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$\mathscr{L}^0$-equivalence of maps

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Abstract

We use the theory of the integral closure of an ideal to study the equivalence of map-germs under C^0 coordinate changes in the target. We also derive a formula for the number of double points of a map germ from $\mathbf{C}^n \to \mathbf{C}^{2n}$ in terms of the Segre number of dimension 0 of an ideal associated to the double point locus of f, and the number of Whitney umbrellas of the composition of f with a generic projection to $\mathbf{C}^{2n-1}$.

Introduction

This paper is part of a long-term project by several mathematicians to describe the C^0-sufficiency of jets under various equivalence relations (cf. [**2**, **3**, **13**–**15**, **23**, **25**–**27**]). The methods originally used in tackling these problems are mostly based on Lojasiewicz inequalities. The study of the equivalence relations involving coordinate changes in the target of a mapping by this method is complicated by the necessity of considering Lojasiewicz inequalities which control multiple point singularities. These inequalities are typically based on some product of the original source with itself with the diagonal removed. Problems arise when points get close to the diagonal.

This paper looks at the simplest such case; it considers the problem of trivializing families of maps using only coordinate changes in the target. This kind of equivalence is known as $\mathscr{L}$-equivalence.

One way around the problem of the diagonal is to use the theory of Hilbert schemes. Hilbert schemes provide a way of combining inequalities which describe behaviour at the multi-jet level, with those that control the behaviour that occurs 'on the diagonal'. This approach was used in the author's notes on $\mathscr{L}^0$-equivalence which were circulated in 1989 ([**9**]).

Reworking the machinery in terms of the theory of the integral closure of modules and ideals is another way around the problem with the diagonal. This machinery also results in easier and more conceptual proofs, which give clearer insight into this equivalence relation. Because the theory of integral closure has an algebraic side, it is possible to prove results describing when a family of sets or maps is equisingular in terms of invariants depending only on the members of the family. The unfolding of this theory is the purpose of the present paper.

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An alternative approach to the question of the $\mathscr{L}$-sufficiency of jets, based on Lojasiewicz inequalities, can be found in [**4**].

Section 1 of this paper reviews fundamental results in the case when the coordinate changes are analytic; our purpose is to underscore the analogy with the C^0 case. Section 2 of the paper reviews some ideas from the theory of the integral closure of ideals, and uses them to get a theorem of Thom–Levine type for left rugose trivializations (Theorem 2·1). This section goes on to give a necessary and sufficient condition for a map-germ to be k-determined with respect to left rugose trivialization. The section concludes with a brief discussion of the results needed to extend the theory to the C^∞ case.

Section 3 of the paper, which only applies to the complex analytic case, develops geometric invariants whose independence from the parameters provides a necessary and sufficient condition for an unfolding to have a *stable rugose trivialization*, a notion defined in Section 2. This result is Theorem 3·6. Along the road to this result, in Theorem 3·2, we develop a computable formula for the number of double points at the origin of a map-germ $f\colon \mathbf{C}^n,0 \to \mathbf{C}^{2n},0$. The formula is given in terms of the Segre number of dimension 0 of the double point ideal of f, which is the ideal generated by $\{f_i(z) - f_i(z')\}, 1 \leqslant i \leqslant p$, where f_i are the components of f, and the number of Whitney umbrella points of $\pi \circ f$, where π is a generic projection of $\mathbf{C}^{2n} \to \mathbf{C}^{2n-1}$. Note that the ideal $\{f_i(z) - f_i(z')\}$ vanishes at all points where $z' = z$. The machinery of Segre numbers enables us to see through these diagonal points to obtain the double point information at $(0,0)$. The connection between the approach by Segre numbers and the approach of Artin and Nagata ([**1**]) to counting double points is briefly discussed. The author thanks the referee for pointing out this reference. The final result in Theorem 3·6 also provides an interesting application of the mixed multiplicities developed by Bernard Teissier.

1. *Analytic left triviality*

In this section we establish some notation and recall a few facts about $\mathscr{L}$-equivalence.

Let $\mathscr{O}_n$ denote the ring of holomorphic germs at the origin in $\mathbf{C}^n$, and m_n the ideal of germs vanishing at the origin in $\mathbf{C}^n$. Given a map-germ $f\colon \mathbf{C}^n,0 \to \mathbf{C}^p,0$, then a k-parameter unfolding F of f is a map-germ $F\colon \mathbf{C}^k \times \mathbf{C}^n,(0,0) \to \mathbf{C}^k \times \mathbf{C}^p,(0,0)$, such that $F(u,z) = (u, \bar{f}(u,z))$, where u and z are coordinates on $\mathbf{C}^k$ and $\mathbf{C}^n$ respectively, and $\bar{f}(0,z) = f(z)$. In what follows, $\bar{f}(u,z)$ will be an element of $\mathscr{O}_{n+k}$; $f_u(z)$ will denote the restriction of $\bar{f}$ to $\{u\} \times \mathbf{C}^n$. Let $I(f)$ (resp. $I(\bar{f})$) denote the ideal generated by the components of f (resp. $\bar{f}$) in $\mathscr{O}_n$, (resp. $\mathscr{O}_{n+k}$), $\mathscr{I}(f)$ the corresponding sheaf.

Let $\mathscr{L}$ denote the group of origin preserving biholomorphic maps on $\mathbf{C}^p$. If the maps are only homeomorphisms then we denote the group by $\mathscr{L}^0$.

The unfolding F is $\mathscr{L}$ trivial if there exists an unfolding L of the identity on $\mathbf{C}^p$, such that $L \circ F = (u, f(z))$ and $l_u(0) = 0$ for all u.

We have the following infinitesimal condition for $\mathscr{L}$-triviality.

THEOREM 1·1. *The unfolding F is $\mathscr{L}$-trivial if and only if*

$$\frac{\partial \bar{f}_i}{\partial u_j} \in F^*(m_{k+p}\mathscr{O}_{k+p})\ \forall i,j.$$

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Proof. This follows easily from the techniques of proof used in [**17**, p. 73].

The condition on the partial derivatives is called an infinitesimal condition, because it allows you to construct a vector field on the target of F, whose integral is the desired unfolding which trivializes F.

Theorem 1·1 is a theorem of Thom–Levine type for this equisingularity condition; namely it provides a necessary and sufficient infinitesimal condition for a family to satisfy a given equisingularity condition. Note that if we specialize $F^*(m_{k+p}\mathscr{O}_{k+p})$ to $u = 0$ we get $f^*(m_p\mathscr{O}_p)$.

We say that f is k-determined with respect to $\mathscr{L}$ if whenever the k-jet of f and another map-germ g agree at the origin, then $g = l \circ f$ for some biholomorphic l. In the case of left equivalence there is a necessary and sufficient condition for a map to be k-determined given in terms of $f^*(m_p\mathscr{O}_p)$.

THEOREM 1·2. *A map germ f is k-determined with respect to $\mathscr{L}$ if and only if m_n^{k+1} is contained in $f^*(m_p^2)$.*

Proof. Cf. [**5**].

2. *The integral closure of ideals, and the order of $\mathscr{L}^0$ determinacy*

Roughly speaking, an element $g \in \mathscr{O}_{X,x}$ is in the integral closure of I_x if g goes to zero as fast as the generators of I go to zero as you approach x. More precisely, we have the following definition theorem.

Let $(X, 0) \subset (\mathbf{C}^N, 0)$ be a reduced analytic space germ. Let I be an ideal in the local ring $\mathscr{O}_{X,0}$ of X at 0 and g an element in this ring. Then g is *integrally dependent* on I if one of the following equivalent conditions obtain.

