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A TIME-DEPENDENT PÓLYA URN WITH MULTIPLE DRAWINGS

MAY-RU CHEN
Department of Applied Mathematics, National Sun Yat-sen University, 70 Lien-hai Rd., Kaohsiung 804, Taiwan, ROC
E-mail: mayru@faculty.nsysu.edu.tw

In this paper, we consider a generalized Pólya urn model with multiple drawings and time-dependent reinforcements. Suppose an urn initially contains w white and r red balls. At the nth action, m balls are drawn at random from the urn, say k white and $m-k$ red balls, and then replaced in the urn along with $c_n k$ white and $c_n(m-k)$ red balls, where $\{c_n\}$ is a given sequence of positive integers. Repeat the above procedure *ad infinitum*. Let X_n be the proportion of the white balls in the urn after the nth action. We first show that X_n converges almost surely to a random variable X. Next, we give a necessary and sufficient condition for X to have a Bernoulli distribution with parameter $w/(w+r)$. Finally, we prove that X is absolutely continuous if $\{c_n\}$ is bounded.

Keywords: absolutely continuous, atoms, martingale, urn model

1. INTRODUCTION

In 1923, Eggenberger and Pólya [4] proposed an urn model, which is well-known as the Pólya urn. The Pólya urn is an urn initially containing w white and r red balls. At each stage, one ball is drawn at random from the urn and then replaced in the urn along with c balls of the same color, where c is a fixed positive integer. Repeat the above procedure *ad infinitum*. It is known that the sequence of the proportions of white balls is a bounded martingale and converges almost surely to a beta-distributed random variable with parameters w/c and r/c. Since then, many generalizations of the Pólya urn have been studied in literature, see [7,9,12,15] for a recent survey.

Recently, some urn models with multiple-drawings are considered, see [2,3,8,10,11,13, 16,17], etc. Here we are interested in a time-dependent version proposed by Pemantle [14] and a multiple-drawings version proposed by Chen and Wei [3]. In 1990, Pemantle [14] extended the Pólya urn with one change that the number of extra balls added in the urn at time n is c_n, where $\{c_n\}$ is a deterministic sequence of nonnegative real numbers. After the nth action, let X_n be the proportion of white balls in the urn. Pemantle showed that $\{X_n\}$ is a bounded martingale and converges almost surely to a random variable, say X, without atoms except possibly at 0 or 1. He also showed that X has a Bernoulli distribution with parameter $w/(w+r)$ if and only if $\sum_{n=1}^{\infty} \rho_n^2 = \infty$, where $\rho_n = c_n/(w+r+\sum_{i=1}^{n-1} c_i)$ is the ratio of the nth reinforcement to the total number of balls in the urn at time $n-1$.

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He furthermore showed that if $\{c_n\}$ is bounded, then the distribution of X has no atoms at all, that is, $\Pr(X = x) = 0$ for all $x \in [0, 1]$. At the end of [14], Pemantle asked when the distribution of X is absolutely continuous. We will give a partial answer to this problem (See Theorem 1.4). On the other hand, Chen and Wei [3] considered a multiple-drawings version of Pólya urn, in which, at the nth step, m ($m > 1$) balls are selected at random, say k white and $m - k$ red balls, and then returned along with ck white and $c(m - k)$ red balls. Let X_n be the proportion of white balls in the urn after the nth adding balls. They showed that $\{X_n\}$ is a bounded martingale and converges almost surely to an absolutely continuous random variable.

In this paper, we consider the following combinatorial model based on Chen and Wei [3] and Pemantle [14]:

Let $m \geq 1$ be a positive integer and $\{c_n\}$ a sequence of positive integers. Assume an urn initially contains w white and r red balls. At the nth action, we draw m balls at random and note their colors, say k white and $m - k$ red balls. Next, return these drawn balls in the urn along with $c_n k$ white and $c_n(m - k)$ red balls. Repeat the above procedure *ad infinitum*.

For our model, after the nth action, let W_n be the number of white balls, T_n be the total number of balls, and $\mathcal{F}_n$ be the $\sigma-$field generated by $W_0, W_1, \ldots W_n$. Then $T_0 = w + r$, $T_{n+1} = T_n + c_{n+1}m$, $W_0 = w$ and $W_{n+1} = W_n + c_{n+1}\xi_{n+1}$, where

$$\Pr(\xi_{n+1} = k \mid \mathcal{F}_n) = \frac{\binom{W_n}{k}\binom{T_n - W_n}{m-k}}{\binom{T_n}{m}}, \quad 0 \leq k \leq m.$$

Also let $X_n = W_n/T_n$ be the proportion of white balls after the nth action. Our main results are the following three theorems.

THEOREM 1.1: *$\{X_n, \mathcal{F}_n, n \geq 1\}$ is a martingale, that is, $E[X_{n+1} \mid \mathcal{F}_n] = X_n$ for all $n \geq 1$. Furthermore, X_n converges almost surely (a.s.) and in L^1 to a random variable X as $n \to \infty$. That is, $\lim_{n\to\infty} X_n = X$ a.s. and in L^1.*

THEOREM 1.2: *The limit X follows a Bernoulli distribution with parameter $w/(w + r)$ if and only if $\sum_{n=1}^{\infty} \rho_n^2 = \infty$, where $\rho_n = c_n m/T_{n-1}$. That is,*

$$\Pr(X = 1) = \frac{w}{w + r} = 1 - \Pr(X = 0) \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} \rho_n^2 = \infty.$$

REMARK 1.3: A possible application of Theorem 1.2 is to consider a contagion process, or epidemics, on a network (see [6]), where an epidemic can be a disease or a computer virus, etc. If the spread of the epidemics satisfies the conditions in Theorem 1.2, then the proportion of the inflected individuals in the network system will converge either to 1 or to 0 with its corresponding probabilities.

THEOREM 1.4: *If $\{c_n\}$ is bounded, then X is absolutely continuous on $[0, 1]$.*

Theorem 1.1 is a natural analog of results of Pemantle [14] and Chen and Wei [3], so we omit its proof. In view of Theorem 2 in [14], Theorem 1.2 seems to hold certainly. In fact, the argument we use to prove Theorem 1.2 is similar to the proof of Theorem 2 in [14]. If $m = 1$, then Theorem 1.4 gives a partial answer for the problem proposed by Pemantle [14]. The proof of Theorem 1.4 is a delicate imitation of the main theorem in Chen and Wei [3]. The proofs of Theorems 1.2 and 1.4 are given in Section 2. At last, we put in Appendix the proofs of Lemmas 2.2 and 2.3, which are needed to prove Theorem 1.4.

