$\int_{\tilde{C}_A} w^a U_{ab}\, dS^b \underset{\times}{=} \int_{C_A} (\exp -2V)\, \omega \underset{\times}{=}$$

$$\underset{\times}{=} -\frac{1}{2}\int_{C_A} (\exp -2V)_{,A}\, dx^A \underset{\times}{=} -\tfrac{1}{2}[\exp(-2V(x)) - \exp(-2V(\bar{x}))] \tag{5·12}$$

with $\omega \underset{\times}{\equiv} x^1[(U_{,1}{}^2 - U_{,2}{}^2)\, dx^1 + 2U_{,1}U_{,2}\, dx^2]$, where x is the beginning point of the curve C_A and $\bar{x}$ the end-point ($x, \bar{x}$ lie on the axis $\mathfrak{A}$). $\underset{\times}{=}$ denotes equality in Weyl coordinates. In the same way we obtain

$$\int_{\tilde{C}_A} u^a U_{ab}\, dS^b \underset{\times}{=} \int_{C_A} \omega \underset{\times}{=} V(x) - V(\bar{x}) \tag{5·13}$$

for

$$u^a \underset{\times}{\equiv} \delta^a_2. \tag{5·14}$$

This and (5·12) mean that (4·5) is *equivalent* to the following *equilibrium* condition:

$$0 = \int_{\tilde{C}_A} u^a U_{ab}\, dS^b. \tag{5·15}$$

We shall rewrite this equilibrium condition in terms of our *flat* auxiliary metric $\bar{g}_{ab}$ of (5·5), which fulfils

$$\bar{g}_{ab}\, dx^a\, dx^b \underset{\times}{=} (dx^1)^2 + (dx^2)^2 + (x^1)^2\, (dx^3)^2 \tag{5·16}$$

† In connexion with (3) it is interesting that (5·11) together with Theorem 2 of (3) prove the following: 'A particle, which is initially held at rest on the surface of the body, will move into the body.'

https://doi.org/10.1017/S0305004100048453 Published online by Cambridge University Press

in the vacuum area $M_0^3 \equiv M^3 \backslash (B_1 \cup B_2)$. This means that we can complete the space $(M_0^3, \bar{g}_{ab})$ in such a way that we obtain an *Euclidean* space $(\hat{M}^3, \bar{g}_{ab})$ with $M_0^3 \subset \hat{M}^3$. With respect to this Euclidean space we *define*: $\bar{r}(x, \tilde{x})$ the distance between two points x and $\tilde{x}$; $d\bar{S}$ the surface area element; $\bar{n}^a$ the unit normal vector of the surface $\dot{B}_A$ ($\bar{n}^a$ points into M_0^3); $\bar{\Delta}$ the (flat) Laplacian operator.

Central part of the proof. We can now write our field equation (2·5) as well as our equilibrium condition (5·15) in terms of our *flat* metric $\bar{g}_{ab}$:

$$\bar{\Delta} U = 0 \quad \text{on} \quad M_0^3 \equiv M^3 \backslash (B_1 \cup B_2), \tag{5·17}$$

$$U = \text{const} \quad \text{on} \quad \dot{M}_0^3 = \dot{B}_1 \cup \dot{B}_2 \tag{5·18}$$

and

$$0 = \underset{\times}{\int_{\dot{B}_A}} U_{,2} \sigma \, d\bar{S}, \quad \sigma \equiv U_{,a} \bar{n}^a. \tag{5·19}$$

In the latter formula, we deliberately choose $\dot{B}_a$ as the surface $\tilde{C}_A$ of our equilibrium condition (5·15), *because* $U = const.$ on $\dot{B}_A$ (see (5·11)). The latter will also be used in (5·20).

This way we can derive the following *integral representation* of our potential U

$$U(\tilde{x}) = \sum_{A=1}^{2} \int_{\dot{B}_A} |\bar{r}(\tilde{x}, x)|^{-1} . \sigma(x) \, dS_x. \tag{5·20}$$

We have used the fact that $|\bar{r}(\tilde{x}, x)|^{-1}$ is a fundamental solution (in the variables $\tilde{x}$ as well as x) of the flat Laplacian equation (5·17). We also used, apart from (5·18), the asymptotic condition† (5·3*c*, *d*).