(i) There exists a positive integer k and elements a_j in I^j, so that g satisfies the relation $g^k + a_1 g^{k-1} + \cdots + a_{k-1}g + a_k = 0$ in $\mathscr{O}_{X,0}$.

(ii) There exists a neighbourhood U of 0 in $\mathbf{C}^N$, a positive real number C, representatives of the germ X, the function germ g and generators $h_1, \ldots, h_m$ of I on U, which we identify with the corresponding germs, so that for all x in X the following equality obtains: $|g(x)| \leqslant C \max\{|h_1(x)|, \ldots, |h_m(x)|\}$.

(iii) For all analytic path germs ϕ: $(\mathbf{C}, 0) \to (X, 0)$ the pullback $\phi^* g$ is contained in the ideal generated by $\phi^*(I)$ in the local ring of $\mathbf{C}$ at 0.
If we consider the normalization $\bar{B}$ of the blow-up B of X along the ideal I we get another equivalent condition for integral dependence. Denote by $\bar{D}$ the pullback of the exceptional divisor D of B to $\bar{B}$.

(iv) For any component C of the underlying set of $\bar{D}$, the order of vanishing of the pullback of f to $\bar{B}$ along C is no smaller than the order of the divisor $\bar{D}$ along C.

The proofs of all of these results can be found in [**16**].

The elements f in $\mathscr{O}_{X,0}$ that are integrally dependent on I form the ideal $\bar{I}$, the *integral closure of* I. Often we are only interested in the properties of the integral closure of an ideal I; so we may replace I by an ideal J contained in I with the same integral closure as I. Such an ideal J is called a *reduction* of I. In some cases it is desirable to study the elements f which go to zero faster than I does as we approach x. We say that f is *strictly dependent* on I if for all holomorphic curves

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ϕ: $(\mathbf{C},0) \to (X,x)$, the pullback $\phi^* f$ is contained in the ideal $m_1\phi^*(I)$ where m_1 is the maximal ideal in $\mathcal{O}_{\mathbf{C},0}$. All such elements form the ideal $\bar{I}^\dagger$.

We can now begin to motivate the kind of morphisms that we will use to trivialize our unfoldings in the C^0 case.

Suppose we are given a 1-parameter unfolding F of f such that each of the map-germs f_u has a representative which is an injective immersion except at the origin on some fixed neighbourhood of the origin. Note that F^{-1} is well defined on the image of F and is holomorphic on the complement of the parameter space in the image of F. If L is a C^0-unfolding which trivializes F, then L restricts to an analytic map on the complement of the parameter space in the image of F. This follows because the restriction of L to the image of F is unique, and equal to $(u,f)\circ F^{-1}$. Further, the curves which are the image of $F|\mathbf{C}\times\{x\}$, $x \neq 0$ are the integral curves of the flow which defines L. The existence of L ensures that this flow extends to the whole ambient space. Given this, it is natural to look for conditions on the vector field determined by the image curves of F described above which ensure that we get a C^0 flow defined on the ambient space. It is known (cf. [**24**]) that this can be done if we ask that the vector field is *rugose* on the image of F. If the coordinates on $\mathbf{C}^p$ are $(y_1,\ldots,y_p)$ then this condition boils down to the inequality

$$\left\| \frac{\partial \bar{f}}{\partial t}(t,y_1,\ldots,y_p) \right\| \leqslant \mathbf{C}\,\|(y_1,\ldots,y_p)\|$$

holding for all (a,y) in a neighbourhood of the origin on the image of F. If the trivialization L comes from integrating a rugose field, then we call it a *rugose* trivialization. (The word 'rugose' was introduced in this context by Verdier ([**24**]), to indicate that the vector fields and associated morphisms were 'rough' as opposed to smooth.)

We can compose the terms of the inequality with F to get an equivalent inequality on $\mathbf{C}^n$:

$$\left\| \frac{\partial \bar{f}}{\partial t}(t,z_1\ldots,z_n) \right\| \leqslant \mathbf{C}\,\left\|\bar{f}(t,z_1,\ldots,z_n)\right\|. \tag{$*$}$$

This condition is very general, ensuring that the trivializing vector field is rugose in the real analytic and $\mathscr{C}^\infty$ cases as well.

Note that condition $(*)$ can be re-phrased as

$$I\left(\frac{\partial \bar{f}}{\partial t}\right) \subset \overline{I(\bar{f})}. \tag{$**$}$$

We can summarize this discussion with the following theorem which is the analogue of Theorem 1·1 in this setting:

THEOREM 2·1. *Suppose F is a k-parameter unfolding of f such that F is an injective immersion at points off the parameter axis. Then F has a rugose trivialization if and only if $I(\partial\bar{f}/\partial u_i) \subset \overline{I(\bar{f})}$ for all i.*

Proof. Clear from the above discussion.

If F has a rugose trivialization, then it follows that the image of F is the total space of a Whitney equisingular family of analytic germs ([**24**]). With this observation it is easy to prove the following result, which gives us more insight into the nature of the maps in a rugose trivialization.

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THEOREM 2·2. *Suppose F is a k-parameter unfolding of f and suppose that F has a rugose trivialization L. Then l_u preserves the multiplicity of every Zariski closed subset of $f_u(\mathbf{C}^n)$.*

Proof. Suppose V is some closed subset of the image of f_u; denote the preimage of V by W. Consider the restriction of F to $\mathbf{C}^k \times W$. Then, this unfolding also has a rugose trivialization again using L. Since the differential of F has maximal rank when restricted to the tangent space of $\mathbf{C}^k \times \{z\}, z \neq 0$, the image of these tangent vectors are tangent to $\mathbf{C}^k \times V$ at all of the smooth points of $\mathbf{C}^k \times V$. It follows that the open stratum of $F(\mathbf{C}^k \times W)$ satisfies the Whitney conditions over the parameter space, hence the multiplicity of $f_u(W)$ is constant (cf. [**22**]).

COROLLARY 2·3. *Suppose F is a k-parameter unfolding of f such that F has a rugose trivialization L. Then $\overline{I(f_u)} = \overline{I(f)}$ for all u.*

Proof. Suppose ϕ: $\mathbf{C} \to \mathbf{C}^n$ is a curve; then the multiplicity of the image of ϕ by f or f_u must be the same and the degree of f_u as a map from the image of ϕ to the image of $f_u \circ \phi$ must be constant because the family is left trivial. This means that the components of $f \circ \phi$ and $f_u \circ \phi$ generate the same ideal in $\mathcal{O}_1$, which means that $\overline{I(f_u)} = \overline{I(f)}$ by the curve criterion for checking integral dependence.

At this point two questions arise. If the image of F is Whitney equisingular, and F is an injective immersion off the parameter axis, does F have a rugose trivialization? Given that F is an injective immersion off the parameter axis, does condition $(*)$ hold generically? As we shall see later, the answer for families of curves is yes to both questions, but in general the answer to both questions is no as the following example shows.

Example 2·4. Consider the family of map-germs given by

$$G(s, x, y) = (s, (1 + s)y - x^2, y^2, xy, x^3).$$

For $s \neq -1$, this family is analytically trivial using coordinate changes in source and target; define unfoldings of the identity by

$$R(s, x, y) = (s, x, (1 + s)^{-1}y), \;\; L(s, Z_1, Z_2, Z_3, Z_4) = (s, Z_1, (1 + s)^2 Z_2, (1 + s)Z_3, Z_4).$$

Then

$$\tilde{G}(s, x, y) = L \circ G \circ R = (s, y - x^2, y^2, xy, x^3).$$

The image of $\tilde{G}$ is a product, so it follows that the image of G is a product up to analytic equivalence, hence is Verdier equisingular.