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2. PROOFS OF THEOREMS 1.2 AND 1.4

Proof of Theorem 1.2: Since x^2 is a convex function and $\{X_n, \mathcal{F}_n, n \geq 1\}$ is a bounded martingale, we see that $\{X_n^2, \mathcal{F}_n, n \geq 1\}$ is a bounded submartingale and so X_n^2 converges a.s. and in L^1 to X^2. Thus $\lim_{n\to\infty} E(X_n^2) = E(X^2)$. For $n \in \mathbb{N}$, let $Y_n = X_n^2 - x_0$, where $x_0 = w/(w+r)$. Since

$$X_{n+1} = \frac{T_n}{T_{n+1}} X_n + \frac{c_{n+1}\xi_{n+1}}{T_{n+1}} = X_n + \frac{c_{n+1}}{T_{n+1}}(\xi_{n+1} - mX_n)$$

and

$$E[\xi_{n+1} \mid \mathcal{F}_n] = \sum_{k=0}^{m} \frac{k\binom{W_n}{k}\binom{T_n - W_n}{m-k}}{\binom{T_n}{m}} = \frac{mW_n}{T_n} = mX_n,$$

it follows that

$$\begin{aligned}
&E[Y_{n+1} \mid \mathcal{F}_n] \\
&\quad = E\left[\left(X_n + \frac{c_{n+1}(\xi_{n+1} - mX_n)}{T_{n+1}}\right)^2 - x_0 \,\middle|\, \mathcal{F}_n\right] \\
&\quad = X_n^2 - x_0 + 2X_n\left(\frac{c_{n+1}}{T_{n+1}}\right) E[\xi_{n+1} - mX_n \mid \mathcal{F}_n] + \left(\frac{c_{n+1}}{T_{n+1}}\right)^2 E[(\xi_{n+1} - mX_n)^2 \mid \mathcal{F}_n] \\
&\quad = Y_n + \left(\frac{c_{n+1}}{T_{n+1}}\right)^2 \mathrm{Var}(\xi_{n+1} \mid \mathcal{F}_n) \\
&\quad = Y_n + \left(\frac{c_{n+1}}{T_{n+1}}\right)^2 mX_n(1 - X_n)\left(\frac{T_n - m}{T_n - 1}\right). \qquad \textbf{(1)}
\end{aligned}$$

The last identity holds since, conditioned on $\mathcal{F}_n$, ξ_n follows the hypergeometric distribution. Moreover, since $\rho_{n+1} = c_{n+1}m/T_n$ and $T_{n+1} = T_n + c_{n+1}m$, it follows that $c_{n+1} = T_n\rho_{n+1}/m$ and $T_{n+1} = T_n(1+\rho_{n+1})$. Thus $c_{n+1}/T_{n+1} = \rho_{n+1}/[m(1+\rho_{n+1})]$ and so (1) becomes

$$\begin{aligned}
&E[Y_{n+1} \mid \mathcal{F}_n] \\
&\quad = Y_n + \left(\frac{\rho_{n+1}}{1+\rho_{n+1}}\right)^2 \frac{X_n(1-X_n)(T_n - m)}{m(T_n - 1)} \\
&\quad = Y_n\left[1 - \left(\frac{\rho_{n+1}}{1+\rho_{n+1}}\right)^2 \frac{(T_n - m)}{m(T_n - 1)}\right] - (x_0 - X_n)\left(\frac{\rho_{n+1}}{1+\rho_{n+1}}\right)^2 \frac{(T_n - m)}{m(T_n - 1)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&E(Y_{n+1}) \\
&\quad = E[E[Y_{n+1} \mid \mathcal{F}_n]] \\
&\quad = E(Y_n)\left[1 - \left(\frac{\rho_{n+1}}{1+\rho_{n+1}}\right)^2 \frac{(T_n - m)}{m(T_n - 1)}\right] - E(x_0 - X_n)\left(\frac{\rho_{n+1}}{1+\rho_{n+1}}\right)^2 \frac{(T_n - m)}{m(T_n - 1)} \\
&\quad = E(Y_n)\left[1 - \left(\frac{\rho_{n+1}}{1+\rho_{n+1}}\right)^2 \frac{(T_n - m)}{m(T_n - 1)}\right]
\end{aligned}$$

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$$= E(Y_{n-1}) \left[1 - \left(\frac{\rho_n}{1+\rho_n}\right)^2 \frac{(T_{n-1}-m)}{m(T_{n-1}-1)}\right] \left[1 - \left(\frac{\rho_{n+1}}{1+\rho_{n+1}}\right)^2 \frac{(T_n-m)}{m(T_n-1)}\right]$$

$$\vdots$$

$$= Y_0 \prod_{j=0}^{n} \left[1 - \left(\frac{\rho_{j+1}}{1+\rho_{j+1}}\right)^2 \frac{(T_j-m)}{m(T_j-1)}\right].$$

Here the third identity holds since $E(X_n) = x_0$. Thus, the sequence $\{E(Y_n)\}$ converges to 0 if and only if

$$\prod_{j=0}^{\infty} \left\{1 - \frac{\rho_{j+1}^2 (T_j - m)}{(1+\rho_{j+1})^2 m(T_j - 1)}\right\} = 0,$$

which is equivalent to

$$\sum_{j=0}^{\infty} \frac{\rho_{j+1}^2 (T_j - m)}{(1+\rho_{j+1})^2 (T_j - 1)} = \infty.$$

This happened exactly when $\sum_{j=1}^{\infty} \rho_j^2 = \infty$ and so $0 = \lim_{n\to\infty} EY_n = \lim_{n\to\infty} E(X_n^2) - x_0$. Hence $E(X^2) = \lim_{n\to\infty} E(X_n^2) = x_0 = EX$.

