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EFFECTIVE BOUNDS ON HOLOMORPHIC MAPPINGS INTO COMPLEX HYPERBOLIC MANIFOLDS

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ABSTRACT

We give an explicit upper bound on the number of non-constant holomorphic maps from a quasi-projective manifold into a complex hyperbolic manifold of finite volume. This gives an effective version of the results of Sunada and Noguchi.

Introduction

Let X be a compact Riemann surface of genus $g \geqslant 0$ and let M be an n-dimensional compact complex hyperbolic manifold. Here M is a complex hyperbolic manifold means that M is a complex manifold endowed with a (necessarily unique) complete complex hyperbolic metric of constant negative holomorphic sectional curvature -1. It is well known that M is necessarily of the form $\mathbb{B}^n/\Gamma$, where $\mathbb{B}^n := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \,|\, |z_1|^2 + \ldots + |z_n|^2 < 1\}$ and Γ is a discrete torsion-free subgroup of the automorphism group of $\mathbb{B}^n$. Denote by $\mathcal{H}(X, M)$ the set of all non-constant holomorphic maps from X into M. In [**18**], Sunada proved that $\mathcal{H}(X, M)$ is finite. This result of Sunada was generalized to the non-compact situation by Noguchi [**14**] in the following form. Let X be an m-punctured Riemann surface of genus g, that is,

$$X = \bar{X} \setminus \{x_1, \ldots, x_m\}$$

for a smooth compact Riemann surface $\bar{X}$ of genus g and m distinct points $x_1, \ldots, x_m$ in $\bar{X}$. Let M be a complex hyperbolic manifold of finite volume. Here the volume of M is measured with respect to the complex hyperbolic metric. Then the set $\mathcal{H}(X, M)$ of non-constant holomorphic maps from X into M is finite. The goal of this paper is to give an effective version of their results as follows.

THEOREM 1. *Let X be an m-punctured Riemann surface of genus g. Let M be a complex hyperbolic manifold of dimension n. Then the number of non-constant holomorphic maps from X into M is bounded above by*

$$\sum_{\ell=\max\{0,3-2g\}}^{m} \binom{m}{\ell} \left(\frac{(5n+4)^n \operatorname{Vol}(M)}{2^{2n-1}\pi^n} \right)^{\ell} \sum_{d=1}^{B} \sum_{j=1}^{2^{\ell d}} \binom{(2n+4)\max(j, C)}{2n+3}^{(2n+4)(j^2+1)}, \tag{0.1.1}$$

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where

$$B = 2g + 1 + \frac{(n^2+3n+4)(n+1)(2g-2+\ell)}{4},$$
$$C = \frac{(2g+1)2^{\ell(n+1)}(n+1)^{n+1}(n^2+3n+4)^n n!}{8^n \pi^n} \operatorname{Vol}(M). \tag{0.1.2}$$

Here $\operatorname{Vol}(M)$ *denotes the hyperbolic volume of* M.

The works of [**14**] and [**18**] did not give any information about the cardinality of $\mathcal{H}(X, M)$. *A priori*, the number may depend on the moduli of X or properties of M other than its volume. Note that if $2g - 2 + m \leqslant 0$, then $\mathcal{H}(X, M)$ is empty. Thus we may assume that $2g - 2 + m > 0$. In this case, X is a hyperbolic Riemann surface and it is well known that the hyperbolic volume of X is $2\pi(2g - 2 + m)$. Hence both m and $2g - 2 + \ell$ (for $\ell \leqslant m$) can be bounded above in terms of the hyperbolic volume of X. Thus Theorem 1 has the following consequence.

COROLLARY 2. *Let* X *be a hyperbolic Riemann surface of finite volume and* M *be a complex hyperbolic manifold of finite volume. Then the number of non-constant holomorphic maps from* X *into* M *is bounded above by a number depending only on the dimension of* M *and the hyperbolic volumes of* X *and* M.

More generally, let X be any quasi-projective manifold. In [**14**], it was proved that $\mathcal{H}(X, M)$ is finite. Using Theorem 1, we can get an effective bound on $\mathcal{H}(X, M)$ as follows.

COROLLARY 3. *Let* X *be a quasi-projective manifold. Suppose that through a general point of* X, *there exists a quasi-projective curve whose normalization is an* m*-punctured Riemann surface of genus* g. *Let* M *be an* n*-dimensional complex hyperbolic manifold of finite volume. Then the number of non-constant holomorphic maps from* X *into* M *is bounded above by the same number as in Theorem* 1.

When X is an n-dimensional projective manifold with a very ample line bundle L, one can find a bound on g with $m = 0$ in terms of $L^{n-1} \cdot K_X$ by the adjunction formula. There are cases when this can also be done for quasi-projective manifolds. In particular, in the course of the proof of Theorem 1, we will show that a complex hyperbolic manifold X of finite volume and of dimension at least 2 contains a smooth irreducible projective curve of genus g through a general point, where g can be explicitly estimated in terms of the dimension and volume of X. Thus we can get the following generalization of Corollary 2.

THEOREM 4. *Let* X *and* M *be two complex hyperbolic manifolds of finite volume. Then the number of non-constant holomorphic maps from* X *into* M *is bounded above by a number* N *depending only on the dimensions and the hyperbolic volumes of* X *and* M. *More explicitly, if* X *and* M *are of dimension* k *and* n *respectively, then* N *can be given as*

$$\sum_{d=1}^{B} \binom{(2n+4)\max(d, C)}{2n+3}^{(2n+4)(d^2+1)}, \tag{0.1.3}$$

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where

$$B = 3 + \frac{4 + (n+1)(n^2+3n+4)}{4} \times \frac{[2 + (k-1)(k^2+3k+4)](k^2+3k+4)^{k-1}(k+1)^k k!}{8^k \pi^k} \operatorname{Vol}(X),$$

$$C = \frac{(n+1)^{n+1}(n^2+3n+4)^n n!}{8^n \pi^n} \operatorname{Vol}(M) \times \left[3 + \frac{[2 + (k-1)(k^2+3k+4)](k^2+3k+4)^{k-1}(k+1)^k k!}{8^k \pi^k} \operatorname{Vol}(X)\right]. \tag{0.1.4}$$

The proof of Theorem 1 is quite straightforward when M is compact. It uses the Schwarz lemma and two effective results in algebraic geometry: the effective bound on the complexity of Chow varieties, and some effective results related to the Fujita conjecture. The bound on the complexity of Chow varieties has been studied in connection with holomorphic maps to surfaces of general type in [**7**] and [**20**]. We just need a slight variation of the result in [**7**]. For the result on the Fujita conjecture, essentially any effective result on the separation of points will suffice here, such as [**1**].