However, $*$ fails to hold for G at every parameter value. This is easy to see using the curve $\phi_a(t) = (\sqrt{1 + at}, t^2)$; for the image of $g_a \circ \phi_a$ has multiplicity 3, while the image of $g_b \circ \phi_a$ has multiplicity 2, $b \neq a$ contradicting Theorem 2·2.

There is a converse to Corollary 2·3, however.

PROPOSITION 2·5. *Suppose F is a r-parameter unfolding of f such that F is an injective immersion off the parameter axis, and $\overline{I(f_u)} = \overline{I(f)}$ for all u close to the origin, then F has a rugose trivialization.*

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Proof. We can expand $\bar{f}(u,z)$ as a power series in u with coefficients $f_K \in \mathcal{O}_n$ where K is the multi-index specifying the term in the Taylor expansion to which the coefficient belongs. We claim $I(f_K)$ is in $\overline{I(f_0)}$ for all K. If not, we can find a curve ϕ and we can index the set of terms $f_K \circ \phi$ whose degree is smallest with respect to ϕ by J, f_0 not being among them. Suppose this degree is h. Then it follows that

$$\sum_{K\in J}((f_K\circ\phi)/t^h)(0)t^h u^K$$

is identically zero for all small values of u, because the multiplicity of $f_u \circ \phi$ is preserved. Hence $((f_K \circ \phi)/t^h)(0))$ are all zero. It now follows that $\partial\bar{f}/\partial u_i \in \overline{I(f_0)}$, and $\overline{I(\bar{f})} = \overline{I(f_0)}$ from which we see that $\partial\bar{f}/\partial u_i \in \overline{I(\bar{f})}$.

COROLLARY 2·6. *Suppose* $f\colon \mathbf{C},0 \to \mathbf{C}^p,0$ *and* F *is a* k*-parameter unfolding of* f *which is an injective immersion off the parameter axis. Then the following are equivalent*:

(A) F *has a rugose trivialization*;

(B) *the image of* F *is the total space of a Whitney equisingular family of curves*;

(C) *the multiplicity of the curves* $f_u(\mathbf{C})$ *at the origin is constant.*

Proof. The implication (A)⇒(B) is discussed after the proof of Theorem 2·1; the implication (B)⇒(C) follows because multiplicity is preserved in a Whitney equisingular family. Assume (C); then this implies that the ideals $f_u^*(m_p)\mathcal{O}_1$ are independent of u, so the result follows by Proposition 2·5.

It now is clear that in a family of curves, $*$ is a generic property, since generically any F is 1–1, and the multiplicity is constant.

These results show that it is important to control the behaviour of f_u at pairs of points to ensure that F is an immersion. We do this by controlling the double point set of f_u which is defined in $\mathbf{C}^n \times \mathbf{C}^n$ by the set of equations $\{f_{ui}(z) - f_{ui}(z') = 0\}$ where the f_{ui} are the component functions of f_u, as long as $z \neq z'$. Denote the corresponding ideal by $I_D(f_u)$. The problem with using this ideal sheaf to try and control the double points is that it vanishes on the diagonal of $\mathbf{C}^n \times \mathbf{C}^n$. This suggests pulling the ideal up to the blow-up of $\mathbf{C}^n \times \mathbf{C}^n$ by the diagonal and looking at the ideal sheaf that you get when you divide by a function which defines the exceptional divisor. We call this ideal sheaf a residual ideal sheaf, because it is obtained by dividing out by the sheaf of the exceptional divisor. The resulting ideal sheaf controls both the double point singularities and non-immersion points; for off the exceptional divisor, the new sheaf vanishes on the double point locus, while at a point of the exceptional divisor (x,l), the values of the generators of the ideal are just the directional derivatives of the f_{ui} in the direction determined by l. Thus the generators vanish at (x,l) if and only if l lies in the kernel of $Df_u(x)$.

However, it would be nice to be able to work directly on $\mathbf{C}^n \times \mathbf{C}^n$ instead of its blow-up. The key step to doing this is to realize that if h is a germ on $\mathbf{C}^n \times \mathbf{C}^n$, then the pull-back of h to the blow-up lies in the integral closure of the residual ideal sheaf if and only if hI_Δ lies in the integral closure of $I_D(f_u)$, where I_Δ denotes the ideal of functions vanishing on the diagonal Δ. This suggests that we study the integral closure of $I_D(f_u)$. We create some notation to facilitate this.

If h is a function in $\mathcal{O}_n$, then we denote $h(z) - h(z')$ in $\mathcal{O}_{2n}$ by h_D. We call $h(z) - h(z')$ the double of h. If I is an ideal in $\mathcal{O}_n$ then we denote by I_D the ideal in $\mathcal{O}_{2n}$ generated

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by the doubles of all the elements of I. (*Warning*: The ideal $I_D(f)$ is not the same as the ideal $I(f)_D$! In general $I_D(f)$ is properly contained in $I(f)_D$.) If we have an r-parameter unfolding $F(u,z) = (u, \bar{f}(u,z))$, then $\bar{f}_D$ is defined slightly differently. In this case, $\bar{f}_D = f(u,z) - f(u,z')$; we do not double the parameter variables.

Our next goal is to give a necessary and sufficient condition for a map-germ f to be finitely k-determined with respect to rugose trivialization. By this we mean that any unfolding of f by elements of m_n^{k+1} has a rugose trivialization. We actually will do more; we will show that our condition is necessary for a much weaker type of equivalence, and sufficient for a much stronger type of equivalence as well. We next describe the weaker form of equivalence.

Given a monomial H in m_n^{k+1} and a vector $\mathbf{a} \in \mathbf{C}^p$ we denote by $f + \mathbf{a}H$, the map-germ obtained by adding the p-tuple $\mathbf{a}H$ to f. We say that the k-jet of f is uniformly 1–1 sufficient if for each H which is a monomial in m_n^{k+1}/m_n^{k+3} and for each $\mathbf{a} \in \mathbf{C}^p$ there exists $\epsilon > 0$ such that $f + \mathbf{a}H$ is 1–1 on $B_\epsilon(0)$ and there exists $\delta_{H,\mathbf{a}} > 0$ such that for all $\mathbf{a}'$ in $B_{\delta_{H,\mathbf{a}}}(\mathbf{a})$, we have $f + H\mathbf{a}'$ is 1–1 on $B_\epsilon(0)$ as well.

If the k-jet of f is uniformly 1–1 sufficient, then for any unfolding F of f by monomial terms in m_n^{k+1}/m_n^{k+3}, the map F will be 1–1 on some polydisk about the origin, and the same property is true for any parameter value, although the polydisk will vary with the parameter. Clearly, if f is finitely k-determined for a C^0-trivialization, then the k-jet of f is uniformly 1–1 sufficient.

Now we describe our stronger equivalence relation. Given an unfolding F of $f\colon \mathbf{C}^n \to \mathbf{C}^p$, where $p \geqslant 2n$, we can consider the unfoldings of $\pi \circ f$ induced from F, where π is any generic linear projection from $\mathbf{C}^p$ to $\mathbf{C}^{p'}$ where $p \geqslant p' \geqslant 2n$. Here a projection is generic, provided $I_D(\pi \circ f)$ is a reduction of $I_D(f)$. Since it is known that any ideal in $\mathscr{O}_{2n}$ has a reduction generated by $2n$ generic linear combinations of generators, generic projections exist, and correspond to choosing such generic linear combinations of generators for $I_D(f)$. We can now ask that all of the induced unfoldings have a rugose trivialization. If all of the induced unfoldings have a rugose trivialization, then we say that the unfolding has a *stable rugose trivialization*.

Before stating the main result, we prove a lemma relating $(m_n^{k+1})_D$ and $m_{2n}^{k+1} I_\Delta$.