Moreover, since $0 \le X \le 1$, if there exists any subinterval $I \subset (0,1)$ satisfying $\Pr(X \in I) > 0$, then $EX - EX^2 = E[X(1-X)1_{\{x\in I\}}] + E[X(1-X)1_{\{X\notin I\}}] \ge E[X(1-X)1_{\{X\in I\}}] > 0$ which contracts to $EX = EX^2$. Thus, $\Pr(X = 0 \text{ or } 1) = 1$ and then $\Pr(X = 1) = EX = x_0$. Hence X follows a Bernoulli distribution with parameter $x_0 = w/(w+r)$. ■

To prove Theorem 1.4, we adopt the argumentation of Chen and Wei [3]. For the convenience of the reader, we present the needed Proposition of [3].

PROPOSITION 2.1 Chen and Wei [3]: *Let $(\Omega_n)_{n\ge1}$ be a sequence of increasing events such that $\Pr(\cup_{n=1}^{\infty}\Omega_n) = 1$. If there exists nonnegative Borel measurable functions $(f_n)_{n\ge1}$ such that $\Pr(\Omega_n \cap X^{-1}(B)) = \int_B f_n(x)dx$ for all Borel sets B, then $f \equiv \lim_{n\to\infty} f_n$ exists almost everywhere, and f is the density of X.*

To finish the proof of Theorem 1.4, we also need the following two lemmas and their proofs are put in Appendix.

LEMMA 2.2: *For $n \ge 1$ and fixed w and r, let $0 < c = \sup_n c_n < \infty$ and*

$$\Omega_n = \{\omega : cm \le W_n(\omega) \le T_n - cm\}.$$

Then $\Omega_{n+1} \supset \Omega_n$ and $P(\cup_{n=1}^{\infty}\Omega_n) = 1$.

LEMMA 2.3: *For fixed ℓ and j with $cm < j < T_\ell - cm$,*

$$\max_{j\le k\le T_n - T_\ell + j} \Pr(W_n = k \mid W_\ell = j) \le \prod_{t=\ell+1}^{n} \frac{1}{1+\rho_t}, \quad n \ge \ell.$$

Proof of Theorem 1.4: Let $(\Omega, \mathcal{F}, P)$ be the probability space. By Proposition 2.1 and Lemma 2.2, it remains to show that restricted on Ω_n, X has a density. It is clear that

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this will hold if we can show that the restriction of X to $\Omega_{\ell j} = \{\omega \in \Omega_\ell : W_\ell(\omega) = j\}$ has a density for each j with $cm \le j \le T_\ell - cm$, where $0 < c = \sup_n c_n < \infty$.

Since $\{c_n\}$ is bounded by c and $T_n = w + r + m\sum_{i=1}^n c_i \ge n$, it follows that $0 \le \lim_{n\to\infty} \rho_n \le \lim_{n\to\infty}(cm/n) = 0$. Thus we can choose an integer ℓ such that $T_\ell > 2\,\mathrm{cm}$ and $\rho_n < 1$ for all $n \ge \ell$. Next, fix j with $cm < j < T_\ell - cm$. For any given $\epsilon > 0$, let $\delta = \epsilon/T_\ell > 0$. Let also $x_1 < x_1' \le x_2 < x_2' \le \cdots < x_s < x_s'$ and $\sum_{i=1}^s (x_i' - x_i) < \delta$. Then

$$
\begin{aligned}
&\sum_{i=1}^s \Pr\bigl(\{x_i \le X \le x_i'\} \bigcap \Omega_{\ell j}\bigr) \\
&\quad = \sum_{i=1}^s E[1_{\{x_i \le X \le x_i'\}} \mid W_\ell = j] \Pr(\Omega_{\ell j}) \\
&\quad \le \sum_{i=1}^s \liminf_{n\to\infty} E[1_{\{x_i \le X_n \le x_i'\}} \mid W_\ell = j] \Pr(\Omega_{\ell j}) \qquad \text{(by Fatou's lemma)} \\
&\quad = \sum_{i=1}^s \liminf_{n\to\infty} P(x_i \le X_n \le x_i' \mid W_\ell = j) \Pr(\Omega_{\ell j}) \\
&\quad = \sum_{i=1}^s \liminf_{n\to\infty} P(T_n x_i \le W_n \le T_n x_i' \mid W_\ell = j) \qquad \text{(since } X_n = W_n/T_n \text{ and } \Pr(\Omega_{\ell j}) \le 1\text{)} \\
&\quad = \sum_{i=1}^s \liminf_{n\to\infty} \left[\sum_{T_n x_i \le k \le T_n x_i'} \Pr(W_n = k \mid W_\ell = j) \right]. \qquad \textbf{(2)}
\end{aligned}
$$

Since $\Pr(\{W_n < j\} \bigcup \{T_n - W_n < T_\ell - j\} \mid W_\ell = j) = 0$ and for $j \le k \le T_n - T_\ell + j$,

$$
\Pr(W_n = k \mid W_\ell = j) \le \max_{j \le k \le T_n - T_\ell + j} \Pr(W_n = k \mid W_\ell = j),
$$

it follows that

$$
\begin{aligned}
&\sum_{T_n x_i \le k \le T_n x_i'} \Pr(W_n = k \mid W_\ell = j) \\
&\qquad \le \sum_{T_n x_i \le k \le T_n x_i'} \left(\max_{j \le k \le T_n - T_\ell + j} \Pr(W_n = k \mid W_\ell = j) \right) \\
&\qquad \le (x_i' - x_i) T_n \left(\max_{j \le k \le T_n - T_\ell + j} \Pr(W_n = k \mid W_\ell = j) \right)
\end{aligned}
$$

and so (2) implies that

$$
\begin{aligned}
&\sum_{i=1}^s \Pr\bigl(\{x_i \le X \le x_i'\} \bigcap \Omega_{\ell j}\bigr) \\
&\quad \le \sum_{i=1}^s (x_i' - x_i) \left[\liminf_{n\to\infty} T_n \left(\max_{j \le k \le T_n - T_\ell + j} \Pr(W_n = k \mid W_\ell = j) \right) \right]
\end{aligned}
$$

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$$< \delta \liminf_{n\to\infty} \left[T_n \left(\max_{j \le k \le T_n - T_\ell + j} \Pr(W_n = k \mid W_\ell = j) \right) \right]$$

$$= \delta T_\ell \liminf_{n\to\infty} \left(\prod_{t=\ell+1}^{n} (1+\rho_t) \right) \left(\max_{j \le k \le T_n - T_\ell + j} \Pr(W_n = k \mid W_\ell = j) \right). \quad (3)$$