The more interesting case is when M is a non-compact complex hyperbolic manifold (see for example [**5**] for background material on complex hyperbolic manifolds). Here an additional tool is needed, namely, the theory of compactifications of complex hyperbolic manifolds of finite volume. The properties of both their Siu–Yau compactifications and toroidal compactifications play important roles here. Using these results and the work of Kollár [**11**] generalizing [**1**], we will modify the aforementioned effective results of algebraic geometry so that they are applicable to non-compact complex hyperbolic manifolds. In the course of the proof, we establish some effective results concerning pluri-canonical sections of complex hyperbolic manifolds, which are of independent interest.

We remark that our primary goal is just to exhibit the effective nature of our results. As such, the reader will notice that there are numerous places in our arguments where the estimates may possibly be improved. Some of them are rather straightforward, while the others may be more subtle. We will leave them for the interested reader. However, as far as we can see, even when all possible improvements of estimates are elaborated, the results will likely still be quite far from optimal.

In [**14**], the finiteness of $\mathcal{H}(X, M)$ is proved when M is an arbitrary Hermitian locally symmetric manifold of finite volume, under certain restrictions on the rank of the map. It is natural to ask whether our method can be generalized to give an effective version of this result. When M is compact, our argument can be generalized easily. On the other hand, our methods here do not seem to generalize readily to the case when M is non-compact. The main difficulty is that the structure of the compactifications of Hermitian locally symmetric manifolds of rank at least 2 is much more complicated than that of complex hyperbolic manifolds. In particular, results analogous to Proposition 2.2.1 and Proposition 4.1.1 are difficult to obtain.

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1. *Effective bounds on the Chow varieties*

1.1. Let Z be an irreducible projective variety, and let $\mathrm{Chow}_k(Z)$ be the Chow variety parametrizing irreducible and reduced subvarieties of dimension k in Z. It has countably many irreducible components and each irreducible component is a quasi-projective variety. Note that our use of the term 'Chow variety' and the symbol '$\mathrm{Chow}_k(Z)$' are not standard, since we consider only irreducible and reduced cycles. Since general algebraic cycles will play no role in our argument, we will restrict our discussion to this subset of the usual Chow variety.

When L is an ample line bundle on Z, we denote by $\mathrm{Chow}^L_{k,d}(Z)$ the Chow variety parametrizing irreducible and reduced subvarieties X of Z which are of dimension k and degree d with respect to L, that is, $X \cdot L^k = d$. Then $\mathrm{Chow}^L_{k,d}(Z)$ is the union of finitely many irreducible components of $\mathrm{Chow}_k(Z)$. We denote by $\sharp\,\mathrm{Chow}^L_{k,d}(Z)$ the number of irreducible components of $\mathrm{Chow}^L_{k,d}(Z)$. Given finitely many points $z_1, \ldots, z_a$ on Z, we denote by $\mathrm{Chow}_k(Z; z_1, \ldots, z_a)$ the subvariety of $\mathrm{Chow}_k(Z)$ parametrizing members of $\mathrm{Chow}_k(Z)$ which contain all the points $z_1, \ldots, z_a$. The subvariety $\mathrm{Chow}^L_{k,d}(Z; z_1, \ldots, z_a)$ is defined similarly. By the construction of Chow varieties in [**10**], the following lemma is obvious.

LEMMA 1.1.1. *Let $f\colon Z_1 \to Z_2$ be a morphism between two irreducible projective varieties which is set-theoretically bijective. For any ample line bundle L on Z_2 and any integers $k, d > 0$, the push-forward morphism*

$$f_*\colon \mathrm{Chow}^{f^*L}_{k,d}(Z_1) \to \mathrm{Chow}^L_{k,d}(Z_2)$$

is bijective and sends each irreducible component of $\mathrm{Chow}^{f^*L}_{k,d}(Z_1)$ to an irreducible component of $\mathrm{Chow}^L_{k,d}(Z_2)$. Also for any finitely many points $z_1, \ldots, z_a$ on Z_1, f induces

$$f_*\colon \mathrm{Chow}^{f^*L}_{k,d}(Z_1; z_1, \ldots, z_a) \to \mathrm{Chow}^L_{k,d}(Z_2; f(z_1), \ldots, f(z_a)),$$

which is bijective and sends an irreducible component to an irreducible component.

We will need results on effective bounds on the number of irreducible components of Chow varieties. First we quote the following basic result of Guerra.

PROPOSITION 1.1.2 [**7**, Lemma 2.1, Propositions 2.3 and 2.4]. *Let $Z \subset \mathbb{P}^N$ be a projective variety of degree δ, and let H be the hyperplane line bundle on $\mathbb{P}^N$. Then*

$$\sharp\,\mathrm{Chow}^H_{k,d}(Z) \leqslant \binom{(N+1)\max\{d,\delta\}}{N}^{(N+1)\left[d\binom{d+k-1}{k}+\binom{d+k-1}{k-1}\right]}. \tag{1.1.1}$$

We have the following easy consequence.

PROPOSITION 1.1.3. *Let Z be an irreducible projective variety of dimension r, and let L be a line bundle over Z whose sections separate any two points of Z. Let $\delta := L^r$ be the degree of Z with respect to L. Then*

$$\sharp\,\mathrm{Chow}^L_{k,d}(Z) \leqslant \binom{(2r+2)\max\{d,\delta\}}{2r+1}^{(2r+2)\left[d\binom{d+k-1}{k}+\binom{d+k-1}{k-1}\right]}. \tag{1.1.2}$$

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Proof. By Lemma 1.1.1, we may assume that L is very ample. By projection, we may assume that Z is a projective subvariety of $\mathbb{P}^{2r+1}$. Then Proposition 1.1.3 follows readily from Proposition 1.1.2 with $N = 2r + 1$. □

We will need the following simple lemma.