Proposition 2·7. *The ideal $(m_n^{k+1})_D$ contains $m_{2n}^{k+1} I_\Delta$ in its integral closure.*

Proof. It suffices to show that all elements of type $z_i^{k+1}(z_j - z_j')$ lie in the integral closure of $(m_n^{k+1})_D$. We have that

$$z_j'(z_i^{k+1} - z_i'^{k+1}) - (z_j z_i^{k+1} - z_j' z_i'^{k+1}) = -z_i^{k+1}(z_j - z_j')$$

which proves the lemma.

Note that in the proof of the lemma, it was only necessary to use the monomials of type z_i^{k+1} and $z_j z_i^{k+1}$. It may be true that $(m_n^{k+1})_D$ contains $m_{2n}^{k+1} I_\Delta$, but this seems difficult to prove and Proposition 2·7 suffices for our purposes. The next theorem is the analogue of Theorem 1·2.

Theorem 2·8. *Suppose $f\colon \mathbf{C}^n, 0 \to \mathbf{C}^p, 0$ is an analytic map-germ. Then the k-jet of f is uniformly 1–1 sufficient if and only if $(m_n^{k+1})_D \subset \overline{I_D(f)}^\dagger$ if and only if any unfolding of f by elements of m_n^{k+1} has a stable rugose trivialization along the parameter axis.*

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Proof. Assume the k-jet of f is uniformly 1–1 sufficient and suppose that there exists a monomial $H \in m_n^{k+1}/m_n^{k+3}$ such that $H_D \notin \overline{I_D(f)}^{\dagger}$. Then there exists a curve $\phi = (\phi_1, \phi_2)$ such that

$$o(H \circ \phi_1 - H \circ \phi_2) \leqslant o(f_i \circ \phi_1 - f_i \circ \phi_2) \quad \forall i,$$

where $o(h)$ denotes the order in t of any element of $\mathcal{O}_1$. Consider the unfolding $F = f + (u_1 H, \ldots, u_p H)$. Define Φ: $\mathbf{C} \to \mathbf{C}^{2n} \times \mathbf{C}^p$ by $z_i \circ \Phi = z_i \circ \phi_1$, $z_i' \circ \Phi = z_i' \circ \phi_2$ and

$$u_i \circ \Phi = -\frac{f_i \circ \phi_1 - f_i \circ \phi_2}{H \circ \phi_1 - H \circ \phi_2}.$$

The existence of the curve Φ contradicts the assumption that the k-jet of f is uniformly 1–1 sufficient, since as the curve tends to the parameter axis there will be a sequence of points along which F fails to be injective. This shows that H_D is in $\overline{I_D(f)}^{\dagger}$, for all monomials in m_n^{k+1}/m_n^{k+3}. Now the proof of Proposition 2·7 shows that the ideal $I(H)$ generated by the set of such H_D contains $m_{2n}^{k+1} I_\Delta$ in its integral closure, hence $m_{2n}^{k+1} I_\Delta$ is strictly dependent on $I_D(f)$ as well. Since the elements of $(m_n^{k+1})_D \subset m_{2n}^{k+1} I_\Delta + I(H)$, it follows that $(m_n^{k+1})_D \subset \overline{I_D(f)}^{\dagger}$.

Now assume that $(m_n^{k+1})_D \subset \overline{I_D(f)}^{\dagger}$. Let $G(u, z) = (u, f(z))$ be the trivial r-parameter unfolding of f. It is easy to check that

$$\overline{I_D(\bar{g})} = \overline{I_D(f)}\mathcal{O}_{2n+r}$$

and

$$\overline{I_D(\bar{g})}^{\dagger} = \overline{I_D(f)}^{\dagger}\mathcal{O}_{2n+r}.$$

(The content of this observation is that the integral closure of $I_D(\bar{g})$, which is taken in $\mathcal{O}_{2n+r}$ is just the ideal generated in $\mathcal{O}_{2n+r}$ by the integral closure of $I_D(f)$ in $\mathcal{O}_{2n}$.)

The proof proceeds by comparing the trivial unfolding of f with F.

Claim. Suppose F is an unfolding of f by terms in m_n^{k+1} then $\overline{I_D(\bar{f})} = \overline{I_D(\bar{g})}$.

The corresponding generators of the two underlying ideals differ by elements of $(m_n^{k+1})_D$. Since this last ideal is strictly dependent on $I_D(f)$, hence on $I_D(\bar{g})$, it follows from Nakayama's lemma that for any curve ϕ, $\phi^* I_D(\bar{g})\mathcal{O}_1 = \phi^* I_D(\bar{f})\mathcal{O}_1$, hence the two integral closures agree. By Proposition 2·7, $I_D(\bar{f})$ contains $m_{2n}^{k+1} I_\Delta \mathcal{O}_{2n+k}$. This implies that the residual double point ideal of F contains the pullback of $m_{2n}^{k+1}\mathcal{O}_{2n+k}$ to the blow-up of $\mathbf{C}^k \times \mathbf{C}^{2n}$ by $\mathbf{C}^k \times \Delta$, hence F must be an injective immersion off the parameter stratum. If we restrict $I_D(\bar{f})$ and $I_D(\bar{g})$ to the set of points where $z' = 0$, we get $\overline{I(\bar{f})} = \overline{I(f)}\mathcal{O}_{n+k}$. This implies that $\overline{I(f_u)} = \overline{I(f)}$ for all u close to the origin. Therefore the unfolding F has a rugose trivialization by Proposition 2·5. If we choose the projection π from $\mathbf{C}^p$ to $\mathbf{C}^{p'}$ so that $(\pi \circ f)_D$ is a reduction of $I_D(f)$, then the same hypotheses apply to the induced unfolding $(u, \pi \circ \bar{f})$, so F has a stable rugose trivialization.

It is clear that the third condition implies the first.

It is not difficult to see that the condition that $(m_n^{k+1})_D \subset \overline{I_D(f)}^{\dagger}$ holds for some k if and only if f is a 1–1 map germ which is an immersion except at the origin. (If f is such a map-germ then, $m_n^{k+1} \subset f^*(m_p^2)$ holds for some k ([**8**]), from which it is not hard to see that $(m_n^{k+1})_D \subset I_D(f)$.)

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In the real analytic case Theorems 2·1 and 2·8 continue to hold using the notion of real integral closure, which is defined using real-valued curves. The results needed to link the valuative criterion with the criterion in terms of analytic inequalities can be found in [**6**]. In the C^∞ case the main obstruction to developing the analogous theory is the link between the curve criterion and the criterion with inequalities. Indeed, it is not even clear which curves to use! As Theorem 2·10 shows, for our purposes, it turns out that analytic curves are a large enough family of curves to test for integral dependence, so if I is an ideal of $\mathscr{C}_n$, then we say $h \in \mathscr{C}$ is in the *integral closure of* I if $h \circ \phi$ is in the ideal generated by $\phi^*(I)$ in $\mathscr{C}_1$ for all *analytic* curves ϕ. (Here $\mathscr{C}_n$ denotes the ring of germs of C^∞ functions at the origin.)

Turning back to the relation between the curve criterion and analytic inequalities, for results on the order of determinacy, it is only necessary to worry about ideals like $m_n\mathscr{C}_n$ and $m_n\mathscr{C}_{n+k}$.

We next develop the material necessary to work with these ideals.

The first result is the analogue of Nakayama's lemma for analytic inequalities.

LEMMA 2·9. *Suppose* $\{g_1, \ldots, g_k\}$, $\{h_1, \ldots, h_j\}$ *are two collections of continuous functions defined in some neighbourhood of a point* x *in some metric space* X. *Then, the inequality*

$$\sup_i \|g_i(z)\| + \rho(z) \sup_i \|h_i(z)\| \geqslant C \sup_i \|h_i(z)\|,$$

where $C > 0$, ρ *is some continuous function which is* 0 *at* x, *holds on* U, *some neighbourhood of* x, *only if the inequality*

$$\sup_i \|g_i(z)\| \geqslant C_1 \sup_i \|h_i(z)\|$$

holds on some neighbourhood $U_1 \subset U$ *for some constant* $C_1 > 0$.