Here the last identity holds since $T_n = T_{n-1} + c_n m = T_{n-1}(1+\rho_n) = \cdots = T_\ell \prod_{t=\ell+1}^{n} (1+\rho_t)$. Therefore, by Lemma 2.3 and (3), we have that for $n \ge \ell$,

$$\sum_{i=1}^{s} \Pr\left(\{x_i \le X \le x_i'\} \bigcap \Omega_{\ell j}\right) < \delta T_\ell \liminf_{n\to\infty} \left(\prod_{t=\ell+1}^{n} (1+\rho_t) \right) \left(\prod_{t=\ell+1}^{n} \frac{1}{1+\rho_t} \right)$$

$$= \delta T_\ell$$

$$= \epsilon.$$

By Theorem 31.7 of Billingsley ([1], page 413), the restricted on Ω_n, X has a density. Hence the proof is completed. ■

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APPENDIX

Proof of Lemma 2.2: For $n \geq 1$, let $\Omega'_n = \{\omega : W_n(\omega) > cm\}$ and $\Omega''_n = \{\omega : T_n(\omega) - W_n(\omega) > cm\}$. Then for all $n \geq 1$, $\Omega_n = \Omega'_n \cap \Omega''_n$. Since $W_{n+1} = W_n + c_{n+1}\xi_{n+1} \geq W_n$ and $T_{n+1} - W_{n+1} = T_n + c_{n+1}m - (W_n + c_{n+1}\xi_{n+1}) \geq T_n - W_n \geq cm$, it follows that $\Omega'_n \subseteq \Omega'_{n+1}$ and $\Omega''_n \subseteq \Omega''_{n+1}$ and so $\Omega_n \subseteq \Omega_{n+1}$. Since $\cup_{n=1}^{\infty}\Omega_n = (\cup_{n=1}^{\infty}\Omega'_n) \cap (\cup_{n=1}^{\infty}\Omega''_n)$,

$$\begin{aligned}\Pr(\cup_{n=1}^{\infty}\Omega_n) &= 1 - \Pr((\cup_{n=1}^{\infty}\Omega_n)^c) \\ &= 1 - \Pr([(\cup_{n=1}^{\infty}\Omega'_n) \cap (\cup_{n=1}^{\infty}\Omega''_n)]^c) \\ &= 1 - \Pr((\cap_{n=1}^{\infty}(\Omega'_n)^c) \cup (\cap_{n=1}^{\infty}(\Omega''_n)^c)).\end{aligned}$$

If we can prove that $\Pr(\cap_{n=1}^{\infty}(\Omega'_n)^c) = \Pr(\cap_{n=1}^{\infty}(\Omega''_n)^c) = 0$, then $\Pr(\cup_{n=1}^{\infty}\Omega_n) = 1$. Here we only prove that $\Pr(\cap_{n=1}^{\infty}(\Omega'_n)^c) = 0$ because the proof of $\Pr(\cap_{n=1}^{\infty}(\Omega''_n)^c) = 0$ can be done by exchanging the roles of white balls and red balls. Let E_n be the event that at least one white ball is drawn in the sample of size m at time n. Then

$$\Pr(\cap_{n=1}^{\infty}(\Omega'_n)^c) = \Pr(W_n \leq cm \text{ for all } n) \leq 1 - \Pr(E_n \text{ infinitely often}).$$

From the conditional Borel-Cantelli Lemma (cf. [5], page 32), we see that

$$\Pr(E_n \text{ infinitely often}) = 1 \quad \text{if} \quad \sum_n \Pr(E_{n+1} \mid \mathcal{F}_n) = +\infty \quad \text{a.s.}$$

Therefore, if we can prove that $\sum_n \Pr(E_{n+1} \mid \mathcal{F}_n) = +\infty$ a.s., then $\Pr(\cap_{n=1}^{\infty}(\Omega'_n)^c) = 0$ holds.

Observe that

$$\Pr(E_{n+1} \mid \mathcal{F}_n) \geq \Pr(\xi_{n+1} = 1 \mid \mathcal{F}_n) \geq \frac{W_n}{T_n} \geq \frac{w}{T_n} \quad \text{a.s.}$$

and $T_n = T_{n-1} + c_n m = \cdots = T_0 + m\sum_{i=1}^{n} c_i \leq w + r + mnc$. Then $\sum_n \Pr(E_{n+1} \mid \mathcal{F}_n) \geq \sum_{n=1}^{\infty}[w/(w + r + mnc)] = \infty$. Hence the claim is proved. ■

To prove Lemma 2.3, we need the following four lemmas.

Lemma A.4: *For any $u, v \in \mathbb{Z}$ and any positive integer k,*

$$\sum_{i=0}^{k} \binom{u - iv}{i} \binom{iv - u}{k - i} = (-1)^k v(v+1)^{k-1},$$

where $\binom{a}{b} = a(a-1)\cdots(a-b+1)/b!$ for any integers a, b and $b \geq 0$.

Proof: To simplify the notation, let $f(u, v, k) = \sum_{i=0}^{k} \binom{u-iv}{i}\binom{iv-u}{k-i}$ for any $u, v \in \mathbb{Z}$ and any positive integer k. Then $f(u, v, 1) = \binom{u}{0}\binom{-u}{1} + \binom{u-v}{1}\binom{v-u}{0} = -v$ which implies that this lemma holds for $k = 1$ and for any $u, v \in \mathbb{Z}$. According to step 1 in the proof of Lemma 4.1 in [3], we see that $f(u, v, k)$ is only depending on v and k. Then $f(u, v, k) = f(0, v, k) = f(-1, v, k)$ for any $u, v \in \mathbb{Z}$ and any positive integer k. Thus it remains to prove that $f(0, v, k) = (-1)^k v(v+1)^{k-1}$.