LEMMA 1.1.4. *Let $\pi : \hat{\mathbb{P}}^N \to \mathbb{P}^N$ be the blow-up of $\mathbb{P}^N$ at a point and F be the exceptional divisor. Then $2\pi^*L - F$ is very ample on $\hat{\mathbb{P}}^N$, where L denotes the hyperplane line bundle on $\mathbb{P}^N$.*

Proof. Let $x \in \mathbb{P}^N$ be the center of the blow-up. Then $\hat{\mathbb{P}}^N$ is the subvariety of $\mathbb{P}^N \times \mathbb{P}^{N-1}$ which is the closure of the graph of the projection of $\mathbb{P}^N$ to a hyperplane $\mathbb{P}^{N-1}$ with the vertex x. Let $p_1 : \mathbb{P}^N \times \mathbb{P}^{N-1} \to \mathbb{P}^N$ and $p_2 : \mathbb{P}^N \times \mathbb{P}^{N-1} \to \mathbb{P}^{N-1}$ denote the projection maps. Then $\pi = p_1\big|_{\hat{\mathbb{P}}^N}$, so that $F = p_1^{-1}(x) \cap \hat{\mathbb{P}}^N$. Moreover, $p_2\big|_F : F \to \mathbb{P}^{N-1}$ is a biholomorphism. Let H be the hyperplane line bundle on $\mathbb{P}^{N-1}$. It is easy to check that

$$(p_1^*L + p_2^*H)|_{\hat{\mathbb{P}}^N} = 2\pi^*L - F. \tag{1.1.3}$$

Since $p_1^*L + p_2^*H$ is very ample on $\mathbb{P}^N \times \mathbb{P}^{N-1}$, it follows that $2\pi^*L - F$ is also very ample on $\hat{\mathbb{P}}^N$. □

Next we have the following result.

PROPOSITION 1.1.5. *Let Z be an irreducible projective variety of dimension r, and let L be a line bundle over Z whose sections separate any two points of Z. Let $\delta := L^r$ be the degree of Z with respect to L. Fix finitely many points $z_1, \dots, z_a$ on Z. Then*

$$\sharp\,\mathrm{Chow}^L_{k,d}(Z; z_1, \dots, z_a) \leqslant \sum_{j=1}^{2^{ak}d} \binom{(2r+2)\max\{j, 2^{ar}\delta\}}{2r+1}^{(2r+2)\left[j\binom{j+k-1}{k}+\binom{j+k-1}{k-1}\right]}. \tag{1.1.4}$$

Proof. By Lemma 1.1.1, we may assume that $Z \subset \mathbb{P}^N$ for some $N > 0$ and L is the hyperplane line bundle of $\mathbb{P}^N$. Let $\pi : \hat{\mathbb{P}}_N \to \mathbb{P}^N$ be the blow-up of $\mathbb{P}^N$ at the points $z_1, \dots, z_a$, and let $F_i = \pi^{-1}(z_i)$ be the exceptional divisor over z_i, $1 \leqslant i \leqslant a$. Let Z' be the strict transform of Z. First we note that, by repeated application of Lemma 1.1.4, one easily checks that the line bundle

$$L' := 2^a\pi^*L - \sum_{i=1}^{a} 2^{i-1}F_i \tag{1.1.5}$$

is very ample on $\hat{\mathbb{P}}^N$ and thus very ample on Z'. Denote by $\mathrm{Chow}_k(Z'; F_1, \dots, F_a)$ the subvariety of $\mathrm{Chow}_k(Z')$ parametrizing irreducible and reduced subvarieties of dimension k in Z' which are not contained in any of the exceptional divisor F_i but have non-empty intersection with each F_i.

CLAIM. *Each irreducible component of $\mathrm{Chow}_k(Z'; F_1, \dots, F_a)$ is an irreducible component of $\mathrm{Chow}_k(Z')$.*

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Proof of Claim. Observe that for any member V of $\mathrm{Chow}_k(Z';F_1,\dots,F_a)$, one has

$$V\cdot(-F_i)^k<0\quad\text{for each } i=1,\dots,a, \tag{1.1.6}$$

because the restriction of each $\mathcal{O}(-F_i)$ to F_i itself is ample. It is also clear that (1.1.6) actually characterizes members of $\mathrm{Chow}_k(Z';F_1,\dots,F_a)$ among those of $\mathrm{Chow}_k(Z')$. This proves the claim. □

Next we observe that it follows readily from the definition that the morphism

$$\pi_*\colon \mathrm{Chow}_k(Z';F_1,\dots,F_a)\to\mathrm{Chow}_k(Z;z_1,\dots,z_a) \tag{1.1.7}$$

induced by π is bijective. For any $k,d>0$ and $V\in\pi_*^{-1}(\mathrm{Chow}^L_{k,d}(Z;z_1,\dots,z_a))$ corresponding to some $V'\in\mathrm{Chow}^L_{k,d}(Z;z_1,\dots,z_a)$ under the bijection in (1.1.7), one has

$$\begin{aligned}V\cdot(L')^k&=V\cdot\left(2^a\pi^*L-\sum_{i=1}^a 2^{i-1}F_i\right)^k\quad\text{(by (1.1.5))}\\&<2^{ak}V\cdot(\pi^*L)^k\quad\text{(by (1.1.6))}\\&=2^{ak}V'\cdot L^k\\&=2^{ak}d.\end{aligned} \tag{1.1.8}$$

Hence, one has

$$\pi_*^{-1}(\mathrm{Chow}^L_{k,d}(Z;z_1,\dots,z_a))\subset\bigcup_{j=1}^{2^{ak}d}\mathrm{Chow}^{L'}_{k,j}(Z';F_1,\dots,F_a). \tag{1.1.9}$$

Together with the Claim and the fact that π_* in (1.1.7) is a bijection, one has

$$\sharp\,\mathrm{Chow}^L_{k,d}(Z;z_1,\dots,z_a)\leqslant\sum_{j=1}^{2^{ak}d}\sharp\,\mathrm{Chow}^{L'}_{k,j}(Z'). \tag{1.1.10}$$

Note that by a calculation similar to (1.1.8), one sees that the degree of Z' with respect to L' is bounded above by $2^{ar}\delta$. Then by applying Proposition 1.1.3 to Z' and L', one obtains (1.1.4) readily from (1.1.10), and this finishes the proof of Proposition 1.1.5. □

2. *Effective separation of points*

2.1. We recall the following results from [**11**].

PROPOSITION 2.1.1 [**11**, Theorem 2.16]. *Let X be a normal proper variety which has invertible canonical bundle K_X. Let L be an ample line bundle over X. Then there exists an ideal sheaf $J\subset\mathcal{O}_X$ such that the support of $\mathcal{O}_X/J$ is exactly the set of points where X is not klt (Kawamata log terminal) and $H^j(X,\mathcal{O}_X(K_X+L)\otimes J)=0$ for all $j>0$.*

PROPOSITION 2.1.2 [**11**, Theorems 5.5 and 5.9]. *Let Z be a smooth projective variety of dimension n and L be a nef (numerically effective) and big line bundle over Z. Let $x_1\neq x_2\in Z$ be two points such that any irreducible subvariety*

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containing x_1 or x_2 has positive degree with respect to L. Then the sections of $K_Z + \binom{n+2}{2}L$ separate x_1 and x_2.