Proof. Suppose the conclusion is false; then for each constant $B > 0$, we can find a sequence of points $\{z_{B,j}\}$ converging to x such that

$$\sup_i \|g_i(z_{B,j})\| < B \sup_i \|h_i(z_{B,j})\|.$$

Take $B = C/2$, and take $z_{B,j}$ close enough to x so that $\rho(z_{B,j}) < C/2$. This gives a contradiction, for

$$\begin{aligned} C \sup_i \|h_i(z_{B,j})\| &= C/2 \sup_i \|h_i(z_{B,j})\| + C/2 \sup_i \|h_i(z_{B,j})\| \\ &> \sup_i \|g_i(z_{B,j})\| + \rho(z_{B,j}) \sup_i \|h_i(z_{B,j})\| \geqslant C \sup_i \|h_i(z_{B,j})\|. \end{aligned}$$

Before we prove the analogue of Theorem 2·8 in the C^∞ case, note that it is possible to strengthen the notion of stable rugose trivialization even further. In the complex analytic case on a space of dimension d we can always find a reduction of an ideal generated by d elements. If f is complex analytic, and is injective, and an immersion off the origin, then $\overline{I_D(f)}$ does not have a reduction generated by fewer than $2n$ elements, so we cannot project to $\mathbf{C}^p$ if $p < 2n$ and retain the hypothesis of Theorem 2·8. However, in the C^∞ case, in some examples, it is possible to find a reduction of $\overline{I_D(f)}$ with as few as n elements. In the C^∞ case the notion of stable rugose trivialization allows generic projections to spaces of dimension p less than $2n$ as long as we can find a reduction of $I_D(f)$ generated by p elements.

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THEOREM 2·10. *Suppose* $f\colon \mathbf{R}^n,0 \to \mathbf{R}^p,0$ *is a* C^∞ *map-germ. Then the* k*-jet of* f *is uniformly* 1–1 *sufficient if and only if* $(m_n^{k+1})_D \subset \overline{I_D(f)}^\dagger$ *if and only if any unfolding of* f *by elements of* m_n^{k+1} *has a stable rugose trivialization along the parameter axis.*

Proof. The proof of the first implication is the same as in Theorem 2·8. Note that since ϕ is an analytic curve and h is a monomial, $h_D \circ \phi$ vanishes to at most finite order, or is identically zero. If it is identically zero, then there is nothing to prove. If it vanishes to finite order, then we can divide as in the proof of Theorem 2·8 and get a smooth function as before.

Now assume $(m_n^{k+1})_D \subset \overline{I_D(f)}^\dagger$. By an easy argument using Nakayama's lemma, the same is true for the l-jet of f, which we denote by $T^l f$, for $l \geqslant k$. Since the real analytic elements of $(m_n^{k+1})_D \mathscr{C}_{2n+r}$ are in $\overline{I(T^k f)_D}$, it follows that there exists a neighbourhood V of the origin in $\mathbf{R}^r \times \mathbf{R}^{2n}$ and a constant $B > 0$ such that for all analytic elements g of $(m_n^{k+1})_D \mathscr{C}_{2n+r}$ and all $(u,z) \in V$,

$$\sup_i \left\| (T^k f_i)_D(u,z) \right\| \geqslant B \sup_i \|g(u,z)\| .$$

(Note that the above expression makes sense since $(T^k f_i)_D$ is independent of u.) Now we come to the main point of the proof:

Claim. There exists a neighbourhood U of the origin in $\mathbf{R}^r \times \mathbf{R}^{2n}$ and a constant $C > 0$ such that on U we have

$$\sup_i \left\| \bar{f}_{i_D}(u,z) \right\| \geqslant C \sup_i \left\| T^k f_{i_D}(u,z) \right\| .$$

Since $I_D(T^k f)$ and $I_D(T^{k+1} f)$ have the same integral closure, we know that

$$\sup_i \left\| T^{k+1} f_{i_D}(u,z) \right\| \geqslant C_1 \sup_i \left\| T^k f_{i_D}(u,z) \right\| .$$

So we have:

$$\begin{aligned} C_1 \sup_i \left\| T^k f_{i_D}(u,z) \right\| &\leqslant \sup_i \left\| T^{k+1} f_{i_D}(u,z) \right\| = \sup_i \left\| \bar{f}_{i_D}(u,z) - R_D(u,z) \right\| \\ &\leqslant \sup_i \left\| \bar{f}_{i_D}(u,z) \right\| + \|R_D(u,z)\| , \end{aligned}$$

where $R(u,z)$ is the remainder term when we subtract $T^{k+1} f_{i_D}(u,z)$ from $\bar{f}(u,z)$. Because $R(u,z)$ is the remainder term, it follows that

$$R(u,z) \in (m_{n+u}) m_n^{k+1} \mathscr{C}_{n+u}.$$

This means that

$$B \|R_D(u,z)\| \leqslant \|(u,z)\| \sup_i \left\| T^k f_{i_D}(u,z) \right\| .$$

Hence, on some smaller neighbourhood we have,

$$C_1 \left\| T^k f_{i_D}(u,z) \right\| \leqslant \sup_i \left\| \bar{f}_{i_D}(u,z) \right\| + 1/B \, \|(u,z)\| \sup_i \left\| T^k f_{i_D}(u,z) \right\| .$$

The claim then follows from Lemma 2·9.

The claim implies that F is an injective immersion off the parameter axis; for the inequalities lift to the blow-up, and once again we get that the residual ideal

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is supported only over the inverse image of the parameter stratum. Restricting the inequalities to the space $\mathbf{R}^k \times \mathbf{R}^n \times 0$, gives the inequality

$$C_3 \sup_i \|g(u,z)\| \leqslant \sup_i \left\|\bar{f}_i(u,z)\right\|$$

for all elements g of $m_n \mathscr{C}_{2n+u}$. This then, implies that the condition needed for a rugose trivialization of F and $\pi \circ F$ (condition $(**)$ of this section) holds.

As before the third implication is trivial.

As an example of the last theorem, which shows how different the C^∞ case is, consider the map-germ $f = (x, y^3 + x^2 y)$. It is not hard to check that f is an injective map-germ which is an immersion off the origin. We have

$$I_D(f) = (x - x', (y^2 + yy' + y'^2 + x^2)(y - y')),$$

and $(m_2^3)_D \subset \overline{I_D(f)}^\dagger$. Hence, f is 3 $\mathscr{L}^0$-determined.

In the next section we look for invariants whose constancy will be necessary and sufficient for the above three conditions in the complex analytic case.

3. *Analytic invariants and left trivialization*

The first invariant we describe will control the double point set of F.

If f is an injective map-germ of $\mathbf{C}^n$ to $\mathbf{C}^{2n}$, and we take a versal unfolding of f, then the double point set defined in the source of the unfolding has the same dimension as the parameter stratum, so the projection from the double point set to the parameter stratum is a finite map, and the degree of this projection at $(0,0)$ is the number of double points of f hiding at the origin. It is clear that, as we vary the parameter in the base of the versal unfolding, the number of double points, counted with appropriate multiplicities, of the corresponding map-germs remains constant. If the target of f has dimension $p > 2n$ then the double point set no longer surjects onto the base of the versal unfolding, and this property fails to hold. (For more details on the construction of this type of invariant cf. [**10**, p. 190].)

The first goal of this section is to describe a way of calculating this invariant in the case when $p = 2n$.