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Observe that

$$\begin{aligned} f(0,v,k) &= \sum_{i=0}^{k} \binom{-iv}{i}\binom{iv}{k-i} \\ &= \binom{0}{k} + \sum_{i=1}^{k} \binom{-iv}{i}\binom{iv}{k-i} \\ &= \sum_{i=1}^{k} \frac{-iv}{i}\binom{-iv-1}{i-1}\binom{iv}{k-i} \qquad \left(\text{since } \binom{s}{t} = \frac{s}{t}\binom{s-1}{t-1}\right) \\ &= -v\sum_{i=1}^{k} \binom{-iv-1}{i-1}\binom{iv}{k-i} \end{aligned} \tag{A.1}$$

and

$$\begin{aligned} f(-1,v,k) &= \sum_{i=1}^{k} \binom{-1-iv}{i}\binom{iv-(-1)}{k-i} \\ &= \sum_{i=1}^{k} \frac{-iv-1-i+1}{i}\binom{-iv-1}{i-1}\binom{iv+1}{k-i} \qquad \left(\text{since } \binom{s}{t} = \frac{s-t+1}{t}\binom{s}{t-1}\right) \\ &= -(v+1)\sum_{i=1}^{k} \binom{-iv-1}{i-1}\binom{iv+1}{k-i} \\ &= -(v+1)\left[\sum_{i=1}^{k} \binom{-iv-1}{i-1}\binom{iv}{k-i} + \sum_{i=1}^{k-1} \binom{-iv-1}{i-1}\binom{iv}{k-i-1}\right]. \end{aligned} \tag{A.2}$$

The last identity holds because of $\binom{s+1}{t} = \binom{s}{t} + \binom{s}{t-1}$. By (A.1) and (A.2), we see that

$$vf(-1,v,k) = (v+1)[f(0,v,k) + f(0,v,k-1)]. \tag{A.3}$$

Since $f(0,v,k) = f(-1,v,k)$, (A.3) implies that

$$\begin{aligned} f(0,v,k) &= -(v+1)f(0,v,k-1) \\ &= (-1)^2(v+1)^2 f(0,v,k-2) \\ &\vdots \\ &= (-1)^{k-1}(v+1)^{k-1} f(0,v,1) \\ &= (-1)^k v(v+1)^{k-1}. \end{aligned}$$

Hence the proof is complete. ∎

LEMMA A.5: *Assume that m and T are positive integers and $T \geq m$. Then*

$$y\sum_{t=0}^{m-1} (-1)^{m-t} \frac{\binom{T}{t}(y+1)^{m-t-1}}{\binom{T}{m}} = \sum_{s=1}^{m} \frac{(-y)^s\binom{m}{s}}{\binom{T}{s}} = \sum_{s=1}^{m} \frac{(-y)^s m^{\underline{s}}}{T^{\underline{s}}},$$

where $x^{\underline{s}} = x(x-1)\cdots(x-s+1) = x!/(x-s)!$ and y is a variable.

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PROOF: Since $\binom{m}{s}/\binom{T}{s} = \binom{T-s}{m-s}/\binom{T}{m}$, it implies that

$$\sum_{s=1}^{m} \frac{(-y)^s \binom{m}{s}}{\binom{T}{s}} = \sum_{s=1}^{m} \frac{(-y)^s \binom{T-s}{m-s}}{\binom{T}{m}}$$

and so to prove Lemma A.5 is just to prove that

$$y \sum_{t=0}^{m-1} (-1)^{m-t} \binom{T}{t} (y+1)^{m-t-1} = \sum_{s=1}^{m} (-y)^s \binom{T-s}{m-s}.$$

Note that

$$\begin{aligned} y \sum_{t=0}^{m-1} (-1)^{m-t} \binom{T}{t} (y+1)^{m-t-1} &= y \sum_{t=0}^{m-1} (-1)^{m-t} \binom{T}{t} \sum_{s=0}^{m-t-1} \binom{m-t-1}{s} y^s \\ &= \sum_{t=0}^{m-1} \sum_{s=1}^{m-t} (-1)^{m-t} \binom{T}{t} \binom{m-t-1}{s-1} y^s \\ &= \sum_{s=1}^{m} \sum_{t=0}^{m-s} (-1)^{m-t} \binom{T}{t} \binom{m-t-1}{s-1} y^s. \end{aligned}$$

Thus, our goal is to prove that

$$\sum_{s=1}^{m} \sum_{t=0}^{m-s} (-1)^{m-t} \binom{T}{t} \binom{m-t-1}{s-1} y^s = \sum_{s=1}^{m} (-1)^s \binom{T-s}{m-s} y^s.$$

Comparing the coefficients of y^s on both sides of the above identity, it remains to prove that for any positive integer m,

$$\sum_{t=0}^{m-s} (-1)^t \binom{T}{t} \binom{m-t-1}{s-1} = (-1)^{m-s} \binom{T-s}{m-s} \quad \text{for } 1 \le s \le m \text{ and } T \ge m. \tag{A.4}$$

Observe that for any positive integer m, if $s = m$ and $T \ge m$, then

$$\sum_{t=0}^{m-m} (-1)^t \binom{T}{t} \binom{m-t-1}{m-1} = (-1)^0 \binom{T}{0} \binom{m-0-1}{m-1} = 1 = (-1)^{m-m} \binom{T-m}{m-m}.$$

Thus for any positive integer m, (A.4) holds if $s = m$ and $T \ge m$. Therefore, it remains to show that (A.4) holds for $m \ge 2$, $1 \le s \le m-1$ and $T \ge m$. Next, we will prove it by induction on m.

Note that for $m = 2$, $s = 1$ and $T \ge 2$,

$$\sum_{t=0}^{2-1} (-1)^t \binom{T}{t} = 1 - T = (-1)^{2-1} \binom{T-1}{2-1}.$$

Thus (A.4) holds for $m = 2$. Assume (A.4) holds for any integer m, $2 \le m \le k-1$. Replacing m with $k-1$, s with $s-1$ in (A.4), we have that for all $1 \le s-1 \le k-1$ and $T \ge k-1$,

$$\sum_{t=0}^{k-s} (-1)^t \binom{T}{t} \binom{k-t-2}{s-2} = (-1)^{k-s} \binom{T-s+1}{k-s}. \tag{A.5}$$

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Similarly, replacing m with $k-2$, s with $s-1$ and T with $T-1$ in (A.4), we have that for all $1 \le s-1 \le k-2$ and $T-1 \ge k-2$,

$$\sum_{t=0}^{k-s-1} (-1)^t \binom{T-1}{t} \binom{k-t-3}{s-2} = (-1)^{k-s-1} \binom{T-s}{k-s-1}. \tag{A.6}$$