Finally, inserting (5·20) into (5·19) we obtain

$$0 = \underset{\times}{\int_{\dot{B}_2}} \left[\int_{\dot{B}_1} (x^2 - \tilde{x}^2) \, (r(x, \tilde{x}))^{-3} \sigma(x) \, dS_x \right] \sigma(\tilde{x}) \, dS_{\tilde{x}}. \tag{5·21}$$

In electrostatics one calls the right-hand side the force acting between the two *surface layers* $\dot{B}_1$ and $\dot{B}_2$, which are covered with the surface charge σ. (And we shall show that this force is not zero in contradiction to (5·21).)

Now, due to the axisymmetry the plane P, which separates $\dot{B}_1$ and $\dot{B}_2$, fulfils $\underset{\times}{x^2} = \text{const.}$ on P (see Assumption 5·6). This and Assumption 5·6 implies

$$(x^2 - \tilde{x}^2) \quad \text{is } \textit{positive} \text{ for} \quad x \in \dot{B}_1 \quad \text{and} \quad \tilde{x} \in \dot{B}_2 \tag{5·22}$$

(x^2 is one of the Weyl coordinates).

On the other hand, as shown in Theorem 5·23, the *positivity of the* (*active*) *mass density* implies (5·25) (see below). This and (5·11) implies that σ is *not negative* on $\dot{B}_A$ (moreover, there are points on $\dot{B}_A$, where $\sigma \neq 0$).

This and (5·22) imply that the right-hand side of (5·21) is strictly positive. This contradicts (5·21). Hence Theorem 5·10 is proved.∎

Before we state the generalized version of the above Theorem we have to talk

† More precisely: U fulfils $U = O(\rho^{-1})$, $\underset{\times}{U_{,a}} = O(\rho^{-2})$ with $\underset{\times}{\rho^2} = (x^1)^2 + (x^2)^2$, which follows from (5·3) and Assumption 5·4.

https://doi.org/10.1017/S0305004100048453 Published online by Cambridge University Press

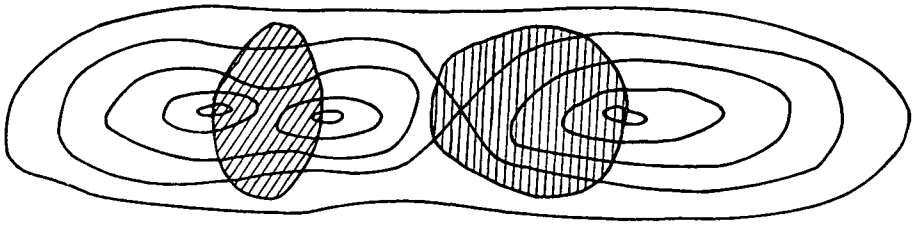


Fig. 1. General case (we have always $E(S) \cap B \neq \varnothing$, see (5·24)).

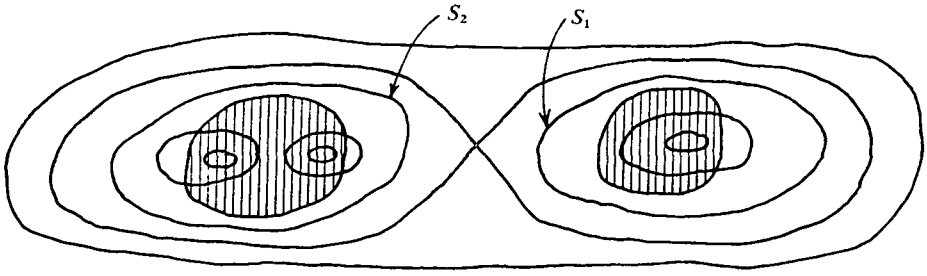


Fig. 2. Special case: there exists S_1, S_2 satisfying (5·27, 28).

about equipotential surfaces. A connected component of $\{x \in M^3;\ U(x) = \text{const.}\}$ will be called an equipotential surface. In (3) the following Theorem has been proved:

THEOREM 5·23. *We assume that our space-time is static, that our space section M^3 is diffeomorphic to R^3 and that the asymptotic assumption* (5·3) *holds. Then every equipotential surface S with $U_{,a} \neq 0$ on S is diffeomorphic† to a sphere with n handles where $n = 0, 1, 2, \ldots$. $M^3 \backslash S$ is split into two non-empty connected components, one of which has compact closure. We shall call this component $E(S)$. (This is the domain 'enclosed' by S.) Furthermore we have*

$$E(S) \cap B \neq \varnothing, \tag{5·24}$$

where $B \equiv \{x \in M^3;\ \tau(x) \neq 0\}$ (see (3·3)*). Moreover,*

$$U_{,a} \text{ points out of } E(S), \tag{5·25}$$

if we assume in addition Assumption 5·1.

We shall now in some sense weaken Assumptions 5·8

Assumption 5·26. There exist two equipotential surfaces S_A $(A = 1, 2)$ such that

$$E(S_A) \supset B_A, \tag{5·27}$$

$$E(S_1) \cap E(S_2) = \varnothing \tag{5·28}$$

and‡ $V_{,a} \neq 0$ everywhere on S_A, where B_A are the two bodies (see Fig. 2).

Remarks on Assumption 5·26. In the case of perfect fluid with $\tau > 0, \mu > 0$ (see (2·3)) the boundary $\dot{B}_A$ of each body B_A is an equipotential surface and therefore Assumption

† This is also true without assuming that M^3 is diffeomorphic to R^3, provided M^3 is oriented. (S is the boundary of a component of the *compact* set $\{x \in M^3;\ U(x) \leqslant \text{const.}\}$.)

‡ 'Most' of the equipotential surfaces S fulfil $V_{,a} \neq 0\ \forall x \in S$. See Theorem 2·11 of (5), which is a consequence of the analyticity of $U(x)$ (see (6)).

https://doi.org/10.1017/S0305004100048453 Published online by Cambridge University Press

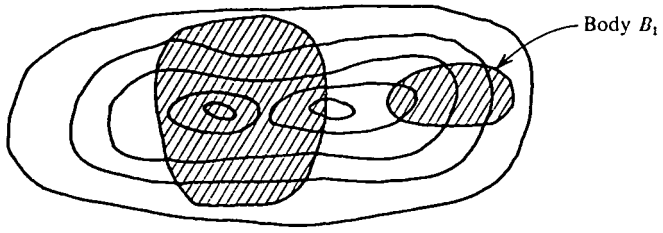


Fig. 3. Special case: $U_{,a} \neq 0$ in B_1.

A is fulfilled for $S_A = \dot{B}_A$ (see below: (5·11)). This means that Assumption 5·26 is also fulfilled if the matter consists of 'almost' perfect fluid. In general the equipotential surfaces look as shown in Fig. 1. This follows from Theorem 5·23. Assumption 5·26 deals with the case sketched in Fig. 2. Physically speaking, bearing (3) in mind, this is the case where the body B_1, say, creates a field, whose strength does not differ too much from the strength of the field created by B_2.

The *opposite case* is the one, where the body B_1 creates a field which is much weaker than the field created by B_2. This case is sketched in Fig. 3. This means, bearing (3) in mind, essentially

$$U_{,a} \neq 0 \quad \text{in} \quad B_1. \tag{5·29}$$

This case is fairly trivial. *Also in this case the two bodies cannot remain in equilibrium,* if we assume,† apart from (5·29), the Assumptions 3·1, 5·1, 5·2 and

$$p_a > 0 \quad (a = 1, 2, 3) \text{ everywhere in } B_1$$
$$p_1 \neq p_2 \neq p_3 \neq p_1 \quad \text{everywhere in } B_1$$

for the three pressures p_a (see (2·3)).

This can be shown by using the *equilibrium condition* (3·5), where we set

$$v^a = U^{,a} . \exp \alpha$$

with‡ α such that

$$\alpha_{,a}(T^a{}_b U^{,b}) = -U_{,a;b} T^{ab}, \tag{5·30}$$

which means $v_{a;b} T^{ab} = 0$. Then the equilibrium condition reads

$$0 = \int_{B_1} (\exp \alpha) U_{,a} U^{,a} \Delta U \, dV,$$

which contradicts $\Delta U = 2\tau \exp(-2U) \geqslant 0$ (assumption 5·1).