This was done by Teissier ([**20**]) in the case when $n = 1$. In this case, it is not necessary to blow-up to get the residual ideal sheaf of $I_D(f)$, because we are working in $\mathbf{C}^2$ and the ideal of the diagonal is principal already. So we just divide $I_D(f)$ by the equation of the diagonal to get the residual sheaf. Teissier showed that the multiplicity of this residual sheaf is the number of double points of the map.

In our case we will use the machinery of Segre cycles and numbers, as developed in [**12**]. These invariants have the advantage that they make it unnecessary to deal directly with the residual sheaf. They also have the advantage that these invariants can be computed in examples without blowing up. We review the definitions briefly here.

Let $(X,0) \subset (\mathbf{C}^N, 0)$ be a reduced, closed analytic space of pure dimension n and $I \subset \mathscr{O}_{X,0}$ an ideal which defines a nowhere dense subspace of $(X,0)$. Consider the blow-up B of X along I:

$$b\colon B = Bl_I X \to X$$

with exceptional divisor D.

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The restriction to B of the projection onto the second factor of $X \times \mathbf{P}^{M-1}$ induces a map p: $B \to \mathbf{P}^{M-1}$. Hence, a hyperplane in the projective space induces a Cartier divisor on B via the pullback p^*, provided B is not contained in the product of X and the hyperplane. We call it a *hyperplane on* B. If V is an analytic subset of B of dimension k, with no component contained in D, then a hyperplane H of B is *general with respect to* V if $H \bigcap V$ is reduced of dimension $k-1$ and none of its components is contained in D.

Consider an m-tuple $g = (g_1, \ldots, g_m)$ of linear combinations of the generators of I. Assume that each hyperplane H_i on B corresponding to g_i is general w.r.t. $H_1 \bigcap \cdots \bigcap H_{i-1} \bigcap Bl_I X$. Then, we also say that g is *general with respect to* X. The *polar varieties* and *Segre cycles* of I on with respect to g are defined as follows:

$$P_0^g(I) \coloneqq X, \quad P_k^g(I) \coloneqq b(H_1 \bigcap \cdots \bigcap H_k \bigcap Bl_I X),$$
$$\Lambda_k^g(I) \coloneqq b_*(H_1 \cdots H_{k-1} \cdot D \cdot Bl_I X).$$

Define the multiplicity of a cycle $S = \sum a_i[V_i]$ at 0 as

$$\operatorname{mult}_0 S = \sum a_i(\operatorname{mult}_0 (V_i)).$$

In [**12**] it is shown that for a generic set of generators of I the multiplicities of the polar cycles and Segre cycles are independent of all the choices made. Assuming that g is a member of this generic set, we drop the superscripts on the polar varieties and Segre cycles and define the kth *polar multiplicity* of I, $k = 0, \ldots, m-1$, as

$$m_k(I) = \operatorname{mult}_0 P_k(I),$$

and for $k = 1, \ldots, m$ the kth *Segre number* of I as

$$e_k(I) = \operatorname{mult}_0 \Lambda_k(I).$$

If we wish to construct the polar varieties of I working directly on X, then we can look at the space defined by intersecting the irreducible components of $I(g)$ which do not contain I. For details on this approach see [**18**, p. 10]. Since the polar varieties are unions of components of complete intersections, they cannot have embedded components.

In our situation, I will be $I_D(f)$, and we will be interested in $e_{2n}(I_D(f))$. Since f is an injective immersion off the origin, the exceptional divisor will have two components; one will surject onto the diagonal of $\mathbf{C}^{2n}$, the other will be $\mathbf{P}^{2n-1}$, counted $e_{2n}(I_D(f))$ times.

In general, the Segre numbers are only lexicographically upper semicontinuous; that is the Segre number of codimension 1 is upper semicontinuous and, if the Segre number of codimension 1 is constant, then the Segre number of codimension 2 is upper-semicontinuous, etc. However in our case we can do better.

PROPOSITION 3·1. *Suppose F is an unfolding of a map-germ f: $\mathbf{C}^n, 0 \to \mathbf{C}^{2n}, 0$, f an injective immersion off the origin. Then, the sum of the Segre numbers of $\mathscr{I}_D(f_u)$ of codimension $2n$ is independent of u, for u small.*

Proof. To prove this it is helpful to have an alternate characterization of the Segre numbers developed in [**12**]. In order to calculate the Segre number of greatest codimension of I, it suffices to take the polar curve of I and to calculate the multiplicity

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of the ideal obtained by restricting I to the polar curve. In our situation, the restriction of $I_D(f)$ to its polar curve is principal, because $I_D(f)$ has $2n$ generators and $2n-1$ of them are zero on the polar curve, by the definition of the polar curve given above. If F is a k-parameter unfolding, consider the polar variety of $I(\bar{f})_D \subset \mathbf{C}^k \times \mathbf{C}^{2n}$ of codimension $2n-1$. We can ensure that the generators chosen to define this set also define the polar curve of f, because this amounts to looking at two generic conditions on the set of sets of linear combinations of the generators of $I(\bar{f})_D$ with $2n-1$ elements. A priori, we do not know that the fibre of $P_{2n-1}(I(\bar{f})_D)$ over $u=0$ is the polar curve of $I_D(f)$, although it contains it set theoretically. However, choosing $2n-1$ generic linear combinations of generators of $I(\bar{f})_D$ to define the polar variety of codimension $2n-1$ amounts to composing $\bar{f}$ with a generic linear projection π to $\mathbf{C}^{2n-1}$; we can choose this projection so that $\pi \circ f$ has the origin as an isolated critical point and away from the origin is an immersion with normal crossings. Now if the fibre of $P_{2n-1}(I(\bar{f})_D)$ over $u=0$ has a curve in the diagonal as a component, then $\pi \circ f$ has a curve of critical points, which contradicts the choice of π. Further, the remaining components of the fibre of $P_{2n-1}(I(\bar{f})_D)$ are generically reduced, since $\pi \circ f$ is an immersion with normal crossings away from the origin. This implies they are reduced, since the fibre has no components of dimension less than 1. So the fibre of $P_{2n-1}(I(\bar{f})_D)$ over $u=0$ is $P_{2n-1}(I(f)_D)$. As remarked above, the restriction of the ideals $I(\bar{f})_D$, $I(f)_D$ to their respective polars is principal; so we can select a generator $\bar{g}$ of the restriction of $I(\bar{f})_D$ whose restriction g to $P_{2n-1}(I(f)_D)$ also generates $I(f)_D$. Consider the map defined by the product of the projection to the base of the unfolding with the map to $\mathbf{C}$ given by the selected generator. Since this is a finite map to a smooth variety, the degree is constant over the points of $\mathbf{C} \times 0$ in the target. Note that $0 \times \mathbf{C}$ is not contained in the discriminant of the map. This follows because g is finite and $u=0$ defines the fibre with reduced structure. This implies that the degree of g at the origin of $P_{2n-1}(I(f)_D)$ is the same as the degree of $(u, \bar{g})$ at the origin of $P_{2n-1}(I(\bar{f})_D)$. This also holds for nearby fibres of $P_{2n-1}(I(\bar{f})_D)$ as well by the same argument. The degree of g is just the multiplicity of the ideal generated by g, and this is just $e_{2n}(I_D(f))$. At $u \neq 0$, it follows that this constant degree is just the sum of the multiplicities of the ideals generated by g_u, sum taken over preimages of 0. This sum is just the sum of the $e_{2n}(I_D(f_u), x)$ where the sum is taken overall x where $e_{2n}(I_D(f_u), x)$ is non-zero.