Then for $m=k$ and for any $2 \le s \le k-1$ and $T \ge k$,

$$\begin{aligned}
&\sum_{t=0}^{k-s} (-1)^t \binom{T}{t} \binom{k-t-1}{s-1} \\
&\quad = \sum_{t=0}^{k-s} (-1)^t \binom{T}{t} \frac{k-t-1}{s-1} \binom{k-t-2}{s-2} \qquad \left(\text{since } \binom{s}{t} = \frac{s}{t} \binom{s-1}{t-1} \right) \\
&\quad = \frac{k-1}{s-1} \sum_{t=0}^{k-s} (-1)^t \binom{T}{t} \binom{k-t-2}{s-2} - \sum_{t=1}^{k-s} (-1)^t \frac{t}{s-1} \binom{T}{t} \binom{k-t-2}{s-2} \\
&\quad = \frac{k-1}{s-1} (-1)^{k-s} \binom{T-s+1}{k-s} - \sum_{t=1}^{k-s} (-1)^t \frac{t}{s-1} \binom{T}{t} \binom{k-t-2}{s-2} \qquad \text{(by (A.5))} \\
&\quad = (-1)^{k-s} \frac{k-1}{s-1} \binom{T-s+1}{k-s} - \frac{T}{s-1} \sum_{t=1}^{k-s} (-1)^t \binom{T-1}{t-1} \binom{k-t-2}{s-2} \\
&\quad = (-1)^{k-s} \frac{k-1}{s-1} \binom{T-s+1}{k-s} - \frac{T}{s-1} \sum_{t=0}^{k-s-1} (-1)^{t+1} \binom{T-1}{t} \binom{k-t-3}{s-2} \\
&\quad = (-1)^{k-s} \frac{k-1}{s-1} \binom{T-s+1}{k-s} - (-1)^{k-s} \frac{T}{s-1} \binom{T-s}{k-s-1} \qquad \text{(by (A.6))} \\
&\quad = (-1)^{k-s} \binom{T-s}{k-s}.
\end{aligned}$$

Thus, (A.4) holds for $m=k$.

Moreover, from the definition of $x^{\underline{s}}$, it follows that

$$\sum_{s=1}^{m} \frac{(-y)^s \binom{m}{s}}{\binom{T}{s}} = \sum_{s=1}^{m} \frac{(-y)^s m^{\underline{s}}}{T^{\underline{s}}}$$

and hence the proof is completed. ■

Lemma A.6: *Assume that a, m and T are positive integers and $T \ge m$. Then*

$$\sum_{i=0}^{m} \frac{\binom{k-ia}{i} \binom{T-k+ia}{m-i}}{\binom{T}{m}} = \sum_{i=0}^{m} \frac{(-a)^i m^{\underline{i}}}{T^{\underline{i}}}.$$

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PROOF: Observe that

$$\begin{aligned}
&\sum_{i=0}^{m}\binom{k-ia}{i}\binom{T-k+ia}{m-i}\\
&\quad=\sum_{s=0}^{m}T^s\sum_{i=0}^{m-s}\binom{k-ia}{i}\sum_{t=s}^{m-i}\frac{\left[{t\atop s}\right](-1)^{t-s}\binom{ia-k}{m-t-i}}{t!} \qquad \text{(by Remark 5 in [2])}\\
&\quad=\sum_{s=0}^{m}\sum_{i=0}^{m-s}\sum_{t=s}^{m-i}\frac{\left[{t\atop s}\right](-1)^{t-s}T^s}{t!}\binom{k-ia}{i}\binom{ia-k}{m-t-i}\\
&\quad=\sum_{s=0}^{m}\sum_{t=s}^{m}\sum_{i=0}^{m-t}\frac{\left[{t\atop s}\right](-1)^{t-s}T^s}{t!}\binom{k-ia}{i}\binom{ia-k}{m-t-i}\\
&\quad=\sum_{t=0}^{m}\sum_{s=0}^{t}\frac{\left[{t\atop s}\right](-1)^{t-s}T^s}{t!}\left(\sum_{i=0}^{m-t}\binom{k-ia}{i}\binom{ia-k}{m-t-i}\right)\\
&\quad=\sum_{t=0}^{m-1}\sum_{s=0}^{t}\frac{\left[{t\atop s}\right](-1)^{t-s}T^s}{t!}\left(\sum_{i=0}^{m-t}\binom{k-ia}{i}\binom{ia-k}{m-t-i}\right)+\sum_{s=0}^{m}\frac{\left[{m\atop s}\right](-1)^{m-s}T^s}{m!}\\
&\quad=\sum_{t=0}^{m-1}\sum_{s=0}^{t}\frac{\left[{t\atop s}\right](-1)^{t-s}T^s}{t!}(-1)^{m-t}a(a+1)^{m-t-1}+\sum_{s=0}^{m}\frac{\left[{m\atop s}\right](-1)^{m-s}T^s}{m!} \qquad \text{(by Lemma A.4)}\\
&\quad=\sum_{s=0}^{m}\frac{\left[{m\atop s}\right](-1)^{m-s}T^s}{m!}+\sum_{t=0}^{m-1}\left(\sum_{s=0}^{t}\left[{t\atop s}\right](-1)^{t-s}T^s\right)\frac{(-1)^{m-t}a(a+1)^{m-t-1}}{t!}, \qquad \textbf{(A.7)}
\end{aligned}$$

where $\left[{t\atop s}\right]$ is the unsigned Stirling numbers of the first kind. Since $\sum_{s=0}^{k}\left[{k\atop s}\right](-1)^{k-s}T^s=T!/(T-k)!$ for any nonnegative integer k, (A.7) becomes that

$$\begin{aligned}
\sum_{i=0}^{m}\binom{k-ia}{i}\binom{T-k+ia}{m-i}&=\frac{T!}{m!(T-m)!}+\sum_{t=0}^{m-1}\frac{T!}{(T-t)!}\frac{(-1)^{m-t}a(a+1)^{m-t-1}}{t!}\\
&=\binom{T}{m}+a\sum_{t=0}^{m-1}(-1)^{m-t}\binom{T}{t}(a+1)^{m-t-1}.
\end{aligned}$$