We shall now modify assumption 5·6 slightly:

Assumption 5·31. Let $E(S_A)$ be as defined in Assumption 5·26. The point sets $E(S_1)$ and $E(S_2)$ are separated in the same way as the point sets B_1 and B_2 (the bodies) in Assumption 5·6, i.e. replace B_A by $E(S_A)$ in Assumption 5·6.

Remark. This means, because $E(S_A) \supset B_A$, that also the bodies B_1 and B_2 are separated in the same way as described in Assumption 5·6. Assumptions 5·31, 5·1

† We neither assume axial symmetry nor an assumption such as Assumption 5·6.

‡ One can show the existence of a solution α of the differential equation (5·30) by constructing a coordinate system such that $T^a{}_b U^{,b} = \delta^a_3$ in B_1.

https://doi.org/10.1017/S0305004100048453 Published online by Cambridge University Press

and 5·2 are absolutely essential: There exist equilibrium configurations where just one of these assumptions is not fulfilled and the other two are fulfilled. This is also the case within Newton Theory.

The following Theorem is, in the sense of the remarks made about Assumption 5·26, a generalization of Theorem 5·10.

THEOREM 5·32. *There exists no static space time which fulfils Assumptions* 3·1, 5·1, 5·2, 5·4, 5·26 *and* 5·31. *This means that no two bodies as described above can remain in equilibrium.*

Proof of Theorem 5·32. The proof runs along the same lines as the proof of Theorem 5·10 if one replaces $\dot{B}_A$ of that proof by S_A of Assumption 5·26, if one ignores the second paragraph and if one replaces (5·11) by the statement

$$U = \text{const on } S_A$$

which follows from the definition of S_A (S_A is an equipotential surface).∎

Remarks on two black holes in static equilibrium:† It is interesting that in case of a static axisymmetric field, which is produced by *two* black holes (instead of two bodies), the above *convexity assumption* (Assumption 5·31) as well as the Assumption 5·26 are automatically fulfilled. (Assumption 5·26 is fulfilled, since $(\exp 2U) \to 0$ near the black hole.)

This means, using a slightly modified version of the proof† of Theorem 5·32, that *there does not exist an axisymmetric static space time with two black holes*; provided one can show that one can cover the vacuum region with a Weyl coordinate system (Assumption 5·4). Luckily the latter is *possible* in the case of two black holes, as has been proved by H. J. Seifert (private communication). A paper on the subject of two static black holes will be published shortly (in *Int. Jour. of Theor. Phys.*).

The author expresses his appreciation to Professor F. A. E. Pirani for the hospitality of King's College, London.

This work has been carried out under a N.A.T.O. research followship.

REFERENCES

(1) HAWKING, S. W. and PENROSE, R. The singularities of gravitational collapse and cosmology. *Proc. Roy. Soc. Ser. A* **314** (1970), 529.

(2) MÜLLER ZUM HAGEN, H. A new physical characterization of stationary space-times. *Proc. Cambridge Philos. Soc.* **71** (1972), 381.

(3) MÜLLER ZUM HAGEN, H. On the attractive force of the gravitational field in static space times. *Proc. Cambridge Philos. Soc.* **68** (1970), 187.

(4) SYNGE, J. L. *Relativity: the general theory* (North-Holland Publishing Company; Amsterdam, 1964).

(5) MÜLLER ZUM HAGEN, H., ROBINSON, D. C. and SEIFERT, H. J. Black holes in static vacuum space-times (to be published in *General Relativity and Gravitation*).

(6) MÜLLER ZUM HAGEN, H. On the analyticity of static vacuum solution of Einstein's equations. *Proc. Cambridge Philos. Soc.* **67** (1970), 415.

(7) BRUHAT, Y. The Cauchy problem, published in: *Gravitation: an introduction to current research*. Editor: Witten, L. W. (Wiley; New York, 1962).

† This result as well as a sketch of the proof has been given in the talk of the author at the 'Conference On Relativity and Related Topics', Brussels, 1971.

https://doi.org/10.1017/S0305004100048453 Published online by Cambridge University Press