Note that the above proof shows that $e_{2n}(I_D(f))$ is just the colength of $I_D(f)$ in the local ring of $P_{2n-1}(I(f)_D)$ at the origin. This follows because, since $P_{2n-1}(I(f)_D)$ is a reduced curve, hence Cohen–Macaulay, the multiplicity of a principal ideal is just the colength of the ideal.

It seems reasonable to hope that $e_{2n}(I_D(f))$ is the number of double points of f at the origin; unfortunately, this turns out not to be true. Recall, that if g: $\mathbf{C}^n \to \mathbf{C}^{2n-1}$ then the number of Whitney umbrella points of g at the origin is just the colength of the ideal of maximal minors of the Jacobian matrix of g ([**19**]). Denote the number of Whitney umbrella points of g by $C(g)$. The number of Whitney umbrella points in an unfolding of g is also constant ([**19**]). Recall also, that if F is a 1-parameter unfolding of f such that for $u \neq 0$, f_u is a stable map-germ on some fixed neighbourhood of the origin, then we say that F is a stabilization of f. Then we have the following result:

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THEOREM 3·2. *Suppose* $f\colon \mathbf{C}^n, 0 \to \mathbf{C}^{2n}, 0$, *f finitely* $\mathscr{L}$*-determined,* F *is a stabilization of* f, $\pi\colon \mathbf{C}^{2n} \to \mathbf{C}^{2n-1}$ *a generic linear projection for* F, *then the sum of* $C(\pi \circ f_u)$ *and the number of double points of* f_u *is equal to* $e_{2n}(I_D(f))$.

Proof. The polar curve of $I(f_u)_D$ consists of the double point curve of $\pi \circ f_u$ in $\mathbf{C}^{2n}$. Let g be a generator of the restriction of $I(\bar{f})_D$ to its polar variety of codimension $2n-1$. Then g_u vanishes at each point where the polar curve of $I(f_u)_D$ intersects the diagonal. These points project to the Whitney umbrella points of $\pi \circ f_u$.

The sets of points of the polar curve where g_u vanishes which are not on the diagonal, are exactly the double points of f_u in $\mathbf{C}^{2n}$. Because f_u is an immersion at the points where $\pi \circ f_u$ has a Whitney umbrella, it follows that the zero set of g is transverse to the polar curve at points where the polar curve intersects the diagonal. (This is easily verified using the normal form for the Whitney umbrella.) This means that each Whitney umbrella point of f_u contributes 1 to the total value of $e_{2n}(I_D(f_u))$. Using local coordinates it is also easy to see that each double point also contributes 1 to the total value of $e_{2n}(I_D(f_u))$.

COROLLARY 3·3. *Suppose* f *and* π *are as above, then the number of double points of* f *at the origin is equal to* $e_{2n}(I_D(f)) - C(\pi \circ f)$.

Proof. All three terms in the above corollary are constant in a stabilization; since equality holds for the general member of a stabilization of f it holds for f.

Note that if the image of f is a plane curve, then the number of Whitney umbrellas of $\pi \circ f$ is just the multiplicity of the curve minus 1. In this case,

$$e_2(I_D(f)) = \mu(f(\mathbf{C})) + m(f(\mathbf{C})) - 1.$$

Example 3·4. In this example, we will show how to calculate $e_4(I_D(f))$ and $C(f)$.

Let $f(x,y) = (x^2, x^3 - xy, y^2, y^3 + 2xy)$. It is easy to check that f is injective and an immersion except at 0. We want to find the polar curve of $I_D(f)$. This is equivalent to finding the double point curve of $\pi \circ f$ where π is a generic projection to $\mathbf{C}^3$ for f. We claim that $\tilde{f}(x,y) = (x^2, x^3 + xy + y^3, y^2)$ is generic enough and prove this at the end of the example.

The number of Whitney umbrellas of $\tilde{f}$ is given by the colength of the ideal of 2×2 minors of $D\tilde{f}$ in $\mathscr{O}_2$; by a theorem of Buchsbaum and Rim ([**7**], [**11**, p. 214]), this is the multiplicity of the module for which the transpose of $D\tilde{f}$ is a matrix of generators. Denote it by $JM(\tilde{f})^T$. (In general, if $JM(\tilde{f})^T$ is a reduction of $JM(f)^T$, then this multiplicity will be as small as possible and hence it will be $C(f)$.)

In our example, this ideal of minors is (xy, x^2, y^2) which has colength 3 and which is as small as possible given that the original map had rank 0, so $C(f) = 3$.

The ideal

$$I_D(\tilde{f}) = (x^2 - x'^2, y^2 - y'^2, x^3 + xy + y^3 - (x'^3 + x'y' + y'^3))$$

defines the polar curves of $I_D(f)$.

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Some computations show that if $x = -x'$ and $y = -y'$, then we get three components of the polar curve with parametrizations

$$(t, -t, t, -t)$$
$$(t, -t, \eta_1 t, -\eta_1 t)$$
$$(t, -t, \eta_2 t, -\eta_2 t),$$

where η_1 and η_2 are non-trivial third roots of unity.

The multiplicity of $I_D(f)$ on each component is 3.

If $x = x'$ and $y = -y'$, then the third generator of $I_D(\tilde{f})$ becomes $2y^3 + 2xy$. With these restrictions, the equation of the diagonal becomes $y = 0$, so the third equation of this component of the polar curve becomes $y^2 + x = 0$. The multiplicity of $I_D(f)$ on this component is 3. The case where $x = -x'$ and $y = y'$ is parallel, and also contributes 3 to the value of $e_4(I_D(f))$. So, given the claim about genericity, $e_4(I_D(f)) = 15$, $C(f) = 3$ and the number of double points of f at the origin is 12.

Now we show that our projection is generic. We can construct a 1-parameter family of projections π_t such that $\pi_0 \circ f = \tilde{f}$, and π_t, $t \neq 0$, is good enough so that we can calculate $C(f)$ and $e_4(I_D(f))$ from $\pi_t \circ f$. Since $C(\tilde{f})$ is already as small as possible, it must be independent of t. Consider the double point set of our one parameter unfolding and the map of it to $\mathbf{C}^2$ with components the projection to the base and $x^3 - x'^3 - xy + x'y'$. This is a finite map to a smooth base, hence the number of preimages of a point in the base is constant when counted with multiplicity. Observe that the fibre of the double point set over the origin in the projection to the base of the unfolding is exactly the five curves identified above; any other points would lie in the diagonal and would give a curve of singular points of $\tilde{f}$. Such curves do not exist. Furthermore, since $\tilde{f}$ is an immersion with normal crossings except at the origin, the fiber over the origin is reduced, so the degree of our map is just the degree of the restriction to the fibre over the origin.

Consider the preimages of the parameter axis in $\mathbf{C}^2$. Suppose these differ from $(0, 0, 0, 0)$. The curve of such points cannot lie outside the diagonal, for then f would not be injective. Nor can they lie inside the diagonal, for then the number of Whitney umbrella points would change. So such a curve does not exist and the multiplicity $x^3 - x'^3 - xy + x'y'$ in the polar curves remains constant. Hence $\tilde{f}$ is generic.

Note that the above discussion shows that $\tilde{f}$ is generic enough if $C(\tilde{f}) = C(f)$ and $\tilde{f}$ is an immersion with normal crossings except at the origin. Thus the computation of $e_{2n}(I_D(f))$ boils down to finding a reduction of $JM(f)^T$ with $2n - 1$ generators, checking that the induced germ from $\mathbf{C}^n \to \mathbf{C}^{2n-1}$ is an immersion with normal crossings except at the origin, and calculating the equations for the double point set of the induced map-germ.