When M is a compact complex hyperbolic manifold, K_M is ample. Thus by Proposition 2.1.2, we get the following result.

PROPOSITION 2.1.3. *Let M be an n-dimensional compact complex hyperbolic manifold. Then the sections of $(1+\binom{n+2}{2})K_M$ separate any two points of M, defining an injection of M into a projective space.*

2.2. Next we consider the case when M is a smooth n-dimensional complex hyperbolic manifold of finite volume. Since such an M is a complete Kähler manifold of bounded negative sectional curvature of finite volume, it follows from a result of Siu and Yau [**17**] that M can be compactified into a compact normal complex space M^* such that the boundary $M^* \setminus M$ consists of finitely many points (cusps), say, $\{P_1, \ldots, P_a\}$. We will simply call M^* the Siu–Yau compactification of M. It is also well known that K_M extends to an ample line bundle K_{M^*} over the normal variety M^* (cf. for example the description from (2.2.1) below and [**21**]). We remark that in the special case when M is defined arithmetically, then M^* coincides with the Baily–Borel compactification of M (see [**3**]).

It is also known (cf. for example [**12**, pp. 175–176], [**19**, pp. 61–62]) that such an M also admits a smooth toroidal compactification $\bar{M}$ together with a morphism $\phi : \bar{M} \to M^*$ such that:

(i) ϕ is biholomorphic on M; and

(ii) each $\phi^{-1}(P_i)$ is an $(n-1)$-dimensional abelian variety E_i with ample conormal bundle in $\bar{M}$ which is blown down to P_i by ϕ.

Moreover, each E_i admits an open neighborhood in $\bar{M}$ which can be identified with an open neighborhood of the zero section of the negative line bundle $[E_i]\big|_{E_i}$. Let $E = \bigcup_{1 \leqslant i \leqslant a} E_i$. Then the above description easily implies that the line bundle $K_{\bar{M}} + E$ is trivial near each E_i, and thus it descends to a line bundle K_{M^*} over M^* extending K_M. In particular, one has

$$\phi^* K_{M^*} = K_{\bar{M}} + E. \tag{2.2.1}$$

We remark that when M is defined arithmetically, the toroidal compactification was constructed in [**2**], and (2.2.1) was proved in [**13**, Proposition 3.4].

PROPOSITION 2.2.1. *Let M^* be the Siu–Yau compactification of an n-dimensional complex hyperbolic manifold M of finite volume. Then the sections of $(1 + \binom{n+2}{2})K_{M^*}$ separate any two points of M^*, defining an injection of M^* into a projective space.*

Proof. Let $\bar{M}$ be the smooth toroidal compactification of M with the morphism $\phi : \bar{M} \to M^*$ and exceptional divisor $E = \bigcup_{1 \leqslant i \leqslant a} E_i$ with each E_i corresponding to a cusp P_i of M^*, $i = 1, \ldots, a$, as described above. By [**11**, 3.8] and (2.2.1), one knows that M^* is not klt at each P_i. Thus by Proposition 2.1.1, we get an ideal $J \subset \mathcal{O}_{M^*}$ such that the support of $\mathcal{O}_{M^*}/J$ is $\{P_1, \ldots, P_a\}$ and $H^1(M^*, \mathcal{O}(mK_{M^*} \otimes J)) = 0$ for any integer $m \geqslant 2$. From the short exact sequence

$$0 \longrightarrow \mathcal{O}(mK_{M^*}) \otimes J \longrightarrow \mathcal{O}(mK_{M^*}) \longrightarrow \mathcal{O}(mK_{M^*}) \otimes \mathcal{O}_{M^*}/J \longrightarrow 0, \tag{2.2.2}$$

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we get the surjectivity of

$$H^0(M^*, \mathcal{O}(mK_{M^*})) \longrightarrow \sum_{i=1}^{a} H^0(P_i, \mathcal{O}(mK_{M^*} \otimes \mathcal{O}_{M^*}/J)). \tag{2.2.3}$$

Thus the sections of mK_{M^*} separate any two cusps if $m \geqslant 2$. From the ampleness of K_{M^*} and (2.2.1), one easily sees that the line bundle $K_{\bar{M}} + E$ is a nef and big line bundle on $\bar{M}$ and has positive intersection number with any irreducible variety not contained in E. Thus by Proposition 2.1.2, the sections of

$$K_{\bar{M}} + \binom{n+2}{2}\phi^* K_{M^*} = \left(1 + \binom{n+2}{2}\right)\phi^* K_{M^*} - E \tag{2.2.4}$$

separate any two points on $M \subset \bar{M}$. By the natural inclusion

$$H^0\left(\bar{M}, \left(1 + \binom{n+2}{2}\right)\phi^* K_{M^*} - E\right) \subset H^0\left(M^*, \left(1 + \binom{n+2}{2}\right)K_{M^*}\right), \tag{2.2.5}$$

we see that the sections of $(1 + \binom{n+2}{2})K_{M^*}$ which vanish at cusps separate any two points on the open subset M. Since they vanish at cusps, they also separate a cusp point from a point of M. This finishes the proof of Proposition 2.2.1. □

We will need the following proposition.

PROPOSITION 2.2.2. *Let M be an n-dimensional complex hyperbolic manifold of finite volume with $n \geqslant 2$. Then through a general point of M, there exists a smooth irreducible projective curve of genus bounded above by*

$$1 + \frac{1}{2}\left(1 + (n-1)\left(1 + \binom{n+2}{2}\right)\right)\left(1 + \binom{n+2}{2}\right)^{n-1} K^n_{M^*}. \tag{2.2.6}$$

Proof. Let C be the irreducible projective curve obtained by the intersection of $(n-1)$ general hyperplanes with M under the injection of Proposition 2.2.1. Since $M^* \setminus M$ is of codimension at least 2 in M^*, it follows from Bertini's theorem that such a general C is smooth and disjoint from $M^* \setminus M$. By the adjunction formula, the canonical line bundle of C is

$$\left(1 + (n-1)\left(1 + \binom{n+2}{2}\right)\right)K_{M*}\Big|_C.$$

In particular, it is a line bundle of degree

$$\left(1 + (n-1)\left(1 + \binom{n+2}{2}\right)\right)\left(1 + \binom{n+2}{2}\right)^{n-1} K^n_{M^*}$$

on C, and thus the genus of C is bounded above by the number given in the proposition. □

3. *Effective bound on the holomorphic maps: the compact case*

3.1. As a motivation and prelude to the proof of Theorem 1, we present in § 3 the proof of an analogous result in the compact case (cf. Theorem 3.1.2 below). First we have the following proposition.

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PROPOSITION 3.1.1. *Let X (respectively Y) be an irreducible projective manifold of dimension k (respectively n) and let H_X (respectively H_Y) be an ample line bundle on X (respectively Y) whose sections separate any two points of X (respectively Y). Then for a fixed integer $B > 0$, the number of components of the space of holomorphic maps $f : X \to Y$ satisfying $(H_X + f^*H_Y)^k \leqslant B$ is bounded above by*

$$\sum_{d=1}^{B} \left(\begin{matrix} (2k+2n+2)\max\{d, \binom{k+n}{k} H_X^k H_Y^n\} \\ 2k+2n+1 \end{matrix} \right)^{(2k+2n+2)\left[d\binom{d+k-1}{k}+\binom{d+k-1}{k-1}\right]}. \tag{3.1.1}$$

Proof. For a fixed positive integer $d \leqslant B$, let $\mathcal{H}_d$ be the space of holomorphic maps $f : X \to Y$ satisfying $(H_X + f^*H_Y)^k = d$, and let $\sharp\,\mathcal{H}_d$ denote the number of irreducible components of $\mathcal{H}_d$. Consider the line bundle $L := p_1^*H_X + p_2^*H_Y$ on $Z := X \times Y$, where p_1 and p_2 denotes the projection maps from Z onto X and Y respectively. Then the sections of L separate any two points of Z. For any $f \in \mathcal{H}_d$, let $G_f \subset Z$ be its graph. Then G_f is an irreducible and reduced subvariety in Z of dimension k; moreover, from the biholomorphism from X to G_f given by $x \to (x, f(x))$, one easily sees that the degree of G_f with respect to L is given by

$$G_f \cdot L^k = (H_X + f^*H_Y)^k = d. \tag{3.1.2}$$

Thus the map $f \to G_f$ identifies $\mathcal{H}_d$ as a subset of $\mathrm{Chow}_{k,d}^L(Z)$. Let $\mathcal{C}$ be a component of $\mathrm{Chow}_{k,d}^L(Z)$ which contains G_f for some $f \in \mathcal{H}_d$. Then a general element of $\mathcal{C}$ is biholomorphic to X by p_1 because one of its element G_f has that property. Thus a general element of $\mathcal{C}$ defines the graph of a holomorphic map in $\mathcal{H}_d$. It follows readily that

$$\sharp\,\mathcal{H}_d \leqslant \sharp\,\mathrm{Chow}_{k,d}^L(Z). \tag{3.1.3}$$

Note that the degree of Z with respect to L is

$$(p_1^*H_X + p_2^*H_Y)^{k+n} = \binom{k+n}{k} H_X^k H_Y^n. \tag{3.1.4}$$

Then by summing both sides of (3.1.3) from $d = 1$ to $d = B$ and using Proposition 1.1.3 (with $r = k + n$ and $\delta = \binom{k+n}{k} H_X^k H_Y^n$), one obtains Proposition 3.1.1 immediately. □

The main result in § 3 is the following.

THEOREM 3.1.2. *Let M be an n-dimensional compact complex hyperbolic manifold, and let X be a smooth compact Riemann surface of genus $g \geqslant 2$. Then the number of non-constant holomorphic maps from X into M is bounded above by*

$$\sum_{d=1}^{B} \left(\begin{matrix} (2n+4)\max\{d, C\} \\ 2n+3 \end{matrix} \right)^{(2n+4)(d^2+1)}, \tag{3.1.5}$$

where

$$\begin{aligned} B &= (2g+1) + \frac{(g-1)(n^2+3n+4)(n+1)}{2}, \\ C &= \frac{(2g+1)(n+1)(n^2+3n+4)^n}{2^n} K_M^n. \end{aligned} \tag{3.1.6}$$

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Proof. By Proposition 2.1.3, the sections of the line bundle

$$H_M := \left(1 + \binom{n+2}{2}\right) K_M \tag{3.1.7}$$

separate any two points on M. Fix any line bundle L over X of degree 3. Then it is well known that the line bundle

$$H_X := K_X + L \tag{3.1.8}$$

is very ample (cf. for example [GH, p. 215]). Note that $\deg H_X = 2g + 1$. Let h_M and h_X denote the complex hyperbolic metrics of constant holomorphic sectional curvature -1 on M and X respectively. Under such normalization, one easily checks that

$$c_1(K_M, h_M) = \frac{n+1}{2\pi}\omega_M \quad \text{and} \quad c_1(K_X, h_X) = \frac{\omega_X}{\pi} \tag{3.1.9}$$

where ω_M and ω_X denote the Kähler forms of h_M and h_X respectively. Without loss of generality, we may assume that the space of non-constant holomorphic maps from X into M is non-empty; and it is well known that this implies readily that $g \geqslant 2$. Then by the Schwarz lemma of [**4**] and [**16**], we have

$$f^*\omega_M \leqslant \omega_X \tag{3.1.10}$$

for any holomorphic map $f\colon X \to M$. Upon integrating over X, we have

$$\begin{aligned} \deg f^*K_M &= \int_X f^* c_1(K_M, h_M) = \frac{n+1}{2\pi}\int_X f^*\omega_M \\ &\leqslant \frac{n+1}{2\pi}\int_X \omega_X = \frac{n+1}{2}\deg K_X = (n+1)(g-1). \end{aligned} \tag{3.1.11}$$

Then one easily checks that

$$\deg(H_X + f^*H_M) \leqslant B, \tag{3.1.12}$$

where B is as in (3.1.6). Thus, all the holomorphic maps f from X into M satisfy $\deg(H_X + f^*H_M) \leqslant B$. One also easily checks that the constant C in (3.1.6) satisfies

$$C = \binom{n+1}{1} \cdot \deg H_X \cdot H_M^n. \tag{3.1.13}$$

Together with Proposition 3.1.1 (with $k = 1$ and H_X, H_M, B as above) and Sunada's result [**18**], which implies that the space of a non-constant holomorphic map from X into M is discrete, one obtains Theorem 3.1.2 immediately. □

4. *Maps into complex hyperbolic space forms of finite volume*

4.1. We will need the following generalization of Proposition 3.1.1.

PROPOSITION 4.1.1. *Let X^* (respectively Y^*) be an irreducible projective variety of dimension k (respectively n) with marked points $x_1, \ldots, x_\ell$ (respectively $y_1, \ldots, y_q$). Let $X = X^* \setminus \{x_1, \ldots, x_\ell\}$, and let $Y = Y^* \setminus \{y_1, \ldots, y_q\}$. We assume that the following property holds:*

(∗) *any proper holomorphic map from X to Y extends to a holomorphic map from X^* to Y^*.*

Let H_{X^*} (respectively H_{Y^*}) be an ample line bundle on X^* (respectively Y^*) whose sections separate any two points of X^* (respectively Y^*). Fix an integer $B > 0$.

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Then the number of irreducible components of the space of proper holomorphic maps $f : X \to Y$ whose extension $\hat{f} : X^* \to Y^*$ (guaranteed by $(*)$) satisfies $(H_{X^*} + \hat{f}^* H_{Y^*})^k \leqslant B$ is bounded above by

$$q^\ell \sum_{d=1}^{B} \sum_{j=1}^{2^{\ell k} d} \left(\begin{matrix} (2k+2n+2) \max\{j, 2^{\ell(k+n)} \binom{k+n}{k} H_{X^*}^k H_{Y^*}^n\} \\ 2k+2n+1 \end{matrix} \right)^{(2k+2n+2)\left[j\binom{j+k-1}{k} + \binom{j+k-1}{k-1}\right]}. \tag{4.1.1}$$

Proof. First we fix a choice of ℓ points $y'_1, \ldots, y'_\ell$, which are not necessarily distinct, from $\{y_1, \ldots, y_q\}$. For a fixed integer $1 \leqslant d \leqslant B$, let $\mathcal{H}_d$ be the space of proper holomorphic maps $f : X \to Y$ whose extension $\hat{f} : X^* \to Y^*$ satisfies $\hat{f}(x_i) = y'_i$ for $1 \leqslant i \leqslant \ell$ and $(H_{X^*} + \hat{f}^* H_{Y^*})^k = d$. For $f \in \mathcal{H}_d$, let $G_{\hat{f}} \subset X^* \times Y^*$ be the graph of its extension $\hat{f}$. Then $G_{\hat{f}}$ is an irreducible and reduced subvariety in $Z := X^* \times Y^*$ of dimension k, and it contains the points $z_i := (x_i, y'_i)$ for each $1 \leqslant i \leqslant \ell$; moreover, as in (3.1.2), $G_{\hat{f}}$ is of degree d with respect to the line bundle $L := p_1^* H_{X^*} + p_2^* H_{Y^*}$ on Z. Thus the map $f \to G_{\hat{f}}$ identifies $\mathcal{H}_d$ as a subset of $\mathrm{Chow}^L_{k,d}(Z; z_1, \ldots, z_\ell)$. The argument leading to (3.1.3) also applies here, so that we get

$$\sharp\, \mathcal{H}_d \leqslant \sharp\, \mathrm{Chow}^L_{k,d}(Z; z_1, \ldots, z_\ell). \tag{4.1.2}$$

Note that as in (3.1.4), the degree of Z with respect to L is $\binom{k+n}{k} H_{X^*}^k H_{Y^*}^n$. By Proposition 1.1.5, $\sharp\, \mathrm{Chow}^L_{k,d}(Z; z_1, \ldots, z_\ell)$ is bounded above by

$$\sum_{j=1}^{2^{\ell k} d} \left(\begin{matrix} (2(k+n)+2) \max\{j, 2^{\ell(k+n)} \binom{k+n}{k} H_{X^*}^k H_{Y^*}^n\} \\ 2k+2n+1 \end{matrix} \right)^{(2k+2n+2)\left[j\binom{j+k-1}{k} + \binom{j+k-1}{k-1}\right]}. \tag{4.1.3}$$

By summing both sides of (4.1.2) and (4.1.3) from $d = 1$ to $d = B$ and noting that there are q^ℓ possible ways to choose $y'_1, \ldots, y'_\ell$ from $y_1, \ldots, y_q$, one obtains Proposition 4.1.1 immediately. □

4.2. For fixed integers $g, \ell \geqslant 0$, we let X be an ℓ-punctured Riemann surface of genus g, so that we may write $X = \bar{X} \setminus \{x_1, \ldots, x_\ell\}$, where $\bar{X}$ is a smooth compact Riemann surface of genus g. Also we let M be an n-dimensional complex hyperbolic space form of finite volume. Let M^* and $\bar{M}$ be the Siu–Yau compactification and the smooth toroidal compactification of M respectively, as described in (2.2). Recall also that we may write $M^* = M \cup \{y_1, \ldots, y_q\}$ and $\bar{M} = M \bigcup_{1 \leqslant j \leqslant q} E_j$ where each E_j is an $(n-1)$-dimensional abelian variety corresponding to the cusp y_j, and there exists a blowing-down holomorphic map $\phi : \bar{M} \to M^*$ extending the identity map on M such that $\phi(E_j) = \{y_j\}$ for each j. Write $E = \bigcup_{1 \leqslant j \leqslant q} E_j$. Recall also from (2.2.1) that the canonical line bundle K_M extends to a line bundle K_{M^*} on M^* such that

$$\phi^* K_{M^*} = K_{\bar{M}} + E. \tag{4.2.1}$$

PROPOSITION 4.2.1. *Let X and M be as above, and let $f : X \to M$ be a proper holomorphic map.*

(i) *Then f extends to a holomorphic map $\hat{f}$ from $\bar{X}$ to M^* sending each x_i to some y_j.*

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(ii) *Also f extends to a holomorphic map $\bar{f}$ from $\bar{X}$ to $\bar{M}$ such that $\phi \circ \bar{f} = \hat{f}$ on $\bar{X}$.*

(iii) *Let h_M be the Hermitian metric on K_M induced by the complex hyperbolic metric on M. Then f^*h_M is a good Hermitian metric, in the sense of Mumford* [**13**], *on f^*K_M with respect to the line bundle extension $\bar{f}^*(K_{\bar{M}} + E)$ over $\bar{X}$.*

Proof. Part (i) follows from a result in [**9**]. See for example [**19**, Proposition 1.4] for a proof of part (ii). Next we describe the proof of part (iii), which is proved in a stronger form in [**19**, Proposition 2.1] for the special (but higher-dimensional) case when f is an immersion. Take a point $x_i \in \bar{X} \setminus X$, and let $y := \bar{f}(x_i) \in E_j$. From the construction of the toroidal compactification (cf. for example [**19**, p. 66]), one can find an open coordinate neighborhood V_y of y in $\bar{M}$ such that $V_y = \Delta^n = \{(w_1, \ldots, w_n) \in \mathbb{C}^n \,\big|\, |w_i| < 1, i = 1, \ldots, n\}$, $V_y \cap E_j = \{(w_1, \ldots, w_{n-1}, w_n) \in \Delta^n \mid w_n = 0\}$, the point y corresponds to the origin in Δ^n, and $w_1, \ldots, w_{n-1}$ give holomorphic flat coordinates on the complex torus E_j. Take an open coordinate neighborhood $V_{x_i} = \Delta = \{z \in \mathbb{C} : |z| < 1\}$ of x_i in $\bar{X}$ with x_i corresponding to the origin. Shrinking V_{x_i} if necessary, we may assume that $\bar{f}(V_{x_i}) \subset\subset V_y$. Also, the hyperbolic metric h_M on M has its Kähler form given by

$$\omega_M = -\sqrt{-1}\,\partial\bar{\partial} \log\left(-c \log |w_n|^2 - \sum_{k=1}^{n-1} |w_k|^2\right) \tag{4.2.2}$$

on V_y for some constant $c > 0$ (cf. for example [**19**, p. 66, equation (1)] and [**5**, § 4.1.2]), from which one can easily check that the volume form of M is given by

$$\frac{1}{n!} \cdot \frac{c^2(\sqrt{-1})^n\, dw_1 \wedge d\bar{w}_1 \wedge \ldots \wedge dw_n \wedge d\bar{w}_n}{|w_n|^2(-c \log |w_n|^2 - \sum_{k=1}^{n-1} |w_k|^2)^{n+1}} \tag{4.2.3}$$

on V_y. Note that a local holomorphic basis of $K_{\bar{M}} + E$ is given by

$$e := \frac{dw_1 \wedge \ldots \wedge dw_n}{w_n} \tag{4.2.4}$$

on V_y, and one sees from (4.2.3) that its norm with respect to the Hermitian metric induced by h_M is given by

$$\|e|_{M \cap V_y}\|^2_{h_M} = C\left(-c \log |w_n|^2 - \sum_{k=1}^{n-1} |w_k|^2\right)^{n+1} \tag{4.2.5}$$

for some constant $C > 0$. As in [To, p. 67], one easily sees that $\bar{f}$ takes the form

$$w_i \circ \bar{f}(z) = z^{\alpha_i}\mu_i(z), \quad i = 1, \ldots, n, \tag{4.2.6}$$

where $\alpha_1, \ldots, \alpha_n$ are positive integers, and $\mu_1, \ldots, \mu_n$ are bounded holomorphic functions in z; moreover, μ_n can be taken to be nowhere vanishing on V_{x_i} since $f(X) \subset M$ (while $\mu_1, \ldots, \mu_{n-1}$ may possibly be identically zero). Now, $\bar{f}^*e$ is a local holomorphic basis of $\bar{f}^*(K_{\bar{M}} + E)$ on V_x, and one sees from (4.2.5) and (4.2.6) that

$$\|\bar{f}^*e|_{X \cap V_x}\|_{f^*h_M} = C\left(-c \log |z^{\alpha_n}\mu_n|^2 - \sum_{k=1}^{n-1} |z^{\alpha_k}\mu_k|^2\right)^{n+1}. \tag{4.2.7}$$

Then by considering such an open neighborhood V_{x_i} for each x_i, one can check as in [To, proof of Proposition 2.1] using (4.2.7) to see that f^*h_M is a good Hermitian metric on f^*K_M with respect to the line bundle extension $\bar{f}^*(K_{\bar{M}} + E)$ over $\bar{X}$. □

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4.3. Now we are ready to finish the proof of Theorem 1.

Proof of Theorem 1. Let X and M be as in Theorem 1. Write $\bar{X} = X \cup \{x_1, \ldots, x_m\}$, where $\bar{X}$ is a smooth compact Riemann surface of genus g such that $2g - 2 + m > 0$. Let $M^* = M \cup \{y_1, \ldots, y_q\}$ and $\bar{M} = M \cup E$ be the Siu–Yau compactification and the smooth toroidal compactification of M respectively, and let $\phi : \bar{M} \to M^*$ be the blowing-up map as before. Consider the index set $\mathcal{I} := \{I \mid I \subset \{1, 2, \ldots, m\}\}$. For each $I \in \mathcal{I}$ with cardinality $|I| = \ell$, we define the ℓ-punctured Riemann surface given by

$$X_I := \bar{X} \setminus \{x_i \mid i \in I\} = X \cup \{x_i \mid i \notin I\}. \tag{4.3.1}$$

Let $\mathcal{H}(X, M)$ denote the space of non-costant holomorphic maps from X into M, and for each $I \in \mathcal{I}$, let $\mathcal{H}_{\text{proper}}(X_I, M)$ denote the space of proper holomorphic maps from X_I into M. By Proposition 4.2.1(i), one knows that each $f \in \mathcal{H}(X, M)$ extends to a holomorphic map from $\bar{X}$ into M^* sending each x_i to M or $M^* \setminus M$, and thus we may regard f as a map in $\mathcal{H}_{\text{proper}}(X_I, M)$ for a unique $I \in \mathcal{I}$ (such that $\{x_i\}_{i \in I}$ are exactly the points mapped to $M^* \setminus M$). Conversely each $f \in \mathcal{H}_{\text{proper}}(X_I, M)$ restricts to a unique map in $\mathcal{H}(X, M)$. Thus $\mathcal{H}(X, M)$ decomposes as a disjoint union given by

$$\mathcal{H}(X, M) \simeq \bigcup_{I \in \mathcal{I}} \mathcal{H}_{\text{proper}}(X_I, M). \tag{4.3.2}$$

Note that Proposition 4.2.1(i) also implies that Property $(*)$ in Proposition 4.1.1 is satisfied for each map in $\mathcal{H}_{\text{proper}}(X_I, M)$. By a result of Noguchi [**14**, Theorem 3.3], the space $\mathcal{H}(X, M)$ is discrete and finite, that is, $\sharp\, \mathcal{H}(X, M) = \text{Card}\,(\mathcal{H}(X, M))$, and the same result also applies to each $\mathcal{H}_{\text{proper}}(X_I, M)$. Note that $\mathcal{H}_{\text{proper}}(X_I, M) = \emptyset$ if $2g - 2 + |I| \leqslant 0$. Thus, we only need to consider the case when X_I is a hyperbolic Riemann surface (or equivalently $\max\{0, 3 - 2g\} \leqslant |I| \leqslant m$). Fix any line bundle L over $\bar{X}$ of degree 3. Then as recalled in Theorem 3.1.2, the line bundle

$$H_{\bar{X}} := K_{\bar{X}} + L \tag{4.3.3}$$

is very ample over $\bar{X}$. Recall also from Proposition 2.2.1 that the sections of the ample line bundle

$$H_{M^*} := \left(1 + \binom{n+2}{2}\right) K_{M^*} \tag{4.3.4}$$

separate any two points of M^*. Now fix an integer ℓ such that $\max\{0, 3 - 2g\} \leqslant \ell \leqslant m$, and fix an X_I with $|I| = \ell$. Let $f \in \mathcal{H}_{\text{proper}}(X_I, M)$. Then as in Theorem 3.1.2, it follows from the Schwarz lemma of [**4**] and [**16**] that

$$f^* \omega_M \leqslant \omega_{X_I}, \tag{4.3.5}$$

where ω_{X_I} and ω_M denote the Kähler forms of the hyperbolic metrics h_{X_I} and h_M on X_I and M (normalized as in (3.1.9)) respectively. Let $\hat{f} : \bar{X} \to M^*$ and $\bar{f} : \bar{X} \to \bar{M}$ be the holomorphic extensions of f satisfying $\phi \circ \bar{f} = \hat{f}$ as guaranteed by Proposition 4.2.1. By Proposition 4.2.1(iii) and [**13**, Theorem 1.4], one has

$$\deg(\bar{f}^*(K_{\bar{M}} + E)) = \int_{X_I} f^* c_1(K_M, h_M). \tag{4.3.6}$$

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Thus, one has

$$\begin{aligned}
&\deg \hat{f}^* H_{M^*} \\
&\quad = \deg \bar{f}^* \phi^* H_{M^*} \quad (\text{since } \hat{f} = \phi \circ \bar{f}) \\
&\quad = \left(1 + \binom{n+2}{2}\right) \deg(\bar{f}^*(K_{\bar{M}} + E)) \quad (\text{by (4.2.1) and (4.3.4)}) \\
&\quad \leqslant \left(1 + \binom{n+2}{2}\right) \frac{n+1}{2\pi} \int_{X_I} \omega_{X_I} \quad (\text{by (4.3.5), (4.3.6) and as in (3.1.11)}) \\
&\quad = \left(1 + \binom{n+2}{2}\right) \frac{n+1}{2} (2g - 2 + \ell). \qquad (4.3.7)
\end{aligned}$$

Together with the fact that $\deg H_{\bar{X}} = 2g + 1$, one easily sees that

$$\deg(H_{\bar{X}} + \hat{f}^* H_{M^*}) \leqslant B, \qquad (4.3.8)$$

where $B = B(g, n, \ell)$ is as in (0.1.2). A consideration similar to (4.3.7) also gives

$$K_{M^*}^n = \frac{n!(n+1)^n}{(4\pi)^n} \operatorname{Vol}(M) \qquad (4.3.9)$$

(which also follows from [**13**, Proportionality Theorem 3.2] in the case when M is defined arithmetically). Then one easily sees that

$$2^{\ell(n+1)}(n+1) \cdot \deg H_{\bar{X}} \cdot H_{M^*}^n = C, \qquad (4.3.10)$$

where $C = C(g, n, \ell, \operatorname{Vol}(M))$ is as in (0.1.2). Then by applying Proposition 4.1.1 to X_I (with $k = 1$, $X^* = \bar{X}$, $Y^* = M^*$ and H_{X^*}, H_{Y^*} as in (4.3.3), (4.3.4) respectively), one has

$$\operatorname{Card}\left(\mathcal{H}_{\text{proper}}(X_I, M)\right) \leqslant q^\ell \sum_{d=1}^{B} \sum_{j=1}^{2^{\ell d}} \binom{(2n+4)\max(j, C)}{2n+3}^{(2n+4)(j^2+1)}. \qquad (4.3.11)$$

By a recent result of [**8**], which is an improvement of an earlier result of Parker [**15**], one has

$$q \leqslant \frac{(5n+4)^n}{2^{2n-1}\pi^n} \operatorname{Vol}(M). \qquad (4.3.12)$$

By summing (4.3.11) over all the hyperbolic X_I (noting that for each $\max\{0, 3 - 2g\} \leqslant \ell \leqslant m$, there are $\binom{m}{\ell}$ distinct X_I with $|I| = \ell$) and using the bound for q given in (4.3.12), one obtains (0.1.1) immediately. This finishes the proof of Theorem 1. □

4.4. Finally we give the proofs of Corollary 3 and Theorem 4.

Proof of Corollary 3. Let X and M be as in Corollary 3. Let ℓ be the cardinality of $\mathcal{H}(X, M)$, and let $X_1, \ldots, X_\ell \subset X \times M$ be the graphs of the maps in $\mathcal{H}(X, M)$. Let $p_X : X \times M \to X$ denote the projection map. Define

$$Z := \bigcup_{1 \leqslant i < j \leqslant \ell} p_X(X_i \cap X_j). \qquad (4.4.1)$$

Then Z is a proper subvariety in X with finitely many components. By the assumption, there exists a non-constant holomorphic map f from an m-punctured Riemann surface X' of genus g into X such that $f(X') \not\subset Z$. Then

$$\{p_X^{-1}(f(X')) \cap X_i\}_{1 \leqslant i \leqslant \ell}$$

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define ℓ distinct elements of $\mathcal{H}(X', M)$. Thus Corollary 3 follows readily from Theorem 1. □

Proof of Theorem 4. Let X, M, k and n be as in Theorem 4. It follows from Proposition 2.2.2 and (4.3.9) that through a general point of X, there exists a smooth irreducible projective curve of genus bounded above by

$$1 + \frac{1}{2} \cdot \frac{[2 + (k-1)(k^2+3k+4)](k^2+3k+4)^{k-1}(k+1)^k k!}{8^k \pi^k} \operatorname{Vol}(X). \qquad (4.4.2)$$

Then Theorem 4 follows readily from Corollary 3 and Theorem 1 (with $m = 0$ and g replaced by the upper bound in (4.4.2)). □

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