Applying Lemma A.5,

$$\sum_{i=0}^{m}\frac{\binom{k-ia}{i}\binom{T-k+ia}{m-i}}{\binom{T}{m}}=1+a\sum_{t=0}^{m-1}(-1)^{m-t}\frac{\binom{T}{t}(a+1)^{m-t-1}}{\binom{T}{m}}=\sum_{s=0}^{m}\frac{(-a)^s m^{\underline{s}}}{T^{\underline{s}}}$$

and hence the proof is completed. ∎

LEMMA A.7: *Assume that a, m and T are positive integers and $T\geq m$. Then*

$$\sum_{i=0}^{m}\binom{m}{i}\left(\frac{k-ia}{T}\right)^i\left(1-\frac{k-ia}{T}\right)^{m-i}=\sum_{i=0}^{m}m^{\underline{i}}\left(\frac{-a}{T}\right)^i.$$

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PROOF: Observe that

$$\begin{aligned}
&\sum_{i=0}^{m}\binom{m}{i}\left(\frac{k-ia}{T}\right)^{i}\left(1-\frac{k-ia}{T}\right)^{m-i}\\
&\quad=\sum_{i=0}^{m}\binom{m}{i}\left(\frac{k-ia}{T}\right)^{i}\left[\sum_{j=0}^{m-i}\binom{m-i}{j}(-1)^{m-i-j}\left(\frac{k-ia}{T}\right)^{m-i-j}\right]\\
&\quad=\sum_{j=0}^{m}\sum_{i=0}^{m-j}\binom{m}{i}\binom{m-i}{j}(-1)^{m-i-j}\left(\frac{k-ia}{T}\right)^{m-j}\\
&\quad=\sum_{j=0}^{m}\binom{m}{j}\sum_{i=0}^{m-j}\binom{m-j}{i}(-1)^{m-i-j}\left(\frac{k-ia}{T}\right)^{m-j}\\
&\quad=1+\sum_{j=0}^{m-1}\binom{m}{j}\sum_{i=0}^{m-j}\binom{m-j}{i}(-1)^{m-i-j}\left(\frac{k-ia}{T}\right)^{m-j}. \qquad \textbf{(A.8)}
\end{aligned}$$

Moreover, for $0\le j\le m-1$,

$$\begin{aligned}
&\sum_{i=0}^{m-j}\binom{m-j}{i}(-1)^{m-i-j}\left(\frac{k-ia}{T}\right)^{m-j}\\
&\quad=\sum_{i=1}^{m-j}\binom{m-j}{i}(-1)^{m-i-j}\left(\frac{k-ia}{T}\right)^{m-j}+(-1)^{m-j}\left(\frac{k}{T}\right)^{m-j}\\
&\quad=\sum_{i=1}^{m-j}\sum_{\ell=0}^{m-j}\binom{m-j}{i}\binom{m-j}{\ell}(-1)^{m-i-j}\left(\frac{k}{T}\right)^{\ell}\left(\frac{-ia}{T}\right)^{m-j-\ell}+(-1)^{m-j}\left(\frac{k}{T}\right)^{m-j}\\
&\quad=\sum_{\ell=0}^{m-j}\left[\sum_{i=1}^{m-j}\binom{m-j}{i}(-1)^{m-j-i}i^{m-j-\ell}\right]\binom{m-j}{\ell}\left(\frac{k}{T}\right)^{\ell}\left(\frac{-a}{T}\right)^{m-j-\ell}+(-1)^{m-j}\left(\frac{k}{T}\right)^{m-j}.
\end{aligned}$$

(A.9)

Let $g(y)=(y-1)^n=\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}y^i$ for all $n\in\mathbb{N}$. Then for $0\le\ell\le n-1$, $\sum_{i=\ell}^{n}\binom{n}{i}(-1)^{n-i}i^{\underline{\ell}}=g^{(\ell)}(1)=0$, where $g^{(\ell)}$ is the ℓth derivative of g. This leads to, by simple calculation,

$$\sum_{i=1}^{n}\binom{n}{i}(-1)^{n-i}=(-1)^{n+1}\quad\text{and}\quad\sum_{i=1}^{n}\binom{n}{i}(-1)^{n-i}i^{\ell}=0,\ \text{for } 1\le\ell\le n-1. \qquad \textbf{(A.10)}$$

Moreover,

$$\begin{aligned}
&\sum_{i=1}^{n}\binom{n}{i}(-1)^{n-i}i^{n}\\
&\quad=n\left[\sum_{i=1}^{n}\binom{n-1}{i-1}(-1)^{n-i}i^{n-1}\right]\\
&\quad=n\left[(-1)^{n-1}+\sum_{i=2}^{n}\binom{n-1}{i-1}(-1)^{n-i}i^{n-1}\right]
\end{aligned}$$

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$$
\begin{aligned}
&= n\left[(-1)^{n-1} + \sum_{i=1}^{n-1}\binom{n-1}{i}(-1)^{n-i-1}(i+1)^{n-1}\right] \\
&= n\left[(-1)^{n-1} + \sum_{i=1}^{n-1}\binom{n-1}{i}(-1)^{n-1-i}\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}i^{\ell}\right] \\
&= n\left[(-1)^{n-1} + \sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\sum_{i=1}^{n-1}\binom{n-1}{i}(-1)^{n-1-i}i^{\ell}\right] \\
&= n\left[(-1)^{n-1} + (-1)^{n} + \sum_{i=1}^{n-1}\binom{n-1}{i}(-1)^{n-1-i}i^{n-1}\right] \qquad \text{(by applying (A.10))} \\
&= n\left[\sum_{i=1}^{n-1}\binom{n-1}{i}(-1)^{n-1-i}i^{n-1}\right].
\end{aligned}
$$

Thus we have that

$$
\begin{aligned}
\sum_{i=1}^{n}\binom{n}{i}(-1)^{n-i}i^{n} &= n\left[\sum_{i=1}^{n-1}\binom{n-1}{i}(-1)^{n-1-i}i^{n-1}\right] \\
&= n(n-1)\left[\sum_{i=1}^{n-2}\binom{n-2}{i}(-1)^{n-2-i}i^{n-2}\right] \\
&\vdots \\
&= n!. \qquad \textbf{(A.11)}
\end{aligned}
$$

Then by (A.9), it follows that

$$
\begin{aligned}
&\sum_{i=0}^{m-j}\binom{m-j}{i}(-1)^{m-i-j}\left(\frac{k-ia}{T}\right)^{m-j} \\
&\quad= \left[\sum_{i=1}^{m-j}\binom{m-j}{i}(-1)^{m-j-i}i^{m-j}\right]\left(\frac{-a}{T}\right)^{m-j} \\
&\qquad+ \sum_{\ell=1}^{m-j-1}\left[\sum_{i=1}^{m-j}\binom{m-j}{i}(-1)^{m-j-i}i^{m-j-\ell}\right]\binom{m-j}{\ell}\left(\frac{k}{T}\right)^{\ell}\left(\frac{-a}{T}\right)^{m-j-\ell} \\
&\qquad+ \left[\sum_{i=1}^{m-j}\binom{m-j}{i}(-1)^{m-j-i}\right]\left(\frac{k}{T}\right)^{m-j} + (-1)^{m-j}\left(\frac{k}{T}\right)^{m-j} \\
&\quad= (m-j)!\left(\frac{-a}{T}\right)^{m-j} + (-1)^{m-j+1}\left(\frac{k}{T}\right)^{m-j} \\
&\qquad+ (-1)^{m-j}\left(\frac{k}{T}\right)^{m-j} \qquad \text{(by using (A.11) and (A.10))} \\
&\quad= (m-j)!\left(\frac{-a}{T}\right)^{m-j}
\end{aligned}
$$

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which implies that (A.8) becomes

$$\begin{aligned}\sum_{i=0}^{m}\binom{m}{i}\left(\frac{k-ia}{T}\right)^{i}\left(1-\frac{k-ia}{T}\right)^{m-i} &= 1+\sum_{j=0}^{m-1}\binom{m}{j}(m-j)!\left(\frac{-a}{T}\right)^{m-j}\\ &= \sum_{j=0}^{m}\frac{m!}{j!}\left(\frac{-a}{T}\right)^{m-j}\\ &= \sum_{i=0}^{m}\frac{m!}{(m-i)!}\left(\frac{-a}{T}\right)^{i}\\ &= \sum_{i=0}^{m}m^{\underline{i}}\left(\frac{-a}{T}\right)^{i}.\end{aligned}$$

Hence the proof is complete. ■

Now we are ready to prove Lemma 2.3.

Proof of Lemma 2.3: Note that for $n \geq \ell$,

$$\begin{aligned}&\max_{j\leq k\leq T_n-T_\ell+j}\Pr(W_n=k\mid W_\ell=j)\\ &\quad=\max_{j\leq k\leq T_n-T_\ell+j}\left\{\sum_{i=0}^{m}\Pr(W_n=k\mid W_{n-1}=k-ic_n)\Pr(W_{n-1}=k-ic_n\mid W_\ell=j)\right\}\\ &\quad\leq\max_{j\leq k\leq T_{n-1}-T_\ell+j}\Pr(W_{n-1}=k\mid W_\ell=j)\max_{j\leq k\leq T_n-T_\ell+j}\left\{\sum_{i=0}^{m}\Pr(W_n=k\mid W_{n-1}=k-ic_n)\right\}.\end{aligned}\tag{A.12}$$

We see that for $j\leq k\leq T_n-T_\ell+j$ and $0\leq i\leq m$,

$$\Pr(W_n=k\mid W_{n-1}=k-ic_n)=\frac{\binom{k-ic_n}{i}\binom{T_{n-1}-k+ic_n}{m-i}}{\binom{T_{n-1}}{m}}$$

and so

$$\begin{aligned}&\max_{j\leq k\leq T_n-T_\ell+j}\left\{\sum_{i=0}^{m}\Pr(W_n=k\mid W_{n-1}=k-ic_n)\right\}\\ &\quad=\max_{j\leq k\leq T_n-T_\ell+j}\left\{\sum_{i=0}^{m}\frac{\binom{k-ic_n}{i}\binom{T_{n-1}-k+ic_n}{m-i}}{\binom{T_{n-1}}{m}}\right\}\\ &\quad=\sum_{s=0}^{m}\frac{(-c_n)^s m^{\underline{s}}}{T_{n-1}^{\underline{s}}}\qquad\text{(by Lemmas A.6)}\\ &\quad=\sum_{s=0}^{m}(-\rho_n)^s\left(\frac{T_{n-1}}{m}\right)^s\frac{m^{\underline{s}}}{T_{n-1}^{\underline{s}}}\qquad\text{(by }\rho_n=c_nm/T_{n-1}).\end{aligned}\tag{A.13}$$

Since $T_{n-1}\geq T_\ell\geq 2\,\text{cm}$ for all $n\geq\ell$, we have that

$$\frac{T_{n-1}(m-s)}{m(T_{n-1}-s)}\leq 1$$

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for all $0 \le s \le m$ and that

$$0 < \rho_n = \frac{c_n m}{T_{n-1}} \le \frac{cm}{T_{n-1}} < 1.$$

Thus from (A.13), we have that for $n \ge \ell$,

$$\max_{j \le k \le T_n - T_\ell + j} \left\{ \sum_{i=0}^{m} \Pr(W_n = k \mid W_{n-1} = k - ic_n) \right\} \le \sum_{s=0}^{m} (-\rho_n)^s \le \frac{1}{1+\rho_n}.$$

Therefore, by (A.12), we have that for $n \ge \ell$,

$$\begin{aligned}
&\max_{j \le k \le T_n - T_\ell + j} \Pr(W_n = k \mid W_\ell = j) \\
&\quad \le \max_{j \le k \le T_{n-1} - T_\ell + j} \Pr(W_{n-1} = k \mid W_\ell = j) \frac{1}{1+\rho_n} \\
&\quad \le \max_{j \le k \le T_{n-2} - T_\ell + j} \Pr(W_{n-2} = k \mid W_\ell = j) \frac{1}{1+\rho_{n-1}} \frac{1}{1+\rho_n} \\
&\quad \vdots \\
&\quad \le \max_{j \le k \le T_\ell - T_\ell + j} \Pr(W_\ell = k \mid W_\ell = j) \prod_{t=\ell+1}^{n} \frac{1}{1+\rho_t} \\
&\quad = \prod_{t=\ell+1}^{n} \frac{1}{1+\rho_t}.
\end{aligned}$$

Hence the proof is completed. ■

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