The referee pointed out to the author that Artin and Nagata ([**1**, p. 320],) had another algebraic invariant which counts the number of the number of double points in the source. In the terminology of this paper, their invariant is $\epsilon(f)$, the dimension over $\mathbf{C}$ of the quotient of I_Δ by $I_D(f)$. If the rank of f is $n - 1$ at the origin, then it is easy to see algebraically that

$$e_{2n}(I_D(f)) = C(\pi \circ f) + \epsilon(f).$$

In general, the existence of an algebraic proof remains an open question, although Corollary 3·3 provides a geometric proof. The calculation of Example 3·4 is easier

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to do using Corollary 3·3 than trying to calculate $\epsilon(f)$ directly because the different components of the polar curve break the calculation into 5 small calculations instead of one large one.

COROLLARY 3·5. *Suppose F is a k-parameter unfolding of f, $f\colon \mathbf{C}^n,0 \to \mathbf{C}^{2n},0$, an injective immersion off the origin. Then F is an injective immersion off the parameter axis if and only if $e_{2n}(I_D(f_u)) - C(\pi \circ f_u)$ is constant along the parameter axis.*

Proof. We know that $e_{2n}(I_D(f_u)) - C(\pi \circ f_u)$ is the number of double points of f_u; this invariant is constant along the parameter stratum if and only if no double point curves split off from the stratum.

The main theorem of this section will give a numerical criterion for a family to have a stable rugose trivialization. Using Corollary 3·5, we can ensure the injectivity of F and all of its generic projections. We can use Proposition 2·5 to ensure the rugose triviality of all of the unfoldings simultaneously provided we have a criterion for $\overline{I(f_u)} = \overline{I(f)}$ for all u close to the origin. We can use the mixed multiplicities developed by Teissier ([**21**]) to provide this criterion.

Given two ideals I_1 and I_2 in $\mathcal{O}_n$ of finite colength, one can produce a sequence of numbers by choosing i generic elements, where $0 \leqslant i \leqslant n$, of I_1 and $n-i$ elements of I_2, then taking the multiplicity of the ideal that they generate. These numbers, denoted $e_{i,n-i}(I_1, I_2)$, are called the mixed multiplicities of I_1 and I_2. Notice that the multiplicity of I_1 and the multiplicity of I_2 are two of the numbers in this sequence.

Teissier showed in [**21**] that two ideals of finite colength have the same integral closure if and only if all of their mixed multiplicities are the same. Using this result we can prove the following:

THEOREM 3·6. *Suppose $f\colon \mathbf{C}^n,0 \to \mathbf{C}^p,0$ is an injective immersion off the origin, $p \geqslant 2n$, F a k-parameter unfolding. Then F has a stable rugose trivialization if and only if $e_{2n}(I_D(f_u)) - C(f_u)$ is constant along the parameter axis, and the mixed multiplicities of $I(f)$ and $I(f_u)$ are equal and independent of u.*

Proof. ($\Rightarrow$) Suppose F has a stable rugose trivialization. This implies that there exists a sequence of generic projections $\pi_{p'}\colon \mathbf{C}^p$ to $\mathbf{C}^{p'}$ where $p \geqslant p' \geqslant 2n$, such that $F_{p'}(u,z) \coloneqq (u, \pi_{p'} \circ \bar{f}(u,z))$ has a rugose trivialization. Here a projection is generic, provided $I_D(\pi \circ f)$ is a reduction of $I_D(f)$. Because f is injective, the existence of a stable rugose trivialization implies that $F_{p'}$ is injective; because f is an immersion off the origin, it follows that $F_{p'}$ is also. (This follows because a rugose trivialization preserves the multiplicity of analytic sets, in particular the singular points of the image of f_u are carried to singular points of f.) Since F_{2n} is an injective immersion, off the parameter axis, $e_{2n}(I_D(\pi_{2n} \circ f_u)) - C(\pi_{2n} \circ f_u)$ is constant along the parameter axis.

We can choose the projection π_{2n} so that on a set which includes a Z-open subset of the parameter axis, $e_{2n}(I_D(\pi_{2n} \circ f_u)) = e_{2n}(I_D(f_u))$, $C(\pi_{2n} \circ f_u) = C(f_u)$. This follows because the analytic spread of of $I_D(\bar{f})$ is generically $2n$, while the analytic spread of $JM_z(\bar{f})^T$ is generically $2n-1$. On this dense set which includes the origin we then have $e_{2n}(I_D(f_u)) - C(f_u)$ is independent of u. Suppose we are at a point u on the complement of this dense set, then the number of double points at the origin of $\pi_{2n} \circ f_u$ must be the same as for some other generic projection, because the number

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of double points is upper semicontinuous. For, if they changed with a new projection, they would have to drop, which is impossible if the complement contains the origin in its closure. So at all points we have $e_{2n}(I_D(f_u)) - C(f_u)$ is constant.

By Corollary 2·3, the integral closure of $I(f)$ and $I(f_u)$ are the same, hence their mixed multiplicities are equal and independent of u.

($\Leftarrow$) If $p = 2n$ then the hypothesis implies that F is an injective immersion off the parameter axis by Corollary 3·5. The equality of the mixed multiplicities implies that the integral closure of $I(f)$ and $I(f_u)$ are the same, so Proposition 2·5 implies the existence of a rugose trivialization.

If $p > 2n$, then we note that the hypotheses of the theorem apply to $\pi_{2n} \circ F$ for a generic projection π_{2n} from $\mathbf{C}^p \to \mathbf{C}^{2n}$. To see this, choose the projection as above, so that at the origin and on a Z-open set dense set of the parameter axis, $e_{2n}(I_D(\pi_{2n} \circ f_u)) = e_{2n}(I_D(f_u))$, $C(\pi_{2n} \circ f_u) = C(f_u)$, $e(I(\pi_{2n} \circ f)) = e(I(f))$, $e(I(\pi_{2n} \circ f_u)) = e(I(f_u))$. Then by upper semi-continuity of the number of double points we have that the difference of $e_{2n}(I_D(\pi_{2n} \circ f_u))$ and $C(\pi_{2n} \circ f_u)$ is independent of u, and $e(I(\pi_{2n} \circ f_u)) = e(I(f_u))$ holds for all u and is independent of u. Since $e(I(\pi_{2n} \circ f_u)) = e(I(f_u))$ for all u, we have that $I(\pi_{2n} \circ f_u)$ is a reduction of $I(f_u)$. Since the integral closures of $I(f_u)$ are independent of u, the same holds then for $I(\pi_{2n} \circ f_u)$.

The above theorems then imply that $\pi_{2n} \circ F$ has a rugose trivialization. Since $\pi_{2n} \circ F$ is an injective immersion off the parameter stratum then F must be also, so we can apply the same theory to this case.

Looking back at Example 2·4, we see that the ideal used to calculate the mixed multiplicity $e_{1,1}$ of the pair of ideals $I(f)$ and $I(f_s)$ for $s \neq 0$ is $(y - x^2, (1+s)y - x^2) = (y, x^2)$. This ideal has multiplicity 2, which is less than the multiplicity of $I(f)$ which is 3. This explains why G does not have a rugose trivialization even though the family G is an injective immersion off the parameter axis.

In the case of plane curves, Theorem 3·6 boils down to the constancy of the multiplicity of the curve and its Milnor number. This constancy is already necessary for the C^0 triviality of a family of plane curves.

Looking back at Example 3·4, consider the set of map-germs of the form $f_h = (x^2, y^2, x^3 + xy + 2y^3, h(x, y))$, where $h \in m_2^3$. For all h, the integral closure of $I(f_h)$ is constant and $C(f_h) = 3$. By Theorem 3·6, studying families of such germs up to rugose trivialization just depends on the values of $e_4(I_D(f_h))$ and this just depends on the contact of h with the five smooth curves described in the example. In particular, the family of map-germs corresponding to plane curves of multiplicity 3 with tangent cones transverse to these five given curves has a rugose trivialization